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Scalars and Vectors

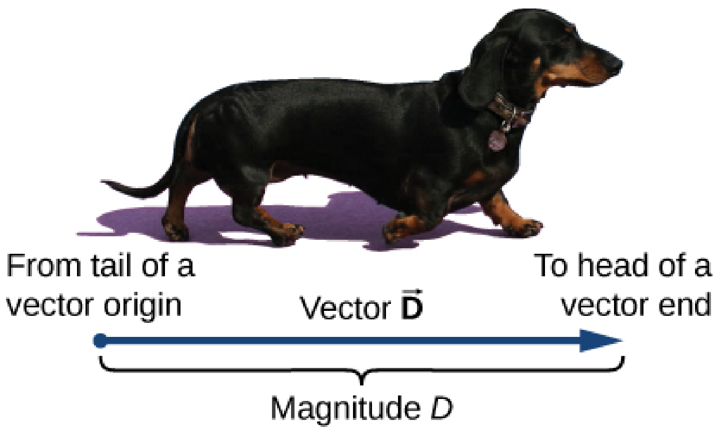
By the end of this section, you will be able to:

- Describe the difference between vector and scalar quantities.
- Identify the magnitude and direction of a vector.
- Explain the effect of multiplying a vector quantity by a scalar.
- Describe how one-dimensional vector quantities are added or subtracted.
- Explain the geometric construction for the addition or subtraction of vectors in a plane.
- Distinguish between a vector equation and a scalar equation.

Many familiar physical quantities can be specified completely by giving a single number and the appropriate unit. For example, “a class period lasts 50 min” or “the gas tank in my car holds 65 L” or “the distance between two posts is 100 m.” A physical quantity that can be specified completely in this manner is called a **scalar quantity**. Scalar is a synonym of “number.” Time, mass, distance, length, volume, temperature, and energy are examples of **scalar** quantities.

Scalar quantities that have the same physical units can be added or subtracted according to the usual rules of algebra for numbers. For example, a class ending 10 min earlier than 50 min lasts $50\text{ min}-10\text{ min}=40\text{ min}$. Similarly, a 60-cal serving of corn followed by a 200-cal serving of donuts gives $60\text{ cal}+200\text{ cal}=260\text{ cal}$ of energy. When we multiply a scalar quantity by a number, we obtain the same scalar quantity but with a larger (or smaller) value. For example, if yesterday’s breakfast had 200 cal of energy and today’s breakfast has four times as much energy as it had yesterday, then today’s breakfast has $4(200\text{ cal})=800\text{ cal}$ of energy. Two scalar quantities can also be multiplied or divided by each other to form a derived scalar quantity. For example, if a train covers a distance of 100 km in 1.0 h, its speed is 100.0 km/1.0 h = 27.8 m/s, where the speed is a derived scalar quantity obtained by dividing distance by time.

Many physical quantities, however, cannot be described completely by just a single number of physical units. For example, when the U.S. Coast Guard dispatches a ship or a helicopter for a rescue mission, the rescue team must know not only the distance to the distress signal, but also the direction from which the signal is coming so they can get to its origin as quickly as possible. Physical quantities specified completely by giving a number of units (magnitude) and a direction are called **vector quantities**. Examples of vector quantities include displacement, velocity, position, force, and torque. In the language of mathematics, physical vector quantities are represented by mathematical objects called **vectors** ([link]). We can add or subtract two vectors, and we can multiply a vector by a scalar or by another vector, but we cannot divide by a vector. The operation of division by a vector is not defined.



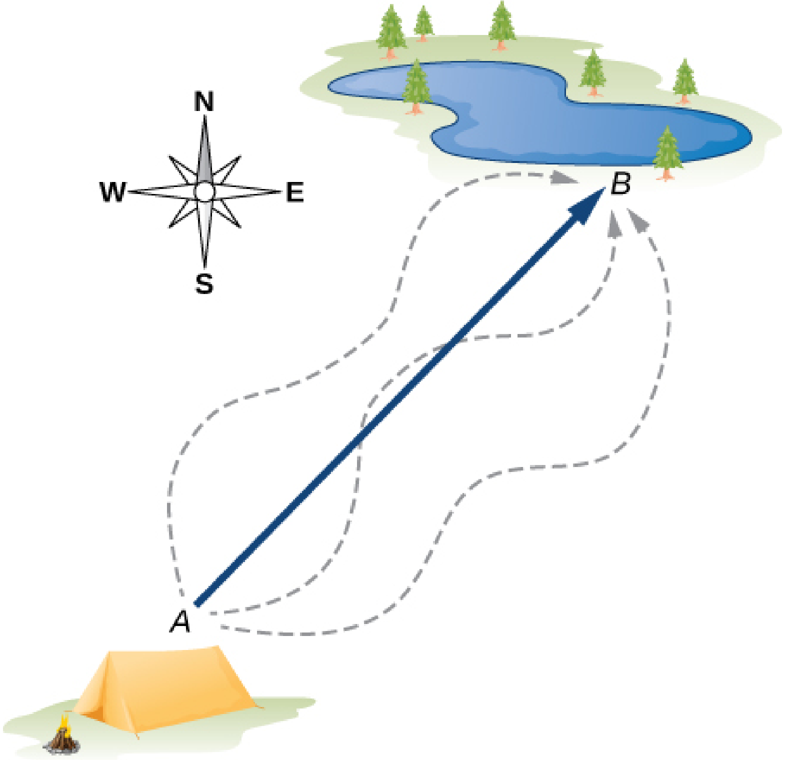
We draw a vector from the initial point or origin (called the “tail” of a vector) to the end or terminal point (called the “head” of a vector), marked by an arrowhead. Magnitude is the length of a vector and is always a positive scalar quantity. (credit "photo": modification of work by Cate Sevilla)

Let’s examine vector algebra using a graphical method to be aware of basic terms and to develop a qualitative understanding. In practice, however, when it comes to solving physics problems, we use analytical methods, which we’ll see in the next section. Analytical methods are more simple computationally and more accurate than graphical methods. From now on, to distinguish between a vector and a scalar quantity, we adopt the common convention that a letter in bold type with an arrow above it denotes a vector, and a letter without an arrow denotes a scalar. For example, a distance of 2.0 km, which is a scalar quantity, is denoted by $d = 2.0$ km, whereas a displacement of 2.0 km in some direction, which is a vector quantity, is denoted by $\vec{\mathbf{d}}$.

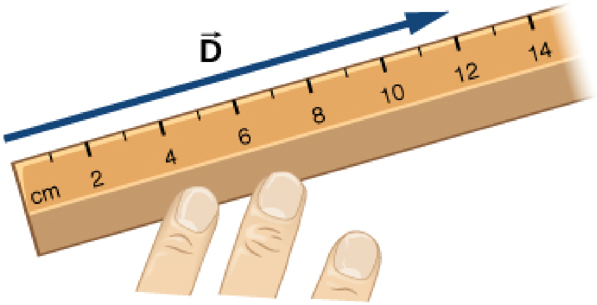
Suppose you tell a friend on a camping trip that you have discovered a terrific fishing hole 6 km from your tent. It is unlikely your friend would be able to find the hole easily unless you also communicate the direction in which it can be found with respect to your campsite. You may say, for example, “Walk about 6 km northeast from my tent.” The key concept here is that you have to give not one but *two* pieces of information—namely, the distance or magnitude (6 km) *and* the direction (northeast).

Displacement is a general term used to describe a *change in position*, such as during a trip from the tent to the fishing hole. Displacement is an example of a vector quantity. If you walk from the tent (location A) to the hole (location B), as shown in [link], the vector $\vec{\mathbf{D}}$, representing your **displacement**, is drawn as the arrow that originates at point

A and ends at point *B*. The arrowhead marks the end of the vector. The direction of the displacement vector $\vec{\mathbf{D}}$ is the direction of the arrow. The length of the arrow represents the **magnitude** *D* of vector $\vec{\mathbf{D}}$. Here, *D* = 6 km. Since the magnitude of a vector is its length, which is a positive number, the magnitude is also indicated by placing the absolute value notation around the symbol that denotes the vector; so, we can write equivalently that $D \equiv \left|\vec{\mathbf{D}}\right|$. To solve a vector problem graphically, we need to draw the vector $\vec{\mathbf{D}}$ to scale. For example, if we assume 1 unit of distance (1 km) is represented in the drawing by a line segment of length *u* = 2 cm, then the total displacement in this example is represented by a vector of length $d = 6u = 6(2\text{ cm}) = 12\text{ cm}$, as shown in [link]. Notice that here, to avoid confusion, we used $D = 6\text{ km}$ to denote the magnitude of the actual displacement and *d* = 12 cm to denote the length of its representation in the drawing.



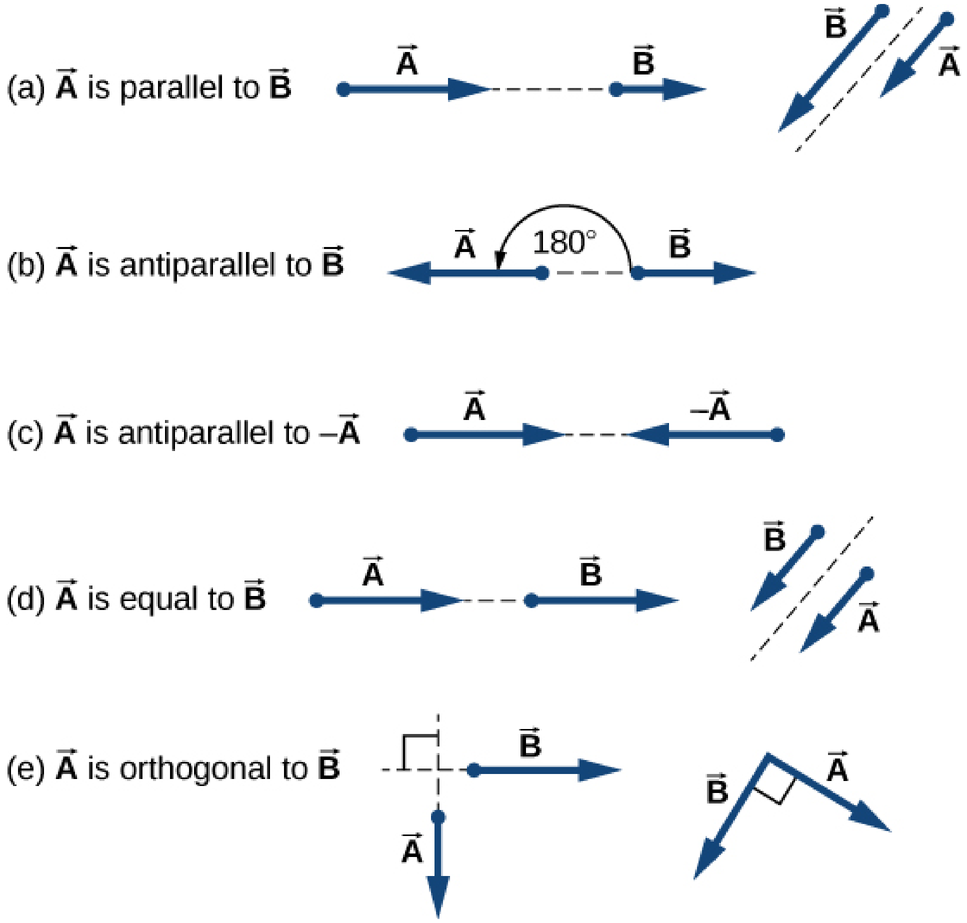
The displacement vector from point *A* (the initial position at the campsite) to point *B* (the final position at the fishing hole) is indicated by an arrow with origin at point *A* and end at point *B*. The displacement is the same for any of the actual paths (dashed curves) that may be taken between points *A* and *B*.



A displacement $\vec{\mathbf{D}}$ of magnitude 6 km is drawn to scale as a vector of length 12 cm when the length of 2 cm represents 1 unit of displacement (which in this case is 1 km).

Suppose your friend walks from the campsite at A to the fishing pond at B and then walks back: from the fishing pond at B to the campsite at A. The magnitude of the displacement vector $\vec{\mathbf{D}}_{AB}$ from A to B is the same as the magnitude of the displacement vector $\vec{\mathbf{D}}_{BA}$ from B to A (it equals 6 km in both cases), so we can write $D_{AB} = D_{BA}$. However, vector $\vec{\mathbf{D}}_{AB}$ is *not* equal to vector $\vec{\mathbf{D}}_{BA}$ because these two vectors have different directions: $\vec{\mathbf{D}}_{AB} \neq \vec{\mathbf{D}}_{BA}$. In [link], vector $\vec{\mathbf{D}}_{BA}$ would be represented by a vector with an origin at point B and an end at point A, indicating vector $\vec{\mathbf{D}}_{BA}$ points to the southwest, which is exactly 180° opposite to the direction of vector $\vec{\mathbf{D}}_{AB}$. We say that vector $\vec{\mathbf{D}}_{BA}$ is **antiparallel** to vector $\vec{\mathbf{D}}_{AB}$ and write $\vec{\mathbf{D}}_{AB} = -\vec{\mathbf{D}}_{BA}$, where the minus sign indicates the antiparallel direction.

Two vectors that have identical directions are said to be **parallel vectors**—meaning, they are *parallel* to each other. Two parallel vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are equal, denoted by $\vec{\mathbf{A}} = \vec{\mathbf{B}}$, if and only if they have equal magnitudes $\left|\vec{\mathbf{A}}\right| = \left|\vec{\mathbf{B}}\right|$. Two vectors with directions perpendicular to each other are said to be **orthogonal vectors**. These relations between vectors are illustrated in [link].


(a) $\vec{\mathbf{A}}$ is parallel to $\vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
$\vec{\mathbf{B}}$
$\vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
(b) $\vec{\mathbf{A}}$ is antiparallel to $\vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
180°
$\vec{\mathbf{B}}$
(c) $\vec{\mathbf{A}}$ is antiparallel to $-\vec{\mathbf{A}}$
$\vec{\mathbf{A}}$
$-\vec{\mathbf{A}}$
(d) $\vec{\mathbf{A}}$ is equal to $\vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
$\vec{\mathbf{B}}$
$\vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
(e) $\vec{\mathbf{A}}$ is orthogonal to $\vec{\mathbf{B}}$
$\vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
$\vec{\mathbf{A}}$
$\vec{\mathbf{B}}$

Various relations between two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. (a) $\vec{\mathbf{A}} \neq \vec{\mathbf{B}}$ because $A \neq B$. (b) $\vec{\mathbf{A}} \neq \vec{\mathbf{B}}$ because they are not parallel and $A \neq B$. (c) $\vec{\mathbf{A}} \neq -\vec{\mathbf{A}}$ because they have different directions (even though $\left|\vec{\mathbf{A}}\right| = \left|-\vec{\mathbf{A}}\right| = A$). (d) $\vec{\mathbf{A}} = \vec{\mathbf{B}}$ because they are parallel *and* have identical magnitudes $A = B$. (e) $\vec{\mathbf{A}} \neq \vec{\mathbf{B}}$ because they have different directions (are not parallel); here, their directions differ by 90°—meaning, they are orthogonal.

Note:
Exercise:

Problem:

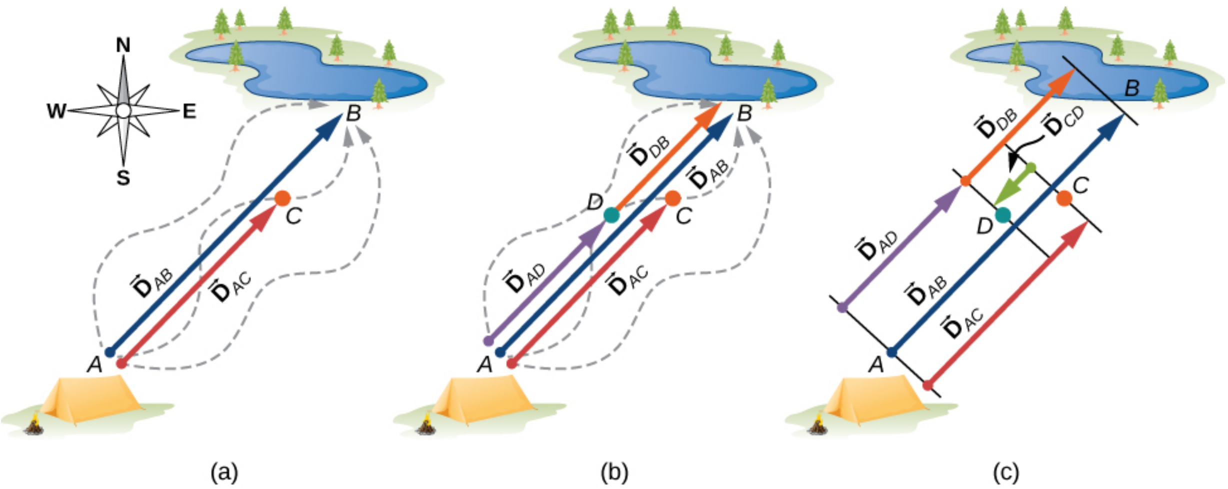
Check Your Understanding Two motorboats named *Alice* and *Bob* are moving on a lake. Given the information about their velocity vectors in each of the following situations, indicate whether their velocity vectors are equal or otherwise. (a) *Alice* moves north at 6 knots and *Bob* moves west at 6 knots. (b) *Alice* moves west at 6 knots and *Bob* moves west at 3 knots. (c) *Alice* moves northeast at 6 knots and *Bob* moves south at 3 knots. (d) *Alice* moves northeast at 6 knots and *Bob* moves southwest at 6 knots. (e) *Alice* moves northeast at 2 knots and *Bob* moves closer to the shore northeast at 2 knots.

Solution:

a. not equal because they are orthogonal; b. not equal because they have different magnitudes; c. not equal because they have different magnitudes and directions; d. not equal because they are antiparallel; e. equal.

Algebra of Vectors in One Dimension

Vectors can be multiplied by scalars, added to other vectors, or subtracted from other vectors. We can illustrate these vector concepts using an example of the fishing trip seen in [link].



Displacement vectors for a fishing trip. (a) Stopping to rest at point *C* while walking from camp (point *A*) to the pond (point *B*). (b) Going back for the dropped tackle box (point *D*). (c) Finishing up at the fishing pond.

Suppose your friend departs from point A (the campsite) and walks in the direction to point B (the fishing pond), but, along the way, stops to rest at some point C located three-quarters of the distance between A and B, beginning from point A ([link](a)). What is his displacement vector $\vec{\mathbf{D}}_{AC}$ when he reaches point C? We know that if he walks all the way to B, his displacement vector relative to A is $\vec{\mathbf{D}}_{AB}$, which has magnitude $D_{AB} = 6\,\text{km}$ and a direction of northeast. If he walks only a 0.75 fraction of the total distance, maintaining the northeasterly direction, at point C he must be $0.75D_{AB} = 4.5\,\text{km}$ away from the campsite at A. So, his displacement vector at the rest point C has magnitude $D_{AC} = 4.5\,\text{km} = 0.75D_{AB}$ and is parallel to the displacement vector $\vec{\mathbf{D}}_{AB}$. All of this can be stated succinctly in the form of the following **vector equation**:

Equation:

$$\vec{\mathbf{D}}_{AC} = 0.75\vec{\mathbf{D}}_{AB}.$$

In a vector equation, both sides of the equation are vectors. The previous equation is an example of a vector multiplied by a positive scalar (number) $\alpha = 0.75$. The result, $\vec{\mathbf{D}}_{AC}$, of such a multiplication is a new vector with a direction parallel to the direction of the original vector $\vec{\mathbf{D}}_{AB}$.

In general, when a vector $\vec{\mathbf{A}}$ is multiplied by a *positive* scalar α, the result is a new vector $\vec{\mathbf{B}}$ that is *parallel* to $\vec{\mathbf{A}}$:

Note:
Equation:

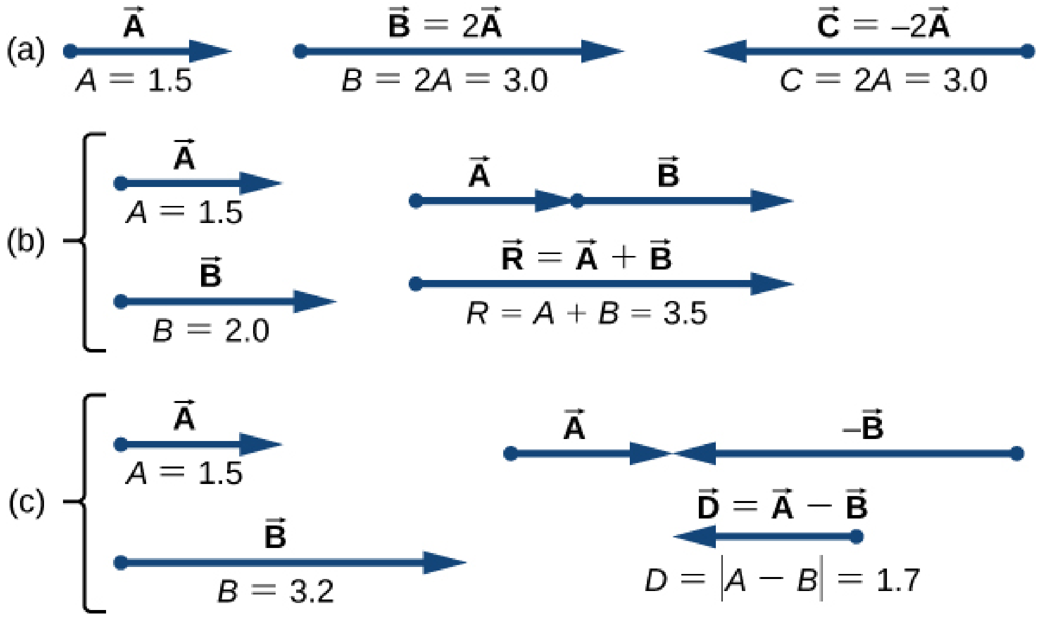
$$\vec{\mathbf{B}} = \alpha\vec{\mathbf{A}}.$$

The magnitude $\left|\vec{\mathbf{B}}\right|$ of this new vector is obtained by multiplying the magnitude $\left|\vec{\mathbf{A}}\right|$ of the original vector, as expressed by the **scalar equation**:

Note:
Equation:

$$B=|\alpha|A.$$

In a scalar equation, both sides of the equation are numbers. [link] is a scalar equation because the magnitudes of vectors are scalar quantities (and positive numbers). If the scalar α is *negative* in the vector equation [link], then the magnitude $\left|\vec{\mathbf{B}}\right|$ of the new vector is still given by [link], but the direction of the new vector $\vec{\mathbf{B}}$ is *antiparallel* to the direction of $\vec{\mathbf{A}}$. These principles are illustrated in [link](a) by two examples where the length of vector $\vec{\mathbf{A}}$ is 1.5 units. When $\alpha=2$, the new vector $\vec{\mathbf{B}}=2\vec{\mathbf{A}}$ has length $B=2A=3.0$ units (twice as long as the original vector) and is parallel to the original vector. When $\alpha=-2$, the new vector $\vec{\mathbf{C}}=-2\vec{\mathbf{A}}$ has length $C=|-2|A=3.0$ units (twice as long as the original vector) and is antiparallel to the original vector.


(a)
$\vec{A}$
$A=1.5$
$\vec{B}=2\vec{A}$
$B=2A=3.0$
$\vec{C}=-2\vec{A}$
$C=2A=3.0$
(b)
$\vec{A}$
$A=1.5$
$\vec{B}$
$B=2.0$
$\vec{A}$
$\vec{B}$
$\vec{R}=\vec{A}+\vec{B}$
$R=A+B=3.5$
(c)
$\vec{A}$
$A=1.5$
$\vec{B}$
$B=3.2$
$\vec{A}$
$-\vec{B}$
$\vec{D}=\vec{A}-\vec{B}$
$D=|A-B|=1.7$

Algebra of vectors in one dimension. (a) Multiplication by a scalar. (b) Addition of two vectors ($\vec{\mathbf{R}}$ is called the *resultant* of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$). (c) Subtraction of two vectors ($\vec{\mathbf{D}}$ is the difference of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$).

Now suppose your fishing buddy departs from point A (the campsite), walking in the direction to point B (the fishing hole), but he realizes he lost his tackle box when he stopped to rest at point C (located three-quarters of the distance between A and B, beginning from point A). So, he turns back and retraces his steps in the direction toward the campsite and finds the box lying on the path at some point D only 1.2 km away from point C (see [link](b)). What is his displacement vector $\vec{\mathbf{D}}_{AD}$ when he finds the box at point D? What is his displacement vector $\vec{\mathbf{D}}_{DB}$ from point D to the hole? We have already established that at rest point C his displacement vector is $\vec{\mathbf{D}}_{AC} = 0.75\vec{\mathbf{D}}_{AB}$. Starting at point C, he walks southwest (toward the campsite), which means his new displacement vector $\vec{\mathbf{D}}_{CD}$ from point C to point D is antiparallel to $\vec{\mathbf{D}}_{AB}$. Its magnitude $\left|\vec{\mathbf{D}}_{CD}\right|$ is $D_{CD} = 1.2\,\text{km} = 0.2D_{AB}$, so his second displacement vector is $\vec{\mathbf{D}}_{CD} = -0.2\vec{\mathbf{D}}_{AB}$. His total displacement $\vec{\mathbf{D}}_{AD}$ relative to the campsite is the **vector sum** of the two displacement vectors: vector $\vec{\mathbf{D}}_{AC}$ (from the campsite to the rest point) and vector $\vec{\mathbf{D}}_{CD}$ (from the rest point to the point where he finds his box):

Note:
Equation:

$$\vec{\mathbf{D}}_{AD} = \vec{\mathbf{D}}_{AC} + \vec{\mathbf{D}}_{CD}.$$

The vector sum of two (or more) vectors is called the **resultant vector** or, for short, the *resultant*. When the vectors on the right-hand-side of [link] are known, we can find the resultant $\vec{\mathbf{D}}_{AD}$ as follows:

Equation:

$$\vec{\mathbf{D}}_{AD} = \vec{\mathbf{D}}_{AC} + \vec{\mathbf{D}}_{CD} = 0.75\vec{\mathbf{D}}_{AB} - 0.2\,\vec{\mathbf{D}}_{AB} = (0.75 - 0.2)\vec{\mathbf{D}}_{AB} = 0.55\vec{\mathbf{D}}_{AB}.$$

When your friend finally reaches the pond at B, his displacement vector $\vec{\mathbf{D}}_{AB}$ from point A is the vector sum of his displacement vector $\vec{\mathbf{D}}_{AD}$ from point A to point D and his displacement vector $\vec{\mathbf{D}}_{DB}$ from point D to the fishing hole: $\vec{\mathbf{D}}_{AB} = \vec{\mathbf{D}}_{AD} + \vec{\mathbf{D}}_{DB}$ (see [link](c)). This means his displacement vector $\vec{\mathbf{D}}_{DB}$ is the **difference of two vectors**:

Equation:

$$\vec{\mathbf{D}}_{DB} = \vec{\mathbf{D}}_{AB} - \vec{\mathbf{D}}_{AD} = \vec{\mathbf{D}}_{AB} + (-\vec{\mathbf{D}}_{AD}).$$

Notice that a difference of two vectors is nothing more than a vector sum of two vectors because the second term in [link] is vector $-\vec{\mathbf{D}}_{AD}$ (which is antiparallel to $\vec{\mathbf{D}}_{AD}$). When we substitute [link] into [link], we obtain the second displacement vector:
Equation:

$$\vec{\mathbf{D}}_{DB} = \vec{\mathbf{D}}_{AB} - \vec{\mathbf{D}}_{AD} = \vec{\mathbf{D}}_{AB} - 0.55\vec{\mathbf{D}}_{AB} = (1.0 - 0.55)\vec{\mathbf{D}}_{AB} = 0.45\vec{\mathbf{D}}_{AB}.$$

This result means your friend walked $D_{DB} = 0.45 D_{AB} = 0.45(6.0\,\text{km}) = 2.7\,\text{km}$ from the point where he finds his tackle box to the fishing hole.

When vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ lie along a line (that is, in one dimension), such as in the camping example, their resultant $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ and their difference $\vec{\mathbf{D}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$ both lie along the same direction. We can illustrate the addition or subtraction of vectors by drawing the corresponding vectors to scale in one dimension, as shown in [link].

To illustrate the resultant when $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are two parallel vectors, we draw them along one line by placing the origin of one vector at the end of the other vector in head-to-tail fashion (see [link](b)). The magnitude of this resultant is the sum of their magnitudes: $R = A + B$. The direction of the resultant is parallel to both vectors. When vector $\vec{\mathbf{A}}$ is antiparallel to vector $\vec{\mathbf{B}}$, we draw them along one line in either head-to-head fashion ([link](c)) or tail-to-tail fashion. The magnitude of the vector difference, then, is the *absolute value* $D = |A - B|$ of the difference of their magnitudes. The direction of the difference vector $\vec{\mathbf{D}}$ is parallel to the direction of the longer vector.

In general, in one dimension—as well as in higher dimensions, such as in a plane or in space—we can add any number of vectors and we can do so in any order because the addition of vectors is **commutative**,

Note:
Equation:

$$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}},$$

and **associative**,

Note:
Equation:

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}}).$$

Moreover, multiplication by a scalar is **distributive**:

Note:
Equation:

$$\alpha_1\vec{\mathbf{A}} + \alpha_2\vec{\mathbf{A}} = (\alpha_1 + \alpha_2)\vec{\mathbf{A}}.$$

We used the distributive property in [link] and [link].

When adding many vectors in one dimension, it is convenient to use the concept of a **unit vector**. A unit vector, which is denoted by a letter symbol with a hat, such as $\widehat{\mathbf{u}}$, has a magnitude of one and does not have any physical unit so that $\left|\widehat{\mathbf{u}}\right| \equiv u = 1$. The only role of a unit vector is to specify direction. For example, instead of saying vector $\vec{\mathbf{D}}_{AB}$ has a magnitude of 6.0 km and a direction of northeast, we can introduce a unit vector $\widehat{\mathbf{u}}$ that points to the northeast and say succinctly that $\vec{\mathbf{D}}_{AB} = (6.0\,\text{km})\widehat{\mathbf{u}}$. Then the southwesterly direction is simply given by the unit vector $-\widehat{\mathbf{u}}$. In this way, the displacement of 6.0 km in the southwesterly direction is expressed by the vector
Equation:

$$\vec{\mathbf{D}}_{BA} = (-6.0\,\text{km})\widehat{\mathbf{u}}.$$

Example:
A Ladybug Walker

A long measuring stick rests against a wall in a physics laboratory with its 200-cm end at the floor. A ladybug lands on the 100-cm mark and crawls randomly along the stick. It first walks 15 cm toward the floor, then it walks 56 cm toward the wall, then it walks 3 cm toward the floor again. Then, after a brief stop, it continues for 25 cm toward the floor and then, again, it crawls up 19 cm toward the wall before coming to a complete rest ([link]). Find the vector of its total displacement and its final resting position on the stick.

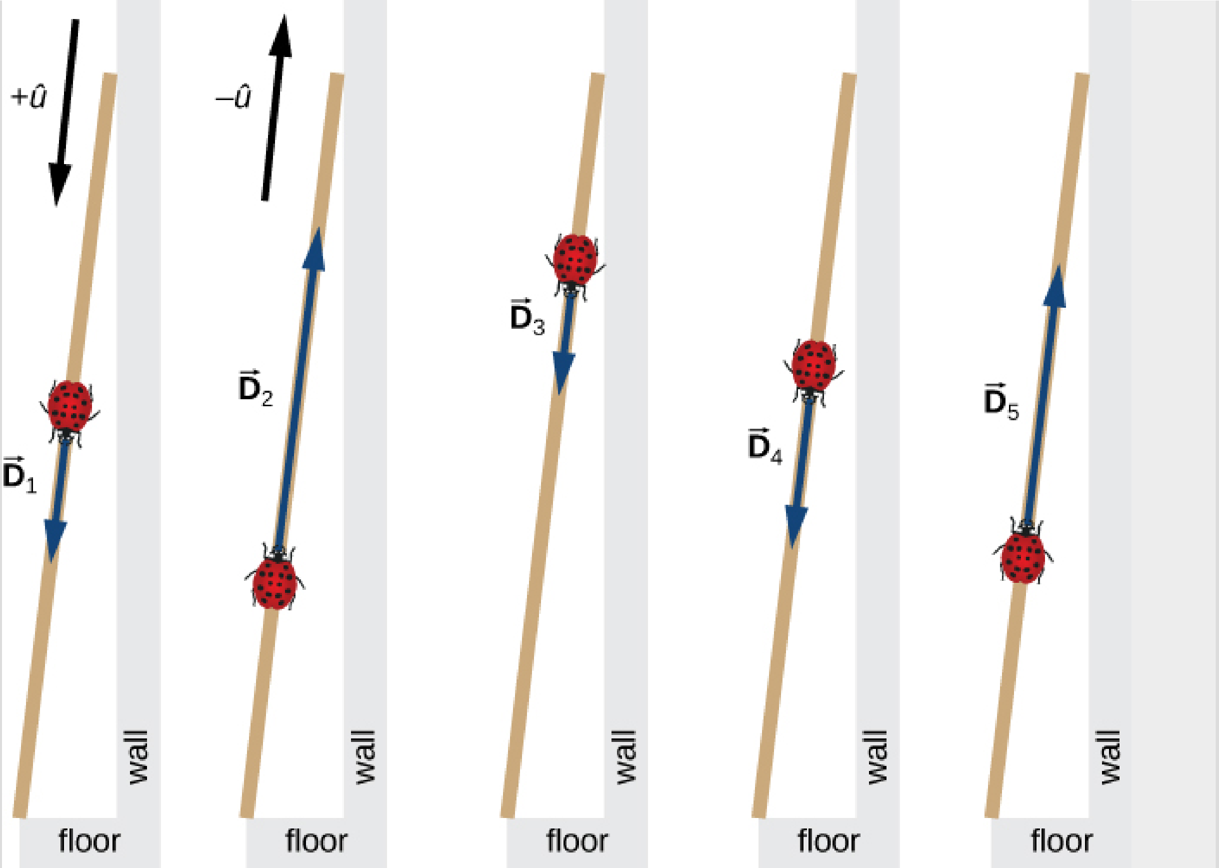
Strategy

If we choose the direction along the stick toward the floor as the direction of unit vector $\mathbf{\hat{u}}$, then the direction toward the floor is $+\mathbf{\hat{u}}$ and the direction toward the wall is $-\mathbf{\hat{u}}$. The ladybug makes a total of five displacements:

Equation:

$$\begin{aligned}
\vec{\mathbf{D}}_1 &= (15\text{ cm})(+\mathbf{\hat{u}}), \\
\vec{\mathbf{D}}_2 &= (56\text{ cm})(-\mathbf{\hat{u}}), \\
\vec{\mathbf{D}}_3 &= (3\text{ cm})(+\mathbf{\hat{u}}), \\
\vec{\mathbf{D}}_4 &= (25\text{ cm})(+\mathbf{\hat{u}}),\text{ and} \\
\vec{\mathbf{D}}_5 &= (19\text{ cm})(-\mathbf{\hat{u}}).
\end{aligned}$$

The total displacement $\vec{\mathbf{D}}$ is the resultant of all its displacement vectors.



Five displacements of the ladybug. Note that in this schematic drawing, magnitudes of displacements are not drawn to scale. (credit "ladybug": modification of work by “Persian Poet Gal”/Wikimedia Commons)

Solution
The resultant of all the displacement vectors is
Equation:

$$\begin{aligned}\vec{\mathbf{D}} &= \vec{\mathbf{D}}_1 + \vec{\mathbf{D}}_2 + \vec{\mathbf{D}}_3 + \vec{\mathbf{D}}_4 + \vec{\mathbf{D}}_5 \\ &= (15\text{ cm})(+\hat{\mathbf{u}}) + (56\text{ cm})(-\hat{\mathbf{u}}) + (3\text{ cm})(+\hat{\mathbf{u}}) + (25\text{ cm})(+\hat{\mathbf{u}}) + (19\text{ cm})(-\hat{\mathbf{u}}) \\ &= (15 - 56 + 3 + 25 - 19)\text{cm}\hat{\mathbf{u}} \\ &= -32\text{ cm}\hat{\mathbf{u}}.\end{aligned}$$

In this calculation, we use the distributive law given by [link]. The result reads that the total displacement vector points away from the 100-cm mark (initial landing site) toward the end of the meter stick that touches the wall. The end that touches the wall is marked 0 cm, so the final position of the ladybug is at the (100 – 32)cm = 68-cm mark.

Note:
Exercise:

Problem:

Check Your Understanding A cave diver enters a long underwater tunnel. When her displacement with respect to the entry point is 20 m, she accidentally drops her camera, but she doesn’t notice it missing until she is some 6 m farther into the tunnel. She swims back 10 m but cannot find the camera, so she decides to end the dive. How far from the entry point is she? Taking the positive direction out of the tunnel, what is her displacement vector relative to the entry point?

Solution:

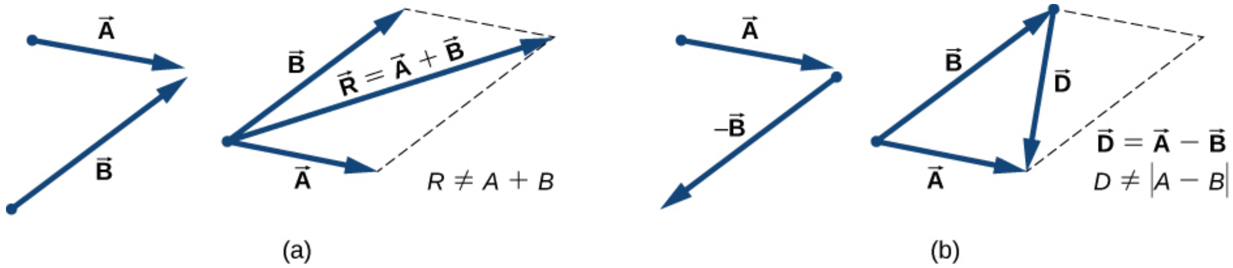
16 m; $\vec{\mathbf{D}} = -16\text{ m}\hat{\mathbf{u}}$

Algebra of Vectors in Two Dimensions

When vectors lie in a plane—that is, when they are in two dimensions—they can be multiplied by scalars, added to other vectors, or subtracted from other vectors in accordance with the general laws expressed by [link], [link], [link], and [link]. However, the addition rule for two vectors in a plane becomes more complicated than the rule for vector addition in one dimension. We have to use the laws of geometry to construct resultant vectors, followed by trigonometry to find vector magnitudes and directions. This geometric approach is commonly used in navigation ([link]). In this section, we need to have at hand two rulers, a triangle, a protractor, a pencil, and an eraser for drawing vectors to scale by geometric constructions.

In navigation, the laws of geometry are used to draw resultant displacements on nautical maps.

For a geometric construction of the sum of two vectors in a plane, we follow **the parallelogram rule**. Suppose two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are at the arbitrary positions shown in [link]. Translate either one of them in parallel to the beginning of the other vector, so that after the translation, both vectors have their origins at the same point. Now, at the end of vector $\vec{\mathbf{A}}$ we draw a line parallel to vector $\vec{\mathbf{B}}$ and at the end of vector $\vec{\mathbf{B}}$ we draw a line parallel to vector $\vec{\mathbf{A}}$ (the dashed lines in [link]). In this way, we obtain a parallelogram. From the origin of the two vectors we draw a diagonal that is the resultant $\vec{\mathbf{R}}$ of the two vectors: $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ ([link](a)). The other diagonal of this parallelogram is the vector difference of the two vectors $\vec{\mathbf{D}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$, as shown in [link](b). Notice that the end of the difference vector is placed at the end of vector $\vec{\mathbf{A}}$.

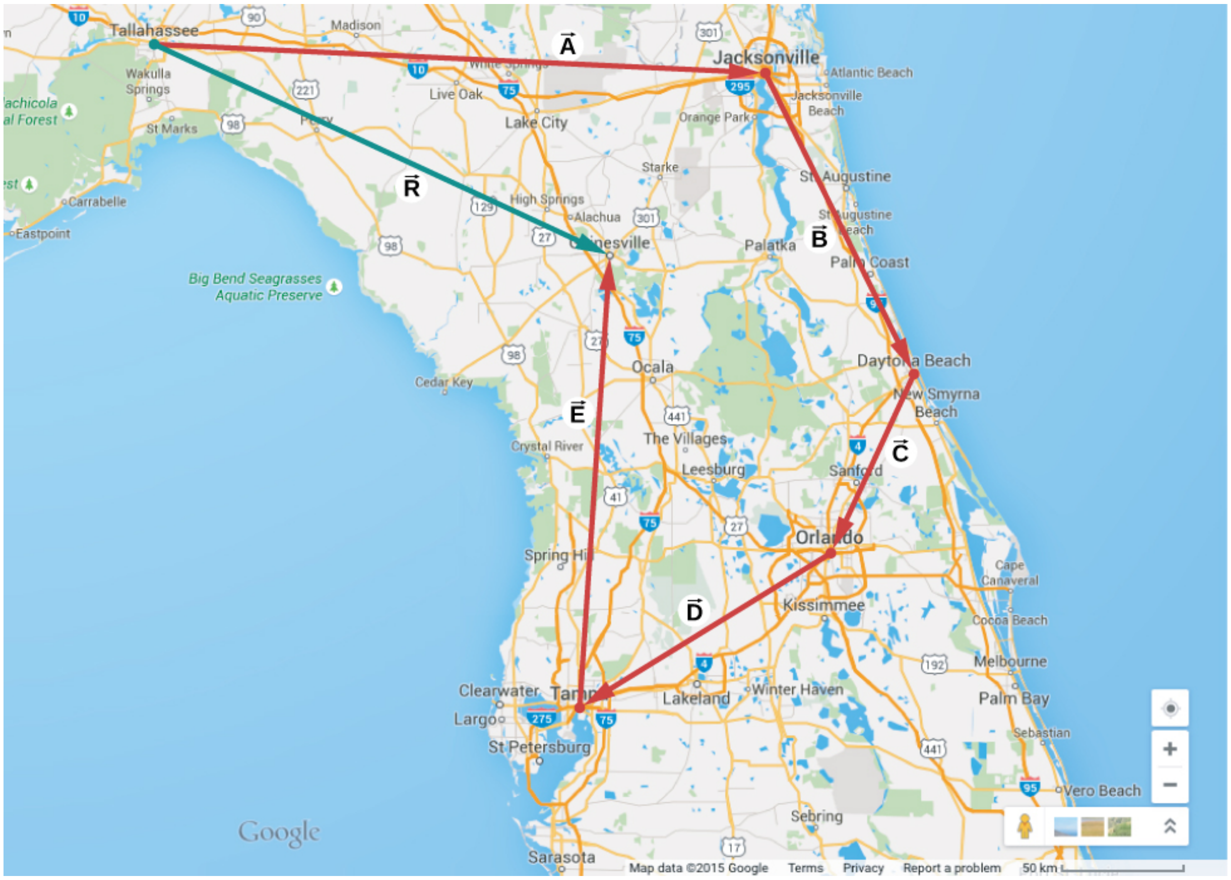


The parallelogram rule for the addition of two vectors. Make the parallel translation of each vector to a point where their origins (marked by the dot) coincide and construct a parallelogram with two sides on the vectors and the other two sides (indicated by dashed lines) parallel to the vectors. (a) Draw the resultant vector $\vec{\mathbf{R}}$ along the diagonal of the parallelogram from the common point to the opposite corner. Length *R* of the resultant vector is *not* equal to the sum of the magnitudes of the two vectors. (b) Draw the difference vector $\vec{\mathbf{D}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$ along the diagonal connecting the ends of the vectors. Place the origin of vector $\vec{\mathbf{D}}$ at the end of vector $\vec{\mathbf{B}}$ and the end (arrowhead) of vector $\vec{\mathbf{D}}$ at the end of vector $\vec{\mathbf{A}}$. Length *D* of the difference vector is *not* equal to the difference of magnitudes of the two vectors.

It follows from the parallelogram rule that neither the magnitude of the resultant vector nor the magnitude of the difference vector can be expressed as a simple sum or difference of magnitudes *A* and *B*, because the length of a diagonal cannot be expressed as a simple sum of side lengths. When using a geometric construction to find magnitudes $\left|\vec{\mathbf{R}}\right|$ and $\left|\vec{\mathbf{D}}\right|$, we have to use trigonometry laws for triangles, which may lead to complicated algebra. There are two ways to circumvent this algebraic complexity. One way is to use the method of components, which we examine in the next section. The other way is to draw the vectors to scale, as is done in navigation, and read approximate vector lengths and angles (directions) from the graphs. In this section we examine the second approach.

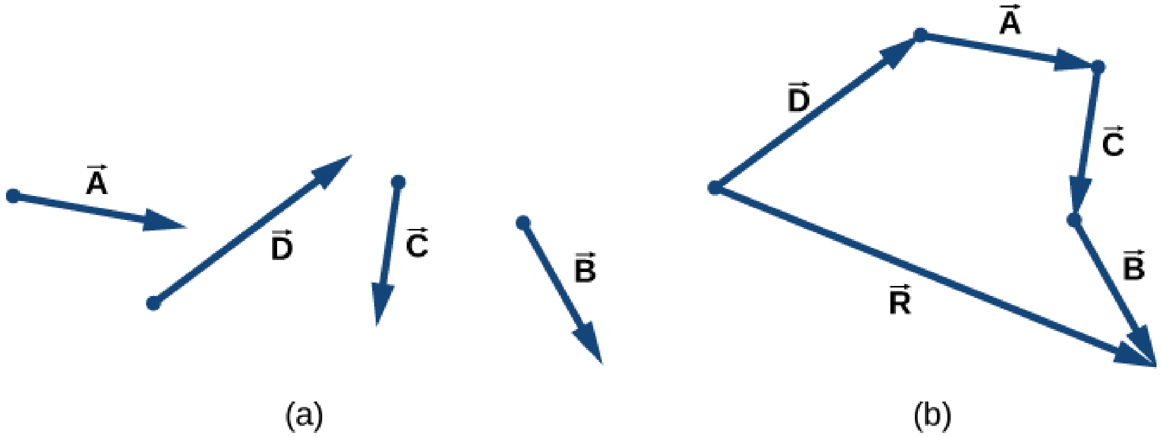
If we need to add three or more vectors, we repeat the parallelogram rule for the pairs of vectors until we find the resultant of all of the resultants. For three vectors, for example, we first find the resultant of vector 1 and vector 2, and then we find the resultant of this resultant and vector 3. The order in which we select the pairs of vectors does not matter because the operation of vector addition is commutative and associative (see [link] and [link]). Before we state a general rule that follows from repetitive applications of the parallelogram rule, let’s look at the following example.

Suppose you plan a vacation trip in Florida. Departing from Tallahassee, the state capital, you plan to visit your uncle Joe in Jacksonville, see your cousin Vinny in Daytona Beach, stop for a little fun in Orlando, see a circus performance in Tampa, and visit the University of Florida in Gainesville. Your route may be represented by five displacement vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$, $\vec{\mathbf{D}}$, and $\vec{\mathbf{E}}$, which are indicated by the red vectors in [link]. What is your total displacement when you reach Gainesville? The total displacement is the vector sum of all five displacement vectors, which may be found by using the parallelogram rule four times. Alternatively, recall that the displacement vector has its beginning at the initial position (Tallahassee) and its end at the final position (Gainesville), so the total displacement vector can be drawn directly as an arrow connecting Tallahassee with Gainesville (see the green vector in [link]). When we use the parallelogram rule four times, the resultant $\vec{\mathbf{R}}$ we obtain is exactly this green vector connecting Tallahassee with Gainesville: $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} + \vec{\mathbf{D}} + \vec{\mathbf{E}}$.



When we use the parallelogram rule four times, we obtain the resultant vector $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} + \vec{\mathbf{D}} + \vec{\mathbf{E}}$, which is the green vector connecting Tallahassee with Gainesville.

Drawing the resultant vector of many vectors can be generalized by using the following **tail-to-head geometric construction**. Suppose we want to draw the resultant vector $\vec{\mathbf{R}}$ of four vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, $\vec{\mathbf{C}}$, and $\vec{\mathbf{D}}$ ([link](a)). We select any one of the vectors as the first vector and make a parallel translation of a second vector to a position where the origin ("tail") of the second vector coincides with the end ("head") of the first vector. Then, we select a third vector and make a parallel translation of the third vector to a position where the origin of the third vector coincides with the end of the second vector. We repeat this procedure until all the vectors are in a head-to-tail arrangement like the one shown in [link]. We draw the resultant vector $\vec{\mathbf{R}}$ by connecting the origin ("tail") of the first vector with the end ("head") of the last vector. The end of the resultant vector is at the end of the last vector. Because the addition of vectors is associative and commutative, we obtain the same resultant vector regardless of which vector we choose to be first, second, third, or fourth in this construction.

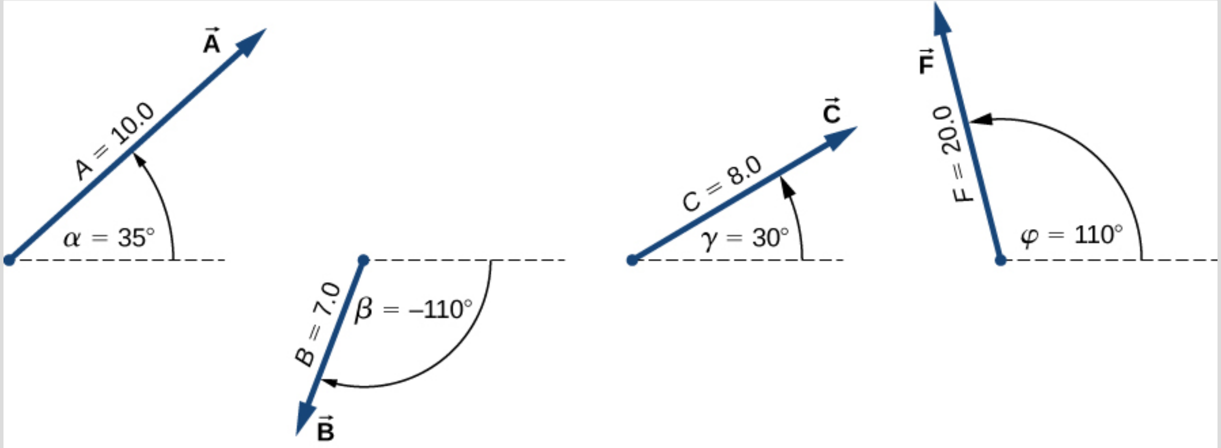


Tail-to-head method for drawing the resultant vector $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}} + \vec{\mathbf{C}} + \vec{\mathbf{D}}$. (a) Four vectors of different magnitudes and directions. (b) Vectors in (a) are translated to new positions where the origin ("tail") of one vector is at the end ("head") of another vector. The resultant vector is drawn from the origin ("tail") of the first vector to the end ("head") of the last vector in this arrangement.

Example:

Geometric Construction of the Resultant

The three displacement vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$ in [link] are specified by their magnitudes $A = 10.0$, $B = 7.0$, and $C = 8.0$, respectively, and by their respective direction angles with the horizontal direction $\alpha = 35^\circ$, $\beta = -110^\circ$, and $\gamma = 30^\circ$. The physical units of the magnitudes are centimeters. Choose a convenient scale and use a ruler and a protractor to find the following vector sums: (a) $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$, (b) $\vec{\mathbf{D}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$, and (c) $\vec{\mathbf{S}} = \vec{\mathbf{A}} - 3\vec{\mathbf{B}} + \vec{\mathbf{C}}$.



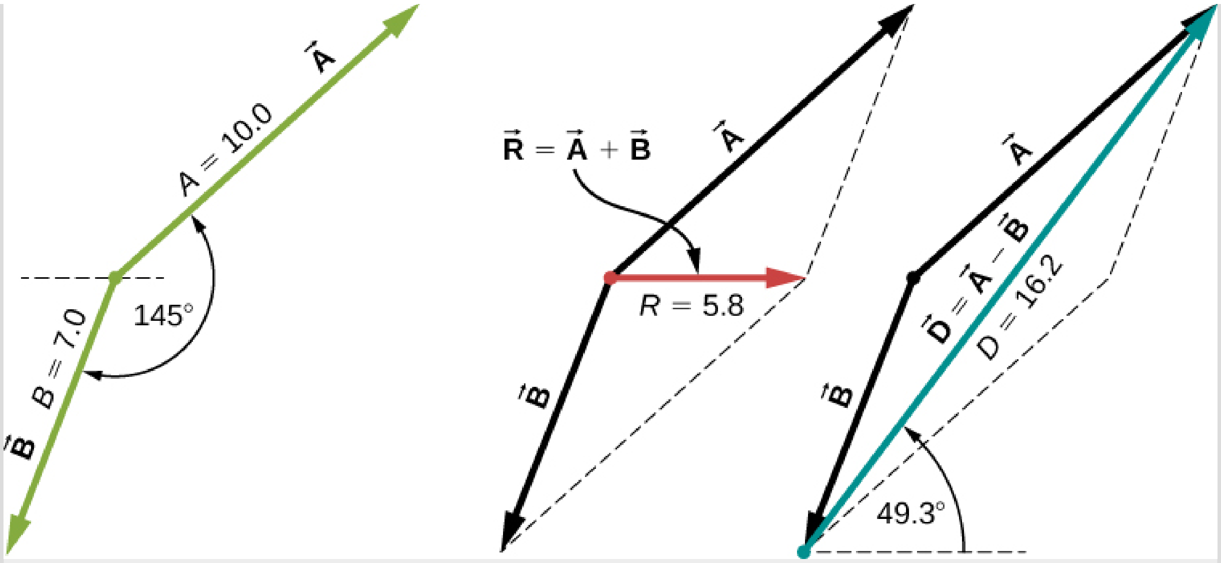
Vectors used in [link] and in the Check Your Understanding feature that follows.

Strategy

In geometric construction, to find a vector means to find its magnitude and its direction angle with the horizontal direction. The strategy is to draw to scale the vectors that appear on the right-hand side of the equation and construct the resultant vector. Then, use a ruler and a protractor to read the magnitude of the resultant and the direction angle. For parts (a) and (b) we use the parallelogram rule. For (c) we use the tail-to-head method.

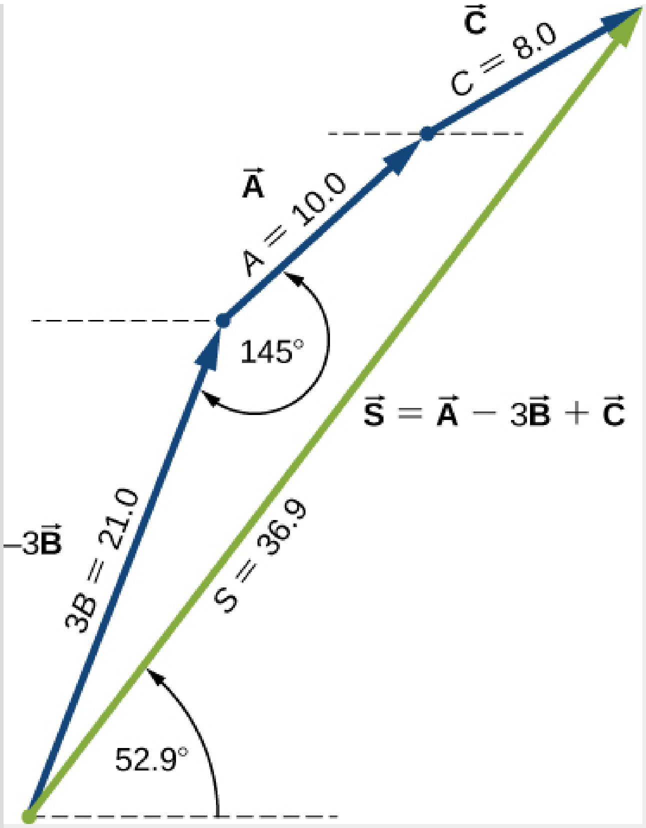
Solution

For parts (a) and (b), we attach the origin of vector $\vec{\mathbf{B}}$ to the origin of vector $\vec{\mathbf{A}}$, as shown in [link], and construct a parallelogram. The shorter diagonal of this parallelogram is the sum $\vec{\mathbf{A}} + \vec{\mathbf{B}}$. The longer of the diagonals is the difference $\vec{\mathbf{A}} - \vec{\mathbf{B}}$. We use a ruler to measure the lengths of the diagonals, and a protractor to measure the angles with the horizontal. For the resultant $\vec{\mathbf{R}}$, we obtain $R = 5.8$ cm and $\theta_R \approx 0^\circ$. For the difference $\vec{\mathbf{D}}$, we obtain $D = 16.2$ cm and $\theta_D = 49.3^\circ$, which are shown in [link].


$\vec{\mathbf{A}}$
$A = 10.0$
145°
$B = 7.0$
$\vec{\mathbf{B}}$
$\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
$R = 5.8$
$\vec{\mathbf{B}}$
$\vec{\mathbf{A}}$
$\vec{\mathbf{D}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$
$D = 16.2$
$\vec{\mathbf{B}}$
49.3°

Using the parallelogram rule to solve (a) (finding the resultant, red) and (b) (finding the difference, blue).

For (c), we can start with vector $-3\vec{\mathbf{B}}$ and draw the remaining vectors tail-to-head as shown in [link]. In vector addition, the order in which we draw the vectors is unimportant, but drawing the vectors to scale is very important. Next, we draw vector $\vec{\mathbf{S}}$ from the origin of the first vector to the end of the last vector and place the arrowhead at the end of $\vec{\mathbf{S}}$. We use a ruler to measure the length of $\vec{\mathbf{S}}$, and find that its magnitude is S = 36.9 cm. We use a protractor and find that its direction angle is $\theta_S = 52.9^\circ$. This solution is shown in [link].


$\vec{C}$
$C = 8.0$
$\vec{A}$
$A = 10.0$
145°
$\vec{S} = \vec{A} - 3\vec{B} + \vec{C}$
$-3\vec{B}$
$3B = 21.0$
$S = 36.9$
52.9°

Using the tail-to-head method to solve (c) (finding vector $\vec{S}$, green).

Note:
Exercise:

Problem:

Check Your Understanding Using the three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{F}$ in [link], choose a convenient scale, and use a ruler and a protractor to find vector $\vec{G}$ given by the vector equation $\vec{G} = \vec{A} + 2\vec{B} - \vec{F}$.

Solution:

G = 28.2 cm, $\theta_G = 291^\circ$

Note:
Observe the addition of vectors in a plane by visiting this vector calculator and this Phet simulation.

Summary

- A vector quantity is any quantity that has magnitude and direction, such as displacement or velocity. Vector quantities are represented by mathematical objects called vectors.
- Geometrically, vectors are represented by arrows, with the end marked by an arrowhead. The length of the vector is its magnitude, which is a positive scalar. On a plane, the direction of a vector is given by the angle the vector makes with a reference direction, often an angle with the horizontal. The direction angle of a vector is a scalar.
- Two vectors are equal if and only if they have the same magnitudes and directions. Parallel vectors have the same direction angles but may have different magnitudes. Antiparallel vectors have direction angles that differ by 180°. Orthogonal vectors have direction angles that differ by 90°.
- When a vector is multiplied by a scalar, the result is another vector of a different length than the length of the original vector. Multiplication by a positive scalar does not change the original direction; only the magnitude is affected. Multiplication by a negative scalar reverses the original direction. The resulting vector is antiparallel to the original vector. Multiplication by a scalar is distributive. Vectors can be divided by nonzero scalars but cannot be divided by vectors.
- Two or more vectors can be added to form another vector. The vector sum is called the resultant vector. We can add vectors to vectors or scalars to scalars, but we cannot add scalars to vectors. Vector addition is commutative and associative.
- To construct a resultant vector of two vectors in a plane geometrically, we use the parallelogram rule. To construct a resultant vector of many vectors in a plane geometrically, we use the tail-to-head method.

Conceptual Questions

Exercise:

Problem:

A weather forecast states the temperature is predicted to be $-5\ ^\circ\mathrm{C}$ the following day. Is this temperature a vector or a scalar quantity? Explain.

Solution:

scalar

Exercise:

Problem:

Which of the following is a vector: a person’s height, the altitude on Mt. Everest, the velocity of a fly, the age of Earth, the boiling point of water, the cost of a book, Earth’s population, or the acceleration of gravity?

Exercise:

Problem:

Give a specific example of a vector, stating its magnitude, units, and direction.

Solution:

answers may vary

Exercise:

Problem: What do vectors and scalars have in common? How do they differ?

Exercise:

Problem:

Suppose you add two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?

Solution:

parallel, sum of magnitudes, antiparallel, zero

Exercise:

Problem: Is it possible to add a scalar quantity to a vector quantity?

Exercise:

Problem:

Is it possible for two vectors of different magnitudes to add to zero? Is it possible for three vectors of different magnitudes to add to zero? Explain.

Solution:

no, yes

Exercise:

Problem:

Does the odometer in an automobile indicate a scalar or a vector quantity?

Exercise:

Problem:

When a 10,000-m runner competing on a 400-m track crosses the finish line, what is the runner's net displacement? Can this displacement be zero? Explain.

Solution:

zero, yes

Exercise:

Problem:

A vector has zero magnitude. Is it necessary to specify its direction? Explain.

Exercise:

Problem: Can a magnitude of a vector be negative?

Solution:

no

Exercise:

Problem:

Can the magnitude of a particle's displacement be greater that the distance traveled?

Exercise:

Problem:

If two vectors are equal, what can you say about their components? What can you say about their magnitudes? What can you say about their directions?

Solution:

equal, equal, the same

Exercise:

Problem:

If three vectors sum up to zero, what geometric condition do they satisfy?

Problems

Exercise:

Problem:

A scuba diver makes a slow descent into the depths of the ocean. His vertical position with respect to a boat on the surface changes several times. He makes the first stop 9.0 m from the boat but has a problem with equalizing the pressure, so he ascends 3.0 m and then continues descending for another 12.0 m to the second stop. From there, he ascends 4 m and then descends for 18.0 m, ascends again for 7 m and descends again for 24.0 m, where he makes a stop, waiting for his buddy. Assuming the positive direction up to the surface, express his net vertical displacement vector in terms of the unit vector. What is his distance to the boat?

Solution:

$\vec{\mathbf{h}} = -49\ \text{m}\hat{\mathbf{u}}$, 49 m

Exercise:

Problem:

In a tug-of-war game on one campus, 15 students pull on a rope at both ends in an effort to displace the central knot to one side or the other. Two students pull with force 196 N each to the right, four students pull with force 98 N each to the left, five students pull with force 62 N each to the left, three students pull with force 150 N each to the right, and one student pulls with force 250 N to the left. Assuming the positive direction to the right, express the net pull on the knot in terms of the unit vector. How big is the net pull on the knot? In what direction?

Exercise:

Problem:

Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point and what is the compass direction of a line connecting your starting point to your final position? Use a graphical method.

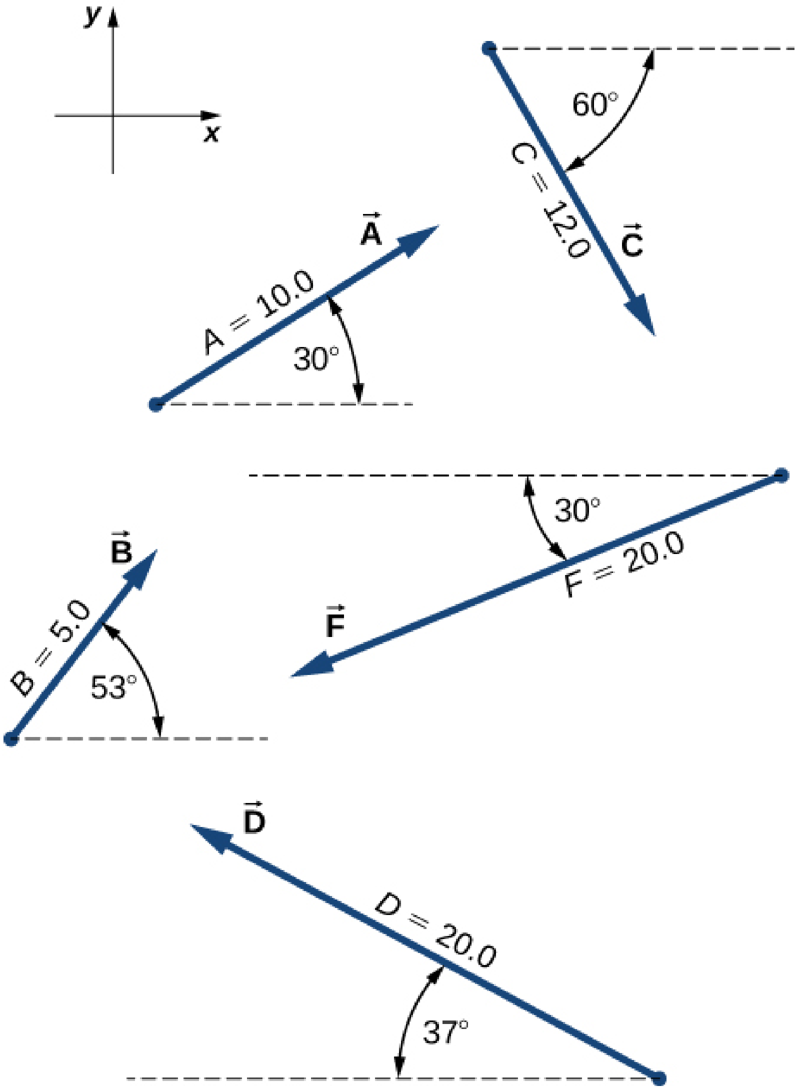
Solution:

30.8 m, 35.7° west of north

Exercise:

Problem:

For the vectors given in the following figure, use a graphical method to find the following resultants: (a) $\vec{\mathbf{A}}+\vec{\mathbf{B}}$, (b) $\vec{\mathbf{C}}+\vec{\mathbf{B}}$, (c) $\vec{\mathbf{D}}+\vec{\mathbf{F}}$, (d) $\vec{\mathbf{A}}-\vec{\mathbf{B}}$, (e) $\vec{\mathbf{D}}-\vec{\mathbf{F}}$, (f) $\vec{\mathbf{A}}+2\vec{\mathbf{F}}$, (g)$\vec{\mathbf{C}}-2\vec{\mathbf{D}}+3\vec{\mathbf{F}}$; and (h) $\vec{\mathbf{A}}-4\vec{\mathbf{D}}+2\vec{\mathbf{F}}$.



Exercise:

Problem:

A delivery man starts at the post office, drives 40 km north, then 20 km west, then 60 km northeast, and finally 50 km north to stop for lunch. Use a graphical method to find his net displacement vector.

Solution:

134 km, 80°

Exercise:

Problem:

An adventurous dog strays from home, runs three blocks east, two blocks north, one block east, one block north, and two blocks west. Assuming that each block is about 100 m, how far from home and in what direction is the dog? Use a graphical method.

Exercise:

Problem:

In an attempt to escape a desert island, a castaway builds a raft and sets out to sea. The wind shifts a great deal during the day and he is blown along the following directions: 2.50 km and 45.0° north of west, then 4.70 km and 60.0° south of east, then 1.30 km and 25.0° south of west, then 5.10 km straight east, then 1.70 km and 5.00° east of north, then 7.20 km and 55.0° south of west, and finally 2.80 km and 10.0° north of east. Use a graphical method to find the castaway's final position relative to the island.

Solution:

7.34 km, 63.5° south of east

Exercise:

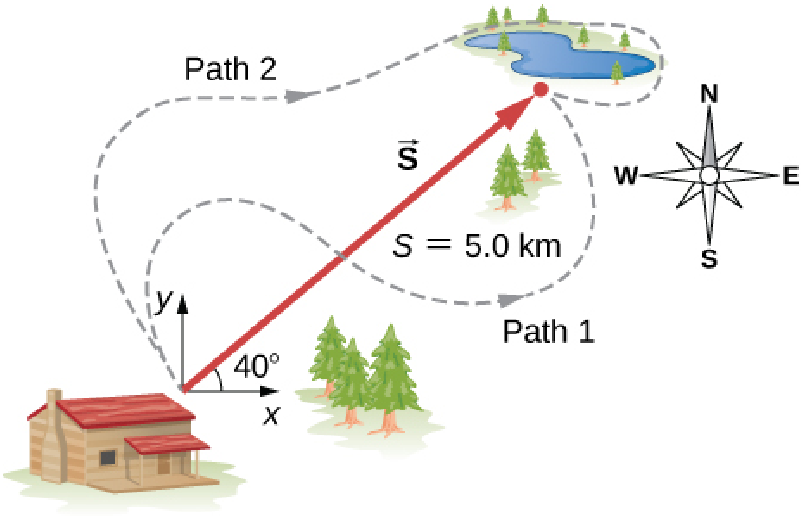
Problem:

A small plane flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east. Use a graphical method to find the total distance the plane covers from the starting point and the direction of the path to the final position.

Exercise:

Problem:

A trapper walks a 5.0-km straight-line distance from his cabin to the lake, as shown in the following figure. Use a graphical method (the parallelogram rule) to determine the trapper's displacement directly to the east and displacement directly to the north that sum up to his resultant displacement vector. If the trapper walked only in directions east and north, zigzagging his way to the lake, how many kilometers would he have to walk to get to the lake?



Solution:

3.8 km east, 3.2 km north, 7.0 km

Exercise:

Problem:

A surveyor measures the distance across a river that flows straight north by the following method. Starting directly across from a tree on the opposite bank, the surveyor walks 100 m along the river to establish a baseline. She then sights across to the tree and reads that the angle from the baseline to the tree is 35°. How wide is the river?

Exercise:

Problem:

A pedestrian walks 6.0 km east and then 13.0 km north. Use a graphical method to find the pedestrian's resultant displacement and geographic direction.

Solution:

14.3 km, 65°

Exercise:

Problem:

The magnitudes of two displacement vectors are A = 20 m and B = 6 m. What are the largest and the smallest values of the magnitude of the resultant $\vec{\mathbf{R}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$?

Glossary

antiparallel vectors
two vectors with directions that differ by 180°

associative
terms can be grouped in any fashion

commutative
operations can be performed in any order

difference of two vectors
vector sum of the first vector with the vector antiparallel to the second

displacement
change in position

distributive
multiplication can be distributed over terms in summation

magnitude
length of a vector

orthogonal vectors
two vectors with directions that differ by exactly 90°, synonymous with perpendicular vectors

parallelogram rule
geometric construction of the vector sum in a plane

parallel vectors
two vectors with exactly the same direction angles

resultant vector
vector sum of two (or more) vectors

scalar
a number, synonymous with a scalar quantity in physics

scalar equation
equation in which the left-hand and right-hand sides are numbers

scalar quantity
quantity that can be specified completely by a single number with an appropriate physical unit

tail-to-head geometric construction
geometric construction for drawing the resultant vector of many vectors

unit vector
vector of a unit magnitude that specifies direction; has no physical unit

vector
mathematical object with magnitude and direction

vector equation
equation in which the left-hand and right-hand sides are vectors

vector quantity
physical quantity described by a mathematical vector—that is, by specifying both its magnitude and its direction; synonymous with a vector in physics

vector sum
resultant of the combination of two (or more) vectors

Coordinate Systems and Components of a Vector

By the end of this section, you will be able to:

- Describe vectors in two and three dimensions in terms of their components, using unit vectors along the axes.
- Distinguish between the vector components of a vector and the scalar components of a vector.
- Explain how the magnitude of a vector is defined in terms of the components of a vector.
- Identify the direction angle of a vector in a plane.
- Explain the connection between polar coordinates and Cartesian coordinates in a plane.

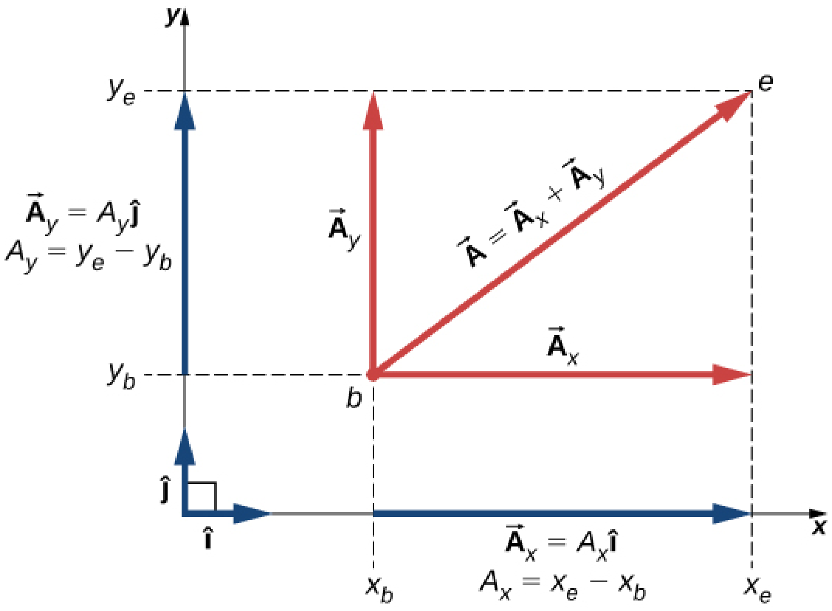
Vectors are usually described in terms of their components in a coordinate system. Even in everyday life we naturally invoke the concept of orthogonal projections in a rectangular coordinate system. For example, if you ask someone for directions to a particular location, you will more likely be told to go 40 km east and 30 km north than 50 km in the direction 37° north of east.

In a rectangular (Cartesian) *xy*-coordinate system in a plane, a point in a plane is described by a pair of coordinates (*x*, *y*). In a similar fashion, a vector $\vec{\mathbf{A}}$ in a plane is described by a pair of its *vector* coordinates. The *x*-coordinate of vector $\vec{\mathbf{A}}$ is called its *x*-component and the *y*-coordinate of vector $\vec{\mathbf{A}}$ is called its *y*-component. The vector *x*-component is a vector denoted by $\vec{\mathbf{A}}_x$. The vector *y*-component is a vector denoted by $\vec{\mathbf{A}}_y$. In the Cartesian system, the *x* and *y* **vector components** of a vector are the orthogonal projections of this vector onto the *x*- and *y*-axes, respectively. In this way, following the parallelogram rule for vector addition, each vector on a Cartesian plane can be expressed as the vector sum of its vector components:

Equation:

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y.$$

As illustrated in [link], vector $\vec{\mathbf{A}}$ is the diagonal of the rectangle where the *x*-component $\vec{\mathbf{A}}_x$ is the side parallel to the *x*-axis and the *y*-component $\vec{\mathbf{A}}_y$ is the side parallel to the *y*-axis. Vector component $\vec{\mathbf{A}}_x$ is orthogonal to vector component $\vec{\mathbf{A}}_y$.



y
y_e
e
$\vec{\mathbf{A}}_y = A_y\hat{\mathbf{j}}$
$A_y = y_e - y_b$
$\vec{\mathbf{A}}_y$
$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$
$\vec{\mathbf{A}}_x$
y_b
b
$\hat{\mathbf{j}}$
$\hat{\mathbf{i}}$
x
$\vec{\mathbf{A}}_x = A_x\hat{\mathbf{i}}$
$A_x = x_e - x_b$
x_b
x_e

Vector $\vec{\mathbf{A}}$ in a plane in the Cartesian coordinate system is the vector sum of its vector *x*- and *y*-components. The *x*-vector component $\vec{\mathbf{A}}_x$ is the orthogonal projection of vector $\vec{\mathbf{A}}$ onto the *x*-axis. The *y*-vector component $\vec{\mathbf{A}}_y$ is the orthogonal projection of vector $\vec{\mathbf{A}}$ onto the *y*-axis. The numbers A_x and A_y that multiply the unit vectors are the scalar components of the vector.

It is customary to denote the positive direction on the *x*-axis by the unit vector $\hat{\mathbf{i}}$ and the positive direction on the *y*-axis by the unit vector $\hat{\mathbf{j}}$. **Unit vectors of the axes**, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, define two orthogonal directions in the plane. As shown in [link], the *x*- and *y*- components of a vector can now be written in terms of the unit vectors of the axes:

Equation:

$$\begin{cases} \vec{\mathbf{A}}_x = A_x\hat{\mathbf{i}} \\ \vec{\mathbf{A}}_y = A_y\hat{\mathbf{j}}. \end{cases}$$

The vectors $\vec{\mathbf{A}}_x$ and $\vec{\mathbf{A}}_y$ defined by [link] are the *vector components* of vector $\vec{\mathbf{A}}$. The numbers A_x and A_y that define the vector components in [link] are the **scalar components** of vector $\vec{\mathbf{A}}$. Combining [link] with [link], we obtain **the component form of a vector**:

Note:

Equation:

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}.$$

If we know the coordinates $b(x_b, y_b)$ of the origin point of a vector (where *b* stands for "beginning") and the coordinates $e(x_e, y_e)$ of the end point of a vector (where *e* stands for "end"), we can obtain the scalar components of a vector simply by subtracting the origin point coordinates from the end point coordinates:

Note:
Equation:

$$\begin{cases} A_x = x_e - x_b \\ A_y = y_e - y_b. \end{cases}$$

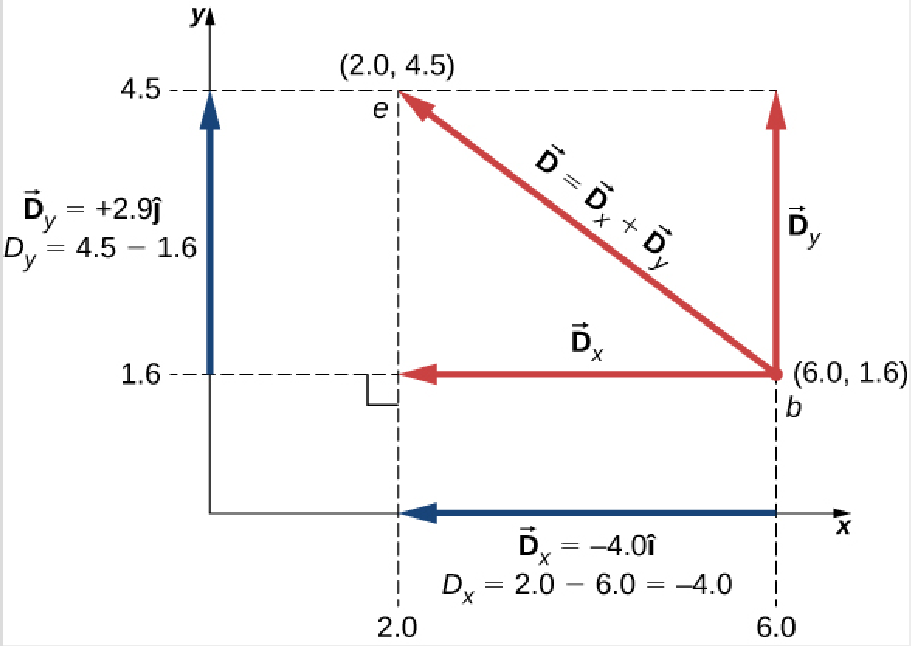
Example:
Displacement of a Mouse Pointer
A mouse pointer on the display monitor of a computer at its initial position is at point (6.0 cm, 1.6 cm) with respect to the lower left-side corner. If you move the pointer to an icon located at point (2.0 cm, 4.5 cm), what is the displacement vector of the pointer?
Strategy
The origin of the *xy*-coordinate system is the lower left-side corner of the computer monitor. Therefore, the unit vector $\hat{\mathbf{i}}$ on the *x*-axis points horizontally to the right and the unit vector $\hat{\mathbf{j}}$ on the *y*-axis points vertically upward. The origin of the displacement vector is located at point *b*(6.0, 1.6) and the end of the displacement vector is located at point *e*(2.0, 4.5). Substitute the coordinates of these points into [link] to find the scalar components D_x and D_y of the displacement vector $\vec{\mathbf{D}}$. Finally, substitute the coordinates into [link] to write the displacement vector in the vector component form.
Solution
We identify $x_b = 6.0$, $x_e = 2.0$, $y_b = 1.6$, and $y_e = 4.5$, where the physical unit is 1 cm. The scalar *x*- and *y*-components of the displacement vector are
Equation:

$$\begin{aligned} D_x &= x_e - x_b = (2.0 - 6.0)\text{cm} = -4.0\text{ cm}, \\ D_y &= y_e - y_b = (4.5 - 1.6)\text{cm} = +2.9\text{ cm}. \end{aligned}$$

The vector component form of the displacement vector is
Equation:

$$\vec{\mathbf{D}} = D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}} = (-4.0\text{ cm})\hat{\mathbf{i}} + (2.9\text{ cm})\hat{\mathbf{j}} = (-4.0\hat{\mathbf{i}} + 2.9\hat{\mathbf{j}})\text{cm}.$$

This solution is shown in [link].



The graph of the displacement vector. The vector points from the origin point at *b* to the end point at *e*.

Significance

Notice that the physical unit—here, 1 cm—can be placed either with each component immediately before the unit vector or globally for both components, as in [link]. Often, the latter way is more convenient because it is simpler.

The vector *x*-component $\vec{\mathbf{D}}_x = -4.0\hat{\mathbf{i}} = 4.0(-\hat{\mathbf{i}})$ of the displacement vector has the magnitude $\left|\vec{\mathbf{D}}_x\right| = \left|-4.0\right|\left|\hat{\mathbf{i}}\right| = 4.0$ because the magnitude of the unit vector is $\left|\hat{\mathbf{i}}\right| = 1$. Notice, too, that the direction of the *x*-component is $-\hat{\mathbf{i}}$, which is antiparallel to the direction of the +*x*-axis; hence, the *x*-component vector $\vec{\mathbf{D}}_x$ points to the left, as shown in [link]. The scalar *x*-component of vector $\vec{\mathbf{D}}$ is $D_x = -4.0$.

Similarly, the vector *y*-component $\vec{\mathbf{D}}_y = +2.9\hat{\mathbf{j}}$ of the displacement vector has magnitude $\left|\vec{\mathbf{D}}_y\right| = \left|2.9\right|\left|\hat{\mathbf{j}}\right| = 2.9$ because the magnitude of the unit vector is $\left|\hat{\mathbf{j}}\right| = 1$. The direction of the *y*-component is $+\hat{\mathbf{j}}$, which is parallel to the direction of the +*y*-axis. Therefore, the *y*-component vector $\vec{\mathbf{D}}_y$ points up, as seen in [link]. The scalar *y*-component of vector $\vec{\mathbf{D}}$ is $D_y = +2.9$. The displacement vector $\vec{\mathbf{D}}$ is the resultant of its two *vector* components.

The vector component form of the displacement vector [link] tells us that the mouse pointer has been moved on the monitor 4.0 cm to the left and 2.9 cm upward from its initial position.

Note:
Exercise:

Problem:

Check Your Understanding A blue fly lands on a sheet of graph paper at a point located 10.0 cm to the right of its left edge and 8.0 cm above its bottom edge and walks slowly to a point located 5.0 cm from the left edge and 5.0 cm from the bottom edge. Choose the rectangular coordinate system with the origin at the lower left-side corner of the paper and find the displacement vector of the fly. Illustrate your solution by graphing.

Solution:

$\vec{\mathbf{D}} = (-5.0\hat{\mathbf{i}} - 3.0\hat{\mathbf{j}})\text{cm}$; the fly moved 5.0 cm to the left and 3.0 cm down from its landing site.

When we know the scalar components A_x and A_y of a vector $\vec{\mathbf{A}}$, we can find its magnitude A and its direction angle θ_A. The **direction angle**—or direction, for short—is the angle the vector forms with the positive direction on the *x*-axis. The angle θ_A is measured in the *counterclockwise direction* from the +*x*-axis to the vector ([link]). Because the lengths A, A_x, and A_y form a right triangle, they are related by the Pythagorean theorem:

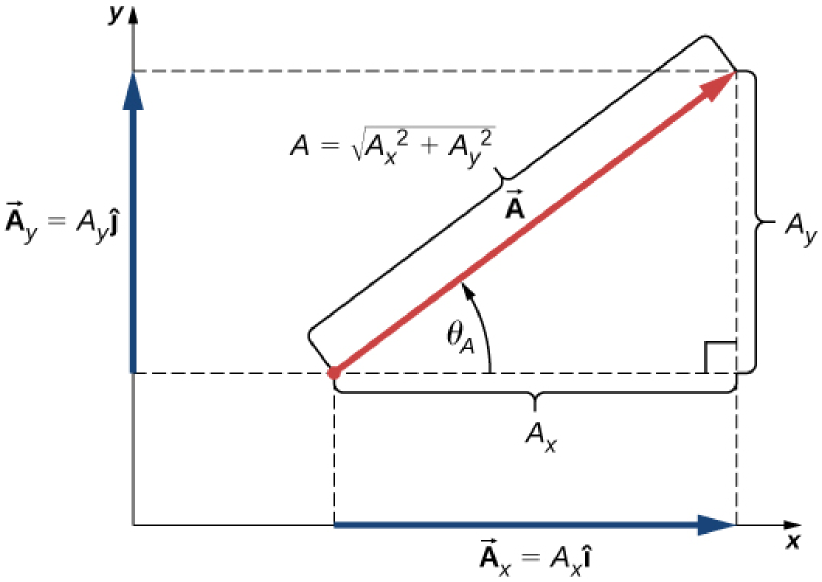
Note:
Equation:

$$A^2 = A_x^2 + A_y^2 \;\Leftrightarrow\; A = \sqrt{A_x^2 + A_y^2}.$$

This equation works even if the scalar components of a vector are negative. The direction angle θ_A of a vector is defined via the tangent function of angle θ_A in the triangle shown in [link]:

Note:
Equation:

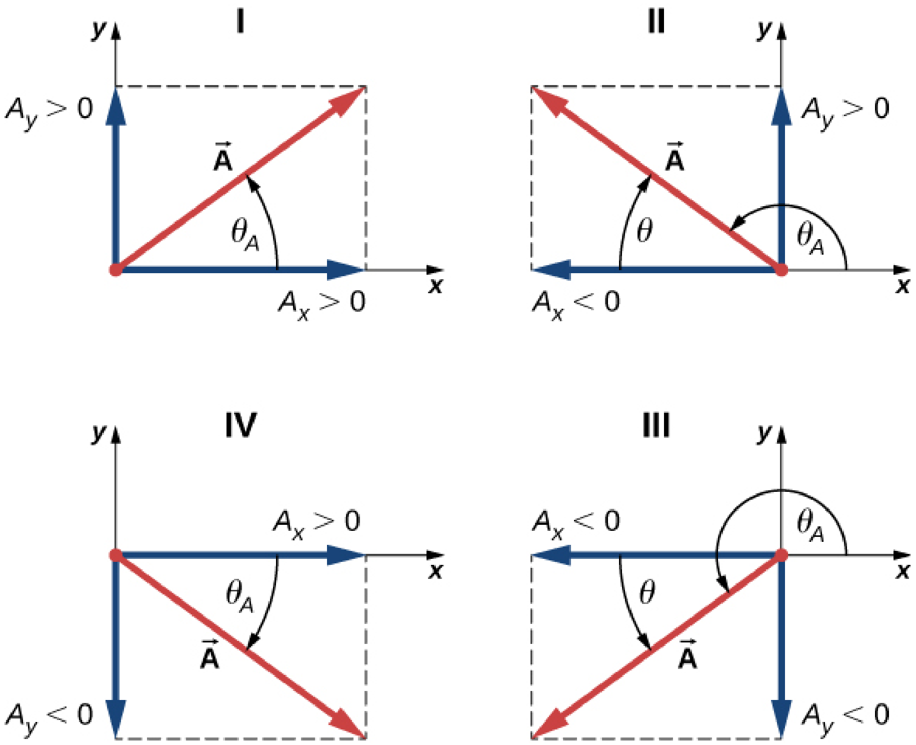
$$\tan\theta = \frac{A_y}{A_x}$$



y
$A = \sqrt{A_x{}^2 + A_y{}^2}$
$\vec{\mathbf{A}}_y = A_y\hat{\mathbf{j}}$
$\vec{\mathbf{A}}$
A_y
θ_A
A_x
x
$\vec{\mathbf{A}}_x = A_x\hat{\mathbf{i}}$

When the vector lies either in the first quadrant or in the fourth quadrant, where component $A_x$is positive (Figure 2.19), the direction angle θ_A in Equation 2.16) is identical to the angle θ.

When the vector lies either in the first quadrant or in the fourth quadrant, where component A_x is positive ([link]), the angle θ in [link] is identical to the direction angle θ_A. For vectors in the fourth quadrant, angle θ is negative, which means that for these vectors, direction angle θ_A is measured *clockwise* from the positive *x*-axis. Similarly, for vectors in the second quadrant, angle θ is negative. When the vector lies in either the second or third quadrant, where component A_x is negative, the direction angle is $\theta_A = \theta + 180^\circ$ ([link]).


I
y
x
$A_y > 0$
$\vec{A}$
θ_A
$A_x > 0$
II
y
x
$A_y > 0$
$\vec{A}$
θ
θ_A
$A_x < 0$
IV
y
x
$A_x > 0$
θ_A
$\vec{A}$
$A_y < 0$
III
y
x
$A_x < 0$
θ_A
θ
$\vec{A}$
$A_y < 0$

Scalar components of a vector may be positive or negative. Vectors in the first quadrant (I) have both scalar components positive and vectors in the third quadrant have both scalar components negative. For vectors in quadrants II and III, the direction angle of a vector is $\theta_A = \theta + 180^\circ$.

Example:

Magnitude and Direction of the Displacement Vector

You move a mouse pointer on the display monitor from its initial position at point (6.0 cm, 1.6 cm) to an icon located at point (2.0 cm, 4.5 cm). What are the magnitude and direction of the displacement vector of the pointer?

Strategy

In [link], we found the displacement vector $\vec{\mathbf{D}}$ of the mouse pointer (see [link]). We identify its scalar components $D_x = -4.0\ \text{cm}$ and $D_y = +2.9\ \text{cm}$ and substitute into [link] and [link] to find the magnitude D and direction θ_D, respectively.

Solution

The magnitude of vector $\vec{\mathbf{D}}$ is

Equation:

$$D = \sqrt{D_x^2 + D_y^2} = \sqrt{(-4.0\ \text{cm})^2 + (2.9\ \text{cm})^2} = \sqrt{(4.0)^2 + (2.9)^2}\ \text{cm} = 4.9\ \text{cm}.$$

The direction angle is

Equation:

$$\tan\theta = \frac{D_y}{D_x} = \frac{+2.9\,\text{cm}}{-4.0\,\text{cm}} = -0.725 \quad \Rightarrow \quad \theta = \tan^{-1}(-0.725) = -35.9^\circ.$$

Vector $\vec{\mathbf{D}}$ lies in the second quadrant, so its direction angle is

Equation:

$$\theta_D = \theta + 180^\circ = -35.9^\circ + 180^\circ = 144.1^\circ.$$

Note:

Exercise:

Problem:

Check Your Understanding If the displacement vector of a blue fly walking on a sheet of graph paper is $\vec{\mathbf{D}} = (-5.00\hat{\mathbf{i}} - 3.00\hat{\mathbf{j}})\text{cm}$, find its magnitude and direction.

Solution:

5.83 cm, 211°

In many applications, the magnitudes and directions of vector quantities are known and we need to find the resultant of many vectors. For example, imagine 400 cars moving on the Golden Gate Bridge in San Francisco in a strong wind. Each car gives the bridge a different push in various directions and we would like to know how big the resultant push can possibly be. We have already gained some experience with the geometric construction of vector sums, so we know the task of finding the resultant by drawing the vectors and measuring their lengths and angles may become intractable pretty quickly, leading to huge errors. Worries like this do not appear when we use analytical methods. The very first step in an analytical approach is to find vector components when the direction and magnitude of a vector are known.

Let us return to the right triangle in [link]. The quotient of the adjacent side A_x to the hypotenuse A is the cosine function of direction angle θ_A, $A_x/A = \cos\theta_A$, and the quotient of the opposite side A_y to the hypotenuse A is the sine function of θ_A, $A_y/A = \sin\theta_A$. When magnitude A and direction θ_A are known, we can solve these relations for the scalar components:

Note:

Equation:

$$\begin{cases} A_x = A\cos\theta_A \\ A_y = A\sin\theta_A \end{cases}.$$

When calculating vector components with [link], care must be taken with the angle. The direction angle θ_A of a vector is the angle measured *counterclockwise* from the positive direction on the *x*-axis to the vector. The clockwise measurement gives a negative angle.

Example:

Components of Displacement Vectors

A rescue party for a missing child follows a search dog named Trooper. Trooper wanders a lot and makes many trial sniffs along many different paths. Trooper eventually finds the child and the story has a happy ending, but his displacements on various legs seem to be truly convoluted. On one of the legs he walks 200.0 m southeast, then he runs north some 300.0 m. On the third leg, he examines the scents carefully for 50.0 m in the direction 30° west of north. On the fourth leg, Trooper goes directly south for 80.0 m, picks up a fresh scent and turns 23° west of south for 150.0 m. Find the scalar components of Trooper's displacement vectors and his displacement vectors in vector component form for each leg.

Strategy

Let's adopt a rectangular coordinate system with the positive *x*-axis in the direction of geographic east, with the positive *y*-direction pointed to geographic north. Explicitly, the unit vector $\hat{\mathbf{i}}$ of the *x*-axis points east and the unit vector $\hat{\mathbf{j}}$ of the *y*-axis points north. Trooper makes five legs, so there are five displacement vectors. We start by identifying their magnitudes and direction angles, then we use [link] to find the scalar components of the displacements and [link] for the displacement vectors.

Solution

On the first leg, the displacement magnitude is $L_1 = 200.0\,\text{m}$ and the direction is southeast. For direction angle θ_1 we can take either 45° measured clockwise from the east direction or $45^\circ + 270^\circ$ measured counterclockwise from the east direction. With the first choice, $\theta_1 = -45^\circ$. With the second choice, $\theta_1 = +315^\circ$. We can use either one of these two angles. The components are

Equation:

$$\begin{aligned} L_{1x} &= L_1\cos\theta_1 = (200.0\,\text{m})\cos 315^\circ = 141.4\,\text{m}, \\ L_{1y} &= L_1\sin\theta_1 = (200.0\,\text{m})\sin 315^\circ = -141.4\,\text{m}. \end{aligned}$$

The displacement vector of the first leg is

Equation:

$$\vec{\mathbf{L}}_1 = L_{1x}\hat{\mathbf{i}} + L_{1y}\hat{\mathbf{j}} = (141.4\hat{\mathbf{i}} - 141.4\hat{\mathbf{j}})\,\text{m}.$$

On the second leg of Trooper's wanderings, the magnitude of the displacement is $L_2 = 300.0\,\text{m}$ and the direction is north. The direction angle is $\theta_2 = +90^\circ$. We obtain the following results:

Equation:

$$\begin{aligned}
L_{2x} &= L_2 \cos\theta_2 = (300.0\,\text{m})\cos 90^\circ = 0.0\,,\\
L_{2y} &= L_2 \sin\theta_2 = (300.0\,\text{m})\sin 90^\circ = 300.0\,\text{m},\\
\vec{\mathbf{L}}_2 &= L_{2x}\hat{\mathbf{i}} + L_{2y}\hat{\mathbf{j}} = (300.0\,\text{m})\hat{\mathbf{j}}.
\end{aligned}$$

On the third leg, the displacement magnitude is $L_3 = 50.0\,\text{m}$ and the direction is 30° west of north. The direction angle measured counterclockwise from the eastern direction is $\theta_3 = 30^\circ + 90^\circ = +120^\circ$. This gives the following answers:

Equation:

$$\begin{aligned}
L_{3x} &= L_3 \cos\theta_3 = (50.0\,\text{m})\cos 120^\circ = -25.0\,\text{m},\\
L_{3y} &= L_3 \sin\theta_3 = (50.0\,\text{m})\sin 120^\circ = +43.3\,\text{m},\\
\vec{\mathbf{L}}_3 &= L_{3x}\hat{\mathbf{i}} + L_{3y}\hat{\mathbf{j}} = (-25.0\hat{\mathbf{i}} + 43.3\hat{\mathbf{j}})\text{m}.
\end{aligned}$$

On the fourth leg of the excursion, the displacement magnitude is $L_4 = 80.0\,\text{m}$ and the direction is south. The direction angle can be taken as either $\theta_4 = -90^\circ$ or $\theta_4 = +270^\circ$. We obtain

Equation:

$$\begin{aligned}
L_{4x} &= L_4 \cos\theta_4 = (80.0\,\text{m})\cos(-90^\circ) = 0\,,\\
L_{4y} &= L_4 \sin\theta_4 = (80.0\,\text{m})\sin(-90^\circ) = -80.0\,\text{m},\\
\vec{\mathbf{L}}_4 &= L_{4x}\hat{\mathbf{i}} + L_{4y}\hat{\mathbf{j}} = (-80.0\,\text{m})\hat{\mathbf{j}}.
\end{aligned}$$

On the last leg, the magnitude is $L_5 = 150.0\,\text{m}$ and the angle is $\theta_5 = -23^\circ + 270^\circ = +247^\circ$ (23° west of south), which gives

Equation:

$$\begin{aligned}
L_{5x} &= L_5 \cos\theta_5 = (150.0\,\text{m})\cos 247^\circ = -58.6\,\text{m},\\
L_{5y} &= L_5 \sin\theta_5 = (150.0\,\text{m})\sin 247^\circ = -138.1\,\text{m},\\
\vec{\mathbf{L}}_5 &= L_{5x}\hat{\mathbf{i}} + L_{5y}\hat{\mathbf{j}} = (-58.6\hat{\mathbf{i}} - 138.1\hat{\mathbf{j}})\text{m}.
\end{aligned}$$

Note:
Exercise:

Problem:

Check Your Understanding If Trooper runs 20 m west before taking a rest, what is his displacement vector?

Solution:

$\vec{\mathbf{D}} = (-20\,\text{m})\hat{\mathbf{j}}$

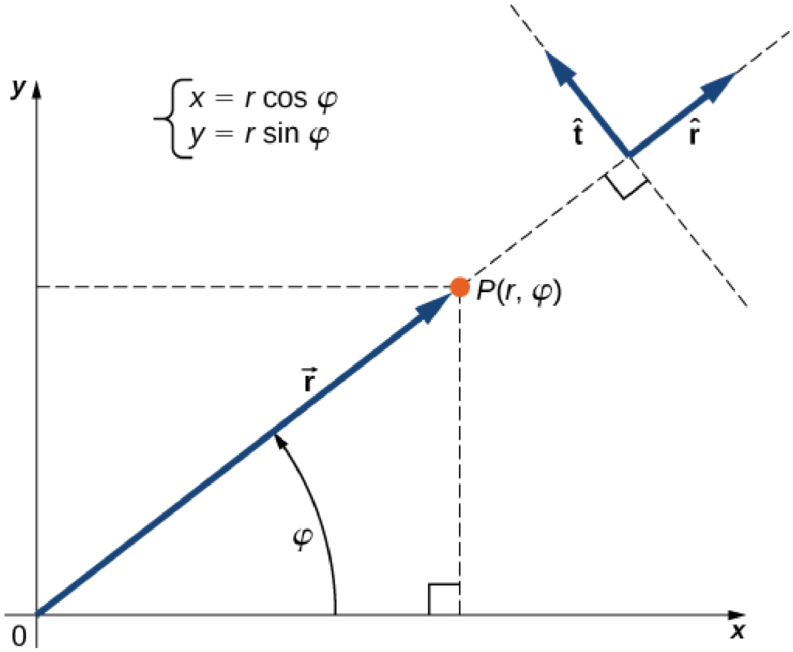
Polar Coordinates

To describe locations of points or vectors in a plane, we need two orthogonal directions. In the Cartesian coordinate system these directions are given by unit vectors $\mathbf{\hat{i}}$ and $\mathbf{\hat{j}}$ along the *x*-axis and the *y*-axis, respectively. The Cartesian coordinate system is very convenient to use in describing displacements and velocities of objects and the forces acting on them. However, it becomes cumbersome when we need to describe the rotation of objects. When describing rotation, we usually work in the **polar coordinate system**.

In the polar coordinate system, the location of point *P* in a plane is given by two **polar coordinates** ([link]). The first polar coordinate is the **radial coordinate** *r*, which is the distance of point *P* from the origin. The second polar coordinate is an angle φ that the radial vector makes with some chosen direction, usually the positive *x*-direction. In polar coordinates, angles are measured in radians, or rads. The radial vector is attached at the origin and points away from the origin to point *P*. This radial direction is described by a unit radial vector $\mathbf{\hat{r}}$. The second unit vector $\mathbf{\hat{t}}$ is a vector orthogonal to the radial direction $\mathbf{\hat{r}}$. The positive $+\mathbf{\hat{t}}$ direction indicates how the angle φ changes in the counterclockwise direction. In this way, a point *P* that has coordinates (*x*, *y*) in the rectangular system can be described equivalently in the polar coordinate system by the two polar coordinates (r,φ). [link] is valid for any vector, so we can use it to express the *x*- and *y*-coordinates of vector $\vec{\mathbf{r}}$. In this way, we obtain the connection between the polar coordinates and rectangular coordinates of point *P*:

Note:
Equation:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}.$$



Using polar coordinates, the unit vector $\mathbf{\hat{r}}$ defines the positive direction along the radius *r* (radial direction) and, orthogonal to it, the unit vector $\mathbf{\hat{t}}$ defines the positive direction of rotation by the angle φ.

Example:

Polar Coordinates

A treasure hunter finds one silver coin at a location 20.0 m away from a dry well in the direction 20° north of east and finds one gold coin at a location 10.0 m away from the well in the direction 20° north of west. What are the polar and rectangular coordinates of these findings with respect to the well?

Strategy

The well marks the origin of the coordinate system and east is the +*x*-direction. We identify radial distances from the locations to the origin, which are $r_S = 20.0\,\text{m}$ (for the silver coin) and $r_G = 10.0\,\text{m}$ (for the gold coin). To find the angular coordinates, we convert 20° to radians: $20^\circ = \pi 20/180 = \pi/9$. We use [link] to find the *x*- and *y*-coordinates of the coins.

Solution

The angular coordinate of the silver coin is $\varphi_S = \pi/9$, whereas the angular coordinate of the gold coin is $\varphi_G = \pi - \pi/9 = 8\pi/9$. Hence, the polar coordinates of the silver coin are $(r_S, \varphi_S) = (20.0\,\text{m}, \pi/9)$ and those of the gold coin are $(r_G, \varphi_G) = (10.0\,\text{m}, 8\pi/9)$. We substitute these coordinates into [link] to obtain rectangular coordinates. For the gold coin, the coordinates are

Equation:

$$\begin{cases} x_G = r_G \cos\varphi_G = (10.0\,\text{m})\cos 8\pi/9 = -9.4\,\text{m} \\ y_G = r_G \sin\varphi_G = (10.0\,\text{m})\sin 8\pi/9 = 3.4\,\text{m} \end{cases} \Rightarrow \ (x_G, y_G) = (-9.4\,\text{m}, 3.4\,\text{m}).$$

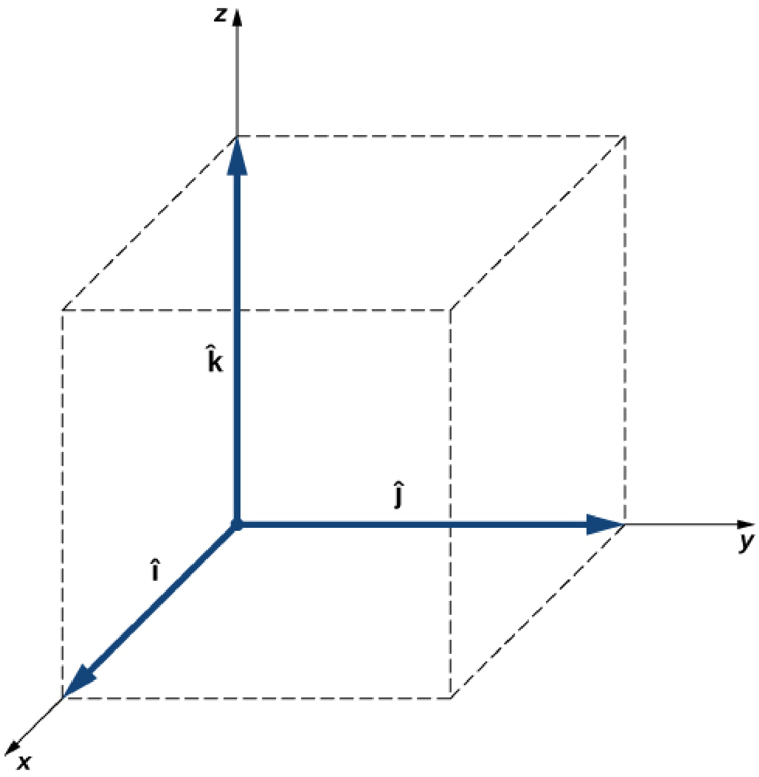
For the silver coin, the coordinates are

Equation:

$$\begin{cases} x_S = r_S \cos\varphi_S = (20.0\,\text{m})\cos \pi/9 = 18.9\,\text{m} \\ y_S = r_S \sin\varphi_S = (20.0\,\text{m})\sin \pi/9 = 6.8\,\text{m} \end{cases} \Rightarrow \ (x_S, y_S) = (18.9\,\text{m}, 6.8\,\text{m}).$$

Vectors in Three Dimensions

To specify the location of a point in space, we need three coordinates (*x*, *y*, *z*), where coordinates *x* and *y* specify locations in a plane, and coordinate *z* gives a vertical position above or below the plane. Three-dimensional space has three orthogonal directions, so we need not two but *three* unit vectors to define a three-dimensional coordinate system. In the Cartesian coordinate system, the first two unit vectors are the unit vector of the *x*-axis $\hat{\mathbf{i}}$ and the unit vector of the *y*-axis $\hat{\mathbf{j}}$. The third unit vector $\widehat{\mathbf{k}}$ is the direction of the *z*-axis ([link]). The order in which the axes are labeled, which is the order in which the three unit vectors appear, is important because it defines the orientation of the coordinate system. The order *x*-*y*-*z*, which is equivalent to the order $\hat{\mathbf{i}}$ - $\hat{\mathbf{j}}$ - $\widehat{\mathbf{k}}$, defines the standard right-handed coordinate system (positive orientation).



Three unit vectors define a Cartesian system in three-dimensional space. The order in which these unit vectors appear defines the orientation of the coordinate system. The order shown here defines the right-handed orientation.

In three-dimensional space, vector $\vec{\mathbf{A}}$ has three vector components: the *x*-component $\vec{\mathbf{A}}_x = A_x\hat{\mathbf{i}}$, which is the part of vector $\vec{\mathbf{A}}$ along the *x*-axis; the *y*-component $\vec{\mathbf{A}}_y = A_y\hat{\mathbf{j}}$, which is the part of $\vec{\mathbf{A}}$ along the *y*-axis; and the *z*-component $\vec{\mathbf{A}}_z = A_z\widehat{\mathbf{k}}$, which is the part of the vector along the *z*-axis. A vector in three-dimensional space is the vector sum of its three vector components ([link]):

Note:
Equation:

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\widehat{\mathbf{k}}.$$

If we know the coordinates of its origin $b(x_b, y_b, z_b)$ and of its end $e(x_e, y_e, z_e)$, its scalar components are obtained by taking their differences: A_x and A_y are given by [link] and the *z*-component is given by

Note:
Equation:

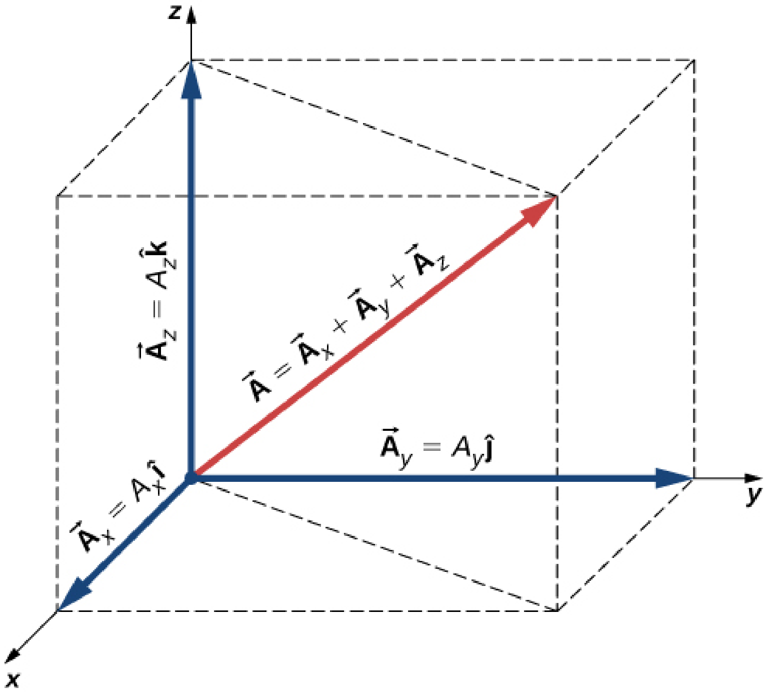
$$A_z = z_e - z_b.$$

Magnitude *A* is obtained by generalizing [link] to three dimensions:

Note:
Equation:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

This expression for the vector magnitude comes from applying the Pythagorean theorem twice. As seen in [link], the diagonal in the *xy*-plane has length $\sqrt{A_x^2 + A_y^2}$ and its square adds to the square A_z^2 to give A^2. Note that when the *z*-component is zero, the vector lies entirely in the *xy*-plane and its description is reduced to two dimensions.


z
$\vec{\mathbf{A}}_z = A_z\hat{\mathbf{k}}$
$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z$
$\vec{\mathbf{A}}_y = A_y\hat{\mathbf{j}}$
y
$\vec{\mathbf{A}}_x = A_x\hat{\mathbf{i}}$
x

A vector in three-dimensional space is the vector sum of its three vector components.

Example:
Takeoff of a Drone
During a takeoff of IAI Heron ([link]), its position with respect to a control tower is 100 m above the ground, 300 m to the east, and 200 m to the north. One minute later, its position is 250 m above the ground, 1200 m to the east, and 2100 m to the north. What is the drone’s displacement vector with respect to the control tower? What is the magnitude of its displacement vector?

The drone IAI Heron in flight. (credit: SSgt Reynaldo Ramon, USAF)

Strategy

We take the origin of the Cartesian coordinate system as the control tower. The direction of the +*x*-axis is given by unit vector $\hat{\mathbf{i}}$ to the east, the direction of the +*y*-axis is given by unit vector $\hat{\mathbf{j}}$ to the north, and the direction of the +*z*-axis is given by unit vector $\widehat{\mathbf{k}}$, which points up from the ground. The drone's first position is the origin (or, equivalently, the beginning) of the displacement vector and its second position is the end of the displacement vector.

Solution

We identify *b*(300.0 m, 200.0 m, 100.0 m) and *e*(1200 m, 2100 m, 250 m), and use [link] and [link] to find the scalar components of the drone's displacement vector:

Equation:

$$\begin{cases} D_x = x_e - x_b = 1200.0\,\text{m} - 300.0\,\text{m} = 900.0\,\text{m}, \\ D_y = y_e - y_b = 2100.0\,\text{m} - 200.0\,\text{m} = 1900.0\,\text{m}, \\ D_z = z_e - z_b = 250.0\,\text{m} - 100.0\,\text{m} = 150.0\,\text{m}. \end{cases}$$

We substitute these components into [link] to find the displacement vector:

Equation:

$$\vec{\mathbf{D}} = D_x\hat{\mathbf{i}} + D_y\hat{\mathbf{j}} + D_z\widehat{\mathbf{k}} = 900.0\,\text{m}\hat{\mathbf{i}} + 1900.0\,\text{m}\hat{\mathbf{j}} + 150.0\,\text{m}\widehat{\mathbf{k}} = (0.90\hat{\mathbf{i}} + 1.90\hat{\mathbf{j}} + 0.15\widehat{\mathbf{k}})\,\text{km}.$$

We substitute into [link] to find the magnitude of the displacement:

Equation:

$$D = \sqrt{D_x^2 + D_y^2 + D_z^2} = \sqrt{(0.90\,\text{km})^2 + (1.90\,\text{km})^2 + (0.15\,\text{km})^2} = 2.11\,\text{km}.$$

Note:

Exercise:

Problem:

Check Your Understanding If the average velocity vector of the drone in the displacement in [link] is $\vec{\mathbf{u}} = (15.0\hat{\mathbf{i}} + 31.7\hat{\mathbf{j}} + 2.5\widehat{\mathbf{k}})\mathrm{m/s}$, what is the magnitude of the drone's velocity vector?

Solution:

35.1 m/s = 126.4 km/h

Summary

- Vectors are described in terms of their components in a coordinate system. In two dimensions (in a plane), vectors have two components. In three dimensions (in space), vectors have three components.
- A vector component of a vector is its part in an axis direction. The vector component is the product of the unit vector of an axis with its scalar component along this axis. A vector is the resultant of its vector components.
- Scalar components of a vector are differences of coordinates, where coordinates of the origin are subtracted from end point coordinates of a vector. In a rectangular system, the magnitude of a vector is the square root of the sum of the squares of its components.
- In a plane, the direction of a vector is given by an angle the vector has with the positive *x*-axis. This direction angle is measured counterclockwise. The scalar *x*-component of a vector can be expressed as the product of its magnitude with the cosine of its direction angle, and the scalar *y*-component can be expressed as the product of its magnitude with the sine of its direction angle.
- In a plane, there are two equivalent coordinate systems. The Cartesian coordinate system is defined by unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ along the *x*-axis and the *y*-axis, respectively. The polar coordinate system is defined by the radial unit vector $\hat{\mathbf{r}}$, which gives the direction from the origin, and a unit vector $\hat{\mathbf{t}}$, which is perpendicular (orthogonal) to the radial direction.

Conceptual Questions

Exercise:

Problem: Give an example of a nonzero vector that has a component of zero.

Solution:

a unit vector of the *x*-axis

Exercise:

Problem: Explain why a vector cannot have a component greater than its own magnitude.

Exercise:

Problem: If two vectors are equal, what can you say about their components?

Solution:

They are equal.

Exercise:

Problem:

If vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are orthogonal, what is the component of $\vec{\mathbf{B}}$ along the direction of $\vec{\mathbf{A}}$? What is the component of $\vec{\mathbf{A}}$ along the direction of $\vec{\mathbf{B}}$?

Exercise:

Problem:

If one of the two components of a vector is not zero, can the magnitude of the other vector component of this vector be zero?

Solution:

yes

Exercise:

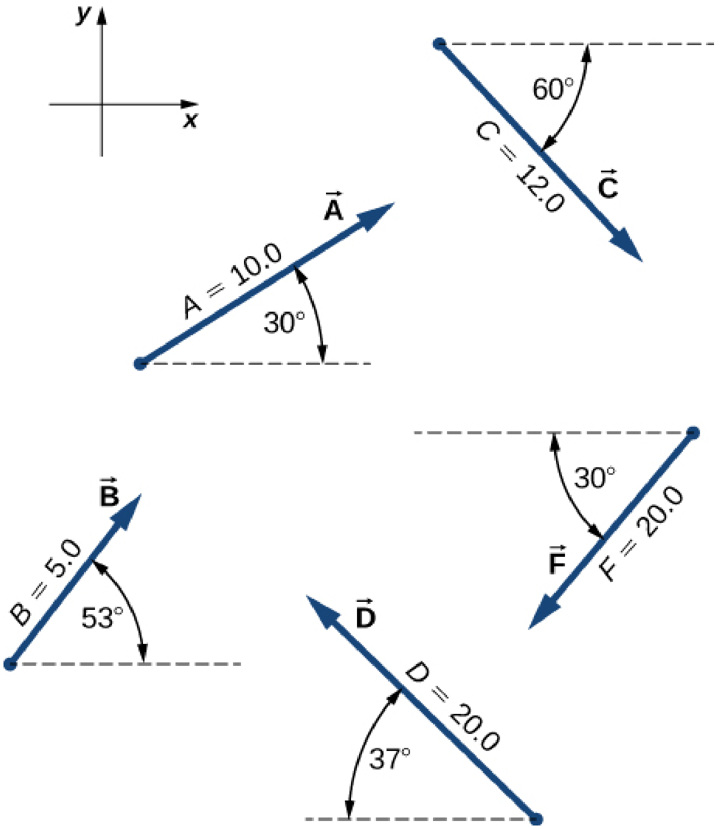
Problem: If two vectors have the same magnitude, do their components have to be the same?

Problems

Exercise:

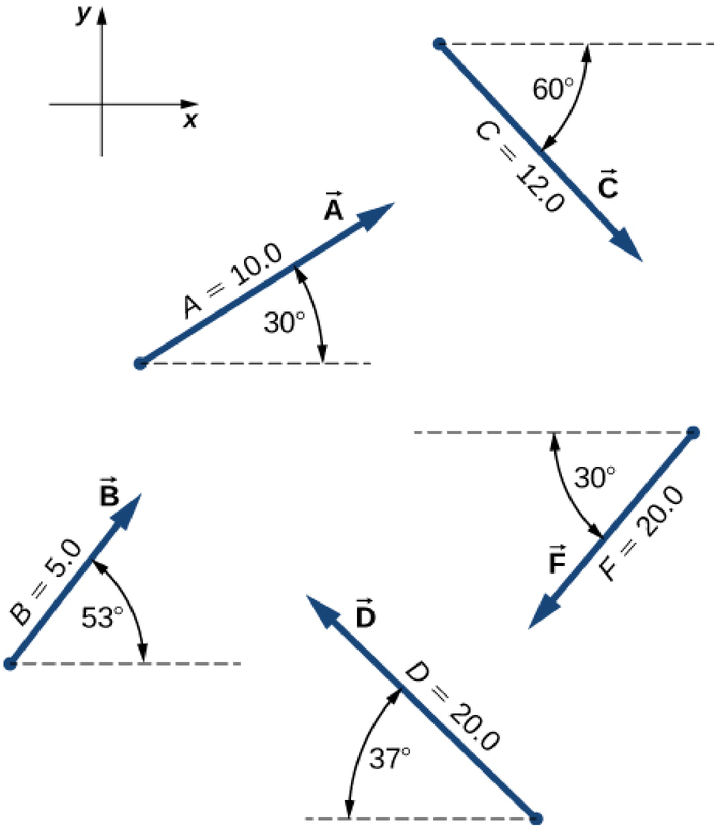
Problem:

Assuming the +*x*-axis is horizontal and points to the right, resolve the vectors given in the following figure to their scalar components and express them in vector component form.


y
x
$\vec{A}$
A = 10.0
30°
$\vec{C}$
C = 12.0
60°
$\vec{B}$
B = 5.0
53°
$\vec{F}$
F = 20.0
30°
$\vec{D}$
D = 20.0
37°

Solution:

a. $\vec{\mathbf{A}} = +8.66\hat{\mathbf{i}} + 5.00\hat{\mathbf{j}}$, b. $\vec{\mathbf{B}} = +3.01\hat{\mathbf{i}} + 3.99\hat{\mathbf{j}}$, c. $\vec{\mathbf{C}} = +6.00\hat{\mathbf{i}} - 10.39\hat{\mathbf{j}}$, d. $\vec{\mathbf{D}} = -15.97\hat{\mathbf{i}} + 12.04\hat{\mathbf{j}}$, f. $\vec{\mathbf{F}} = -17.32\hat{\mathbf{i}} - 10.00\hat{\mathbf{j}}$



Exercise:

Problem:

Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point? What is your displacement vector? What is the direction of your displacement? Assume the +*x*-axis is to the east.

Exercise:

Problem:

You drive 7.50 km in a straight line in a direction 15° east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (b) Show that you still arrive at the same point if the east and north legs are reversed in order. Assume the +*x*-axis is to the east.

Solution:

a. 1.94 km, 7.24 km; b. proof

Exercise:

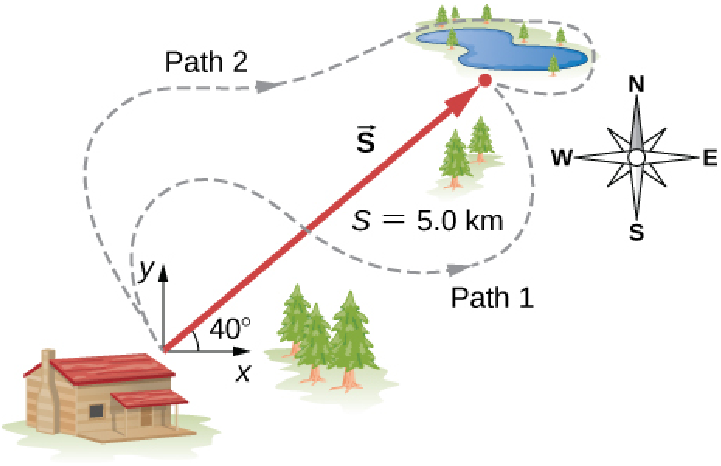
Problem:

A sledge is being pulled by two horses on a flat terrain. The net force on the sledge can be expressed in the Cartesian coordinate system as vector $\vec{\mathbf{F}} = (-2980.0\hat{\mathbf{i}} + 8200.0\hat{\mathbf{j}})\text{N}$, where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ denote directions to the east and north, respectively. Find the magnitude and direction of the pull.

Exercise:

Problem:

A trapper walks a 5.0-km straight-line distance from her cabin to the lake, as shown in the following figure. Determine the east and north components of her displacement vector. How many more kilometers would she have to walk if she walked along the component displacements? What is her displacement vector?



Solution:

3.8 km east, 3.2 km north, 2.0 km, $\vec{\mathbf{D}} = (3.8\hat{\mathbf{i}} + 3.2\hat{\mathbf{j}})\text{km}$

Exercise:

Problem:

The polar coordinates of a point are $4\pi/3$ and 5.50 m. What are its Cartesian coordinates?

Exercise:

Problem:

Two points in a plane have polar coordinates $P_1(2.500\,\text{m}, \pi/6)$ and $P_2(3.800\,\text{m}, 2\pi/3)$. Determine their Cartesian coordinates and the distance between them in the Cartesian coordinate system. Round the distance to a nearest centimeter.

Solution:

$P_1(2.165\,\text{m}, 1.250\,\text{m})$, $P_2(-1.900\,\text{m}, 3.290\,\text{m})$, 5.27 m

Exercise:

Problem:

A chameleon is resting quietly on a lanai screen, waiting for an insect to come by. Assume the origin of a Cartesian coordinate system at the lower left-hand corner of the screen and the horizontal direction to the right as the +*x*-direction. If its coordinates are (2.000 m, 1.000 m), (a) how far is it from the corner of the screen? (b) What is its location in polar coordinates?

Exercise:

Problem:

Two points in the Cartesian plane are *A*(2.00 m, −4.00 m) and *B*(−3.00 m, 3.00 m). Find the distance between them and their polar coordinates.

Solution:

8.60 m, $A(2\sqrt{5}\,\text{m}, 0.647\pi)$, $B(3\sqrt{2}\,\text{m}, 0.75\pi)$

Exercise:

Problem:

A fly enters through an open window and zooms around the room. In a Cartesian coordinate system with three axes along three edges of the room, the fly changes its position from point *b*(4.0 m, 1.5 m, 2.5 m) to point *e*(1.0 m, 4.5 m, 0.5 m). Find the scalar components of the fly's displacement vector and express its displacement vector in vector component form. What is its magnitude?

Glossary

component form of a vector
: a vector written as the vector sum of its components in terms of unit vectors

direction angle
: in a plane, an angle between the positive direction of the *x*-axis and the vector, measured counterclockwise from the axis to the vector

polar coordinate system
: an orthogonal coordinate system where location in a plane is given by polar coordinates

polar coordinates
: a radial coordinate and an angle

radial coordinate
: distance to the origin in a polar coordinate system

scalar component

a number that multiplies a unit vector in a vector component of a vector

unit vectors of the axes
unit vectors that define orthogonal directions in a plane or in space

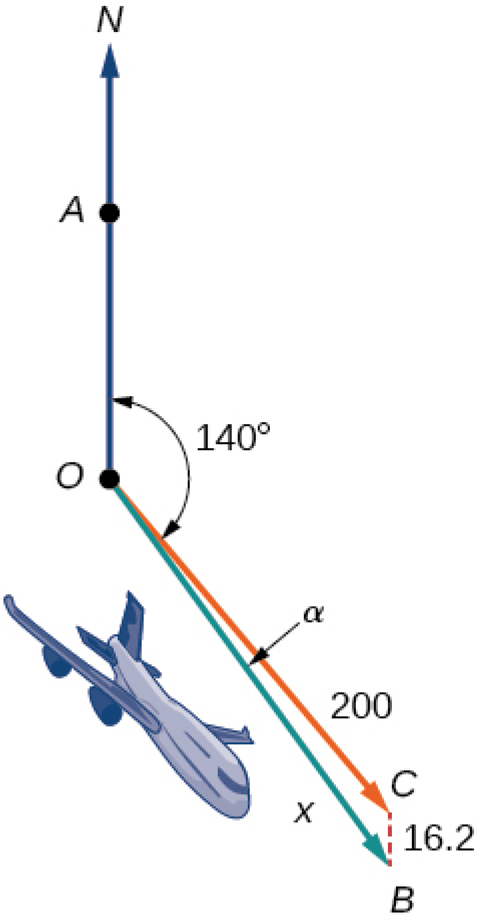
vector components
orthogonal components of a vector; a vector is the vector sum of its vector components.

Vectors

In this section you will:

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of v.
- Perform operations with vectors in terms of i and j.
- Find the dot product of two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in [link]. What are the ground speed and actual bearing of the plane?



Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's groundspeed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A **vector** is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its **magnitude**, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

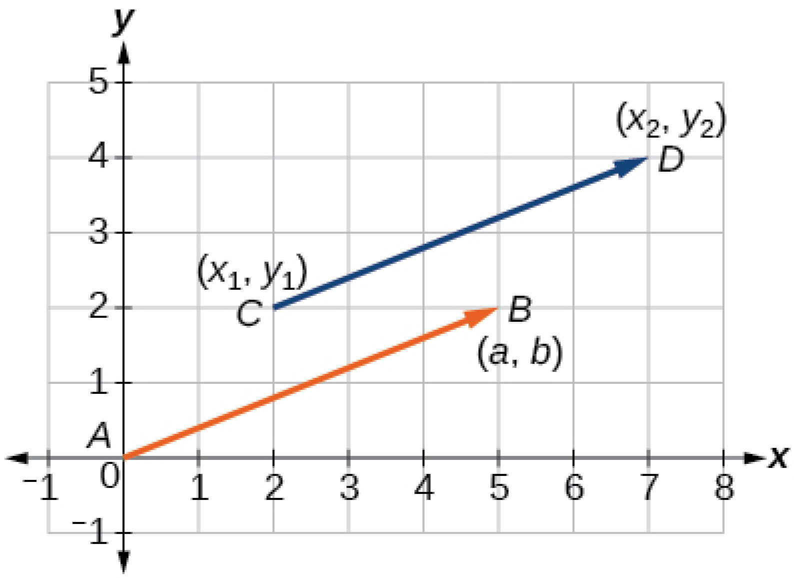
- Lower case, boldfaced type, with or without an arrow on top such as v, $u, w, \vec{v}, \vec{u},\ \vec{w}$.
- Given initial point P and terminal point Q, a vector can be represented as PQ. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0, 0)$ and terminal point (a, b), a vector may be represented as $\langle a, b\rangle$.

This last symbol $\langle a, b\rangle$ has special significance. It is called the **standard position**. The position vector has an initial point $(0, 0)$ and a terminal point (a, b). To change any vector into the position vector, we think about the change in the *x*-coordinates and the change in the *y*-coordinates. Thus, if the initial point of a vector CD is $C\left(x_1, y_1\right)$ and the terminal point is $D\left(x_2, y_2\right)$, then the position vector is found by calculating

Equation:

$$\begin{aligned} AB &= \langle x_2 - x_1, y_2 - y_1 \rangle \\ &= \langle a, b \rangle \end{aligned}$$

In [link], we see the original vector CD and the position vector AB.



Note:
Properties of Vectors
A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at $(0, 0)$ and is identified by its terminal point (a, b).

Example:
Exercise:

Problem:
Find the Position Vector

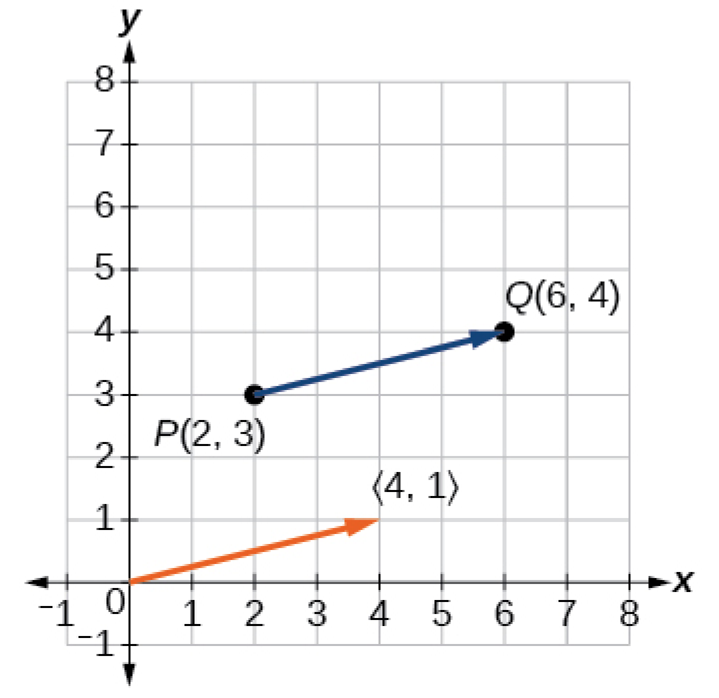
Consider the vector whose initial point is $P\left(2,3\right)$ and terminal point is $Q\left(6,4\right)$. Find the position vector.

Solution:

The position vector is found by subtracting one *x*-coordinate from the other *x*-coordinate, and one *y*-coordinate from the other *y*-coordinate. Thus
Equation:

$$\begin{array}{l}v=\langle 6-2,4-3\rangle \\ =\langle 4,1\rangle \end{array}$$

The position vector begins at $\left(0,0\right)$ and terminates at $\left(4,1\right)$. The graphs of both vectors are shown in [link].



We see that the position vector is $\langle 4,1\rangle$.

Example:
Exercise:

Problem:
Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

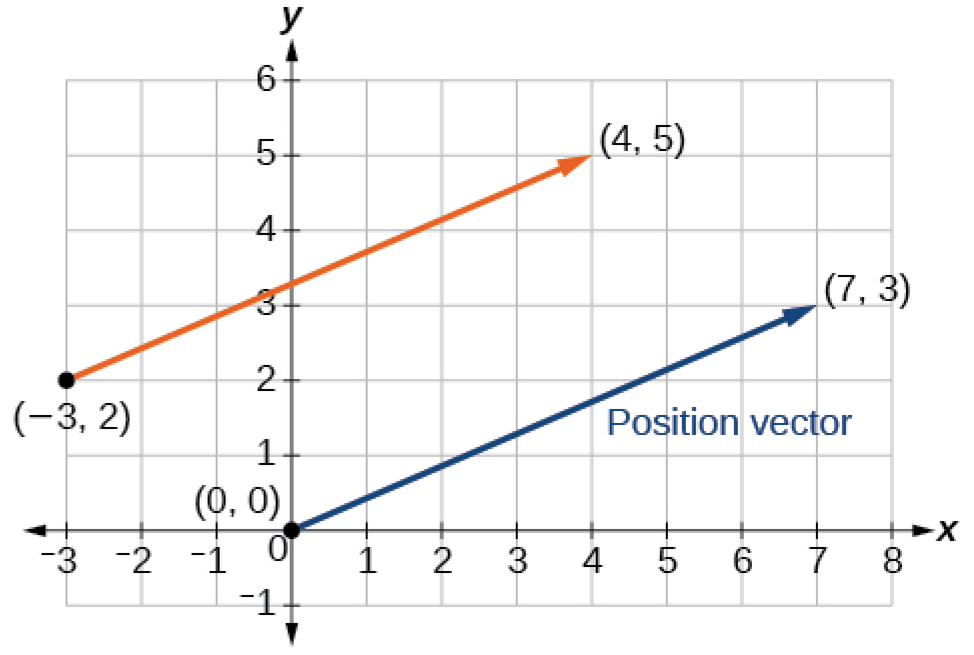
Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$, then graph both vectors in the same plane.

Solution:

The position vector is found using the following calculation:
Equation:

$$\begin{aligned} v &= \langle 4-(-3), 5-2\rangle \\ &= \langle 7, 3\rangle \end{aligned}$$

Thus, the position vector begins at $(0, 0)$ and terminates at $(7, 3)$. See [link].

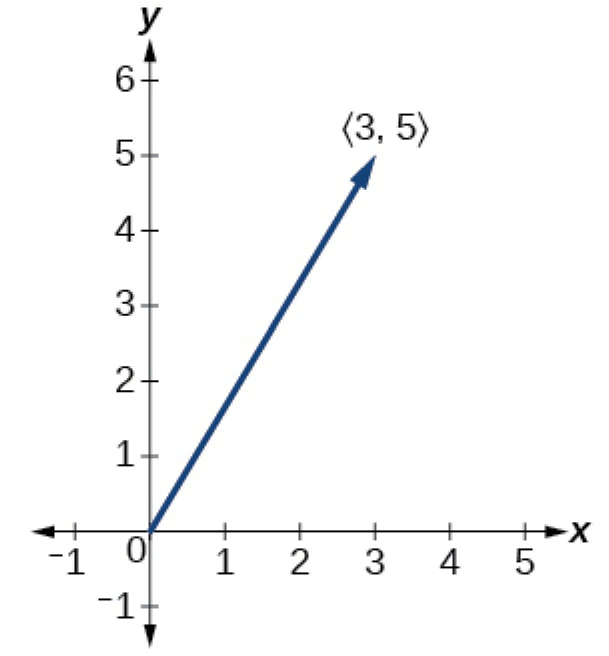


Note:
Exercise:

Problem:

Draw a vector v that connects from the origin to the point $(3, 5)$.

Solution:

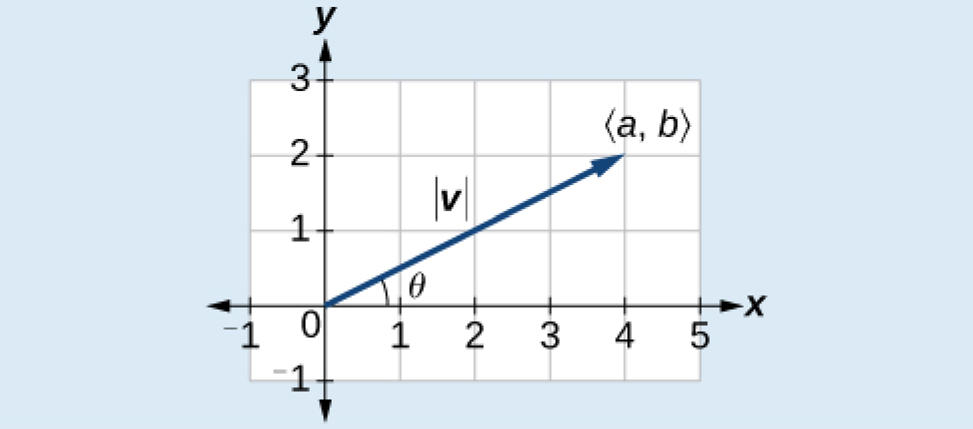


Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

Note:
Magnitude and Direction of a Vector

Given a position vector $v = \langle a, b \rangle$, the magnitude is found by $|v| = \sqrt{a^2 + b^2}$. The direction is equal to the angle formed with the *x*-axis, or with the *y*-axis, depending on the application. For a position vector, the direction is found by $\tan\ \theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in [link].



Two vectors ***v*** and ***u*** are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example:
Exercise:

Problem:
Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P\left(-8, 1\right)$ and terminal point $Q\left(-2, -5\right)$. Draw the vector.

Solution:

First, find the position vector.
Equation:

$$\begin{aligned} u &= \langle -2, -(-8), -5-1 \rangle \\ &= \langle 6, -6 \rangle \end{aligned}$$

We use the Pythagorean Theorem to find the magnitude.

Equation:

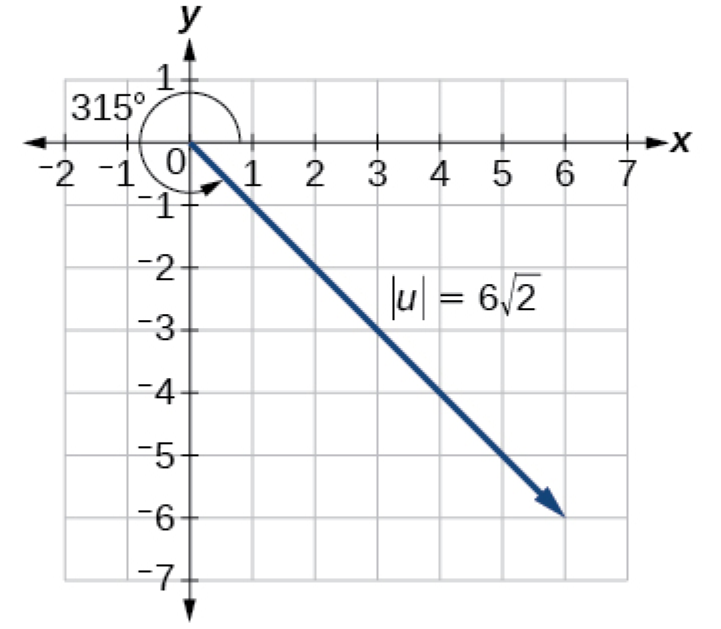
$$\begin{aligned} u &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= 6\sqrt{2} \end{aligned}$$

The direction is given as

Equation:

$$\begin{aligned} &\tan\ \theta = \tfrac{-6}{6} = -1 \Rightarrow \theta = \tan^{-1}(-1) \\ &= -45^\circ \end{aligned}$$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45^\circ + 360^\circ = 315^\circ$. See [link].



Example:

Exercise:

Problem:
Showing That Two Vectors Are Equal

Show that vector ***v*** with initial point at $(5,-3)$ and terminal point at $(-1,2)$ is equal to vector ***u*** with initial point at $(-1,-3)$ and terminal point at $(-7,2)$. Draw the position vector on the same grid as ***v*** and ***u***. Next, find the magnitude and direction of each vector.

Solution:

As shown in [link], draw the vector v starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector u with initial point $(-1,-3)$ and terminal point $(-7,2)$. Find the standard position for each.

Next, find and sketch the position vector for ***v*** and ***u***. We have
Equation:

$$\begin{aligned} v &= \langle -1-5, 2-(-3)\rangle \\ &= \langle -6,5\rangle \end{aligned}$$

$$\begin{aligned} u &= \langle -7-(-1), 2-(-3)\rangle \\ &= \langle -6,5\rangle \end{aligned}$$

Since the position vectors are the same, ***v*** and ***u*** are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.
Equation:

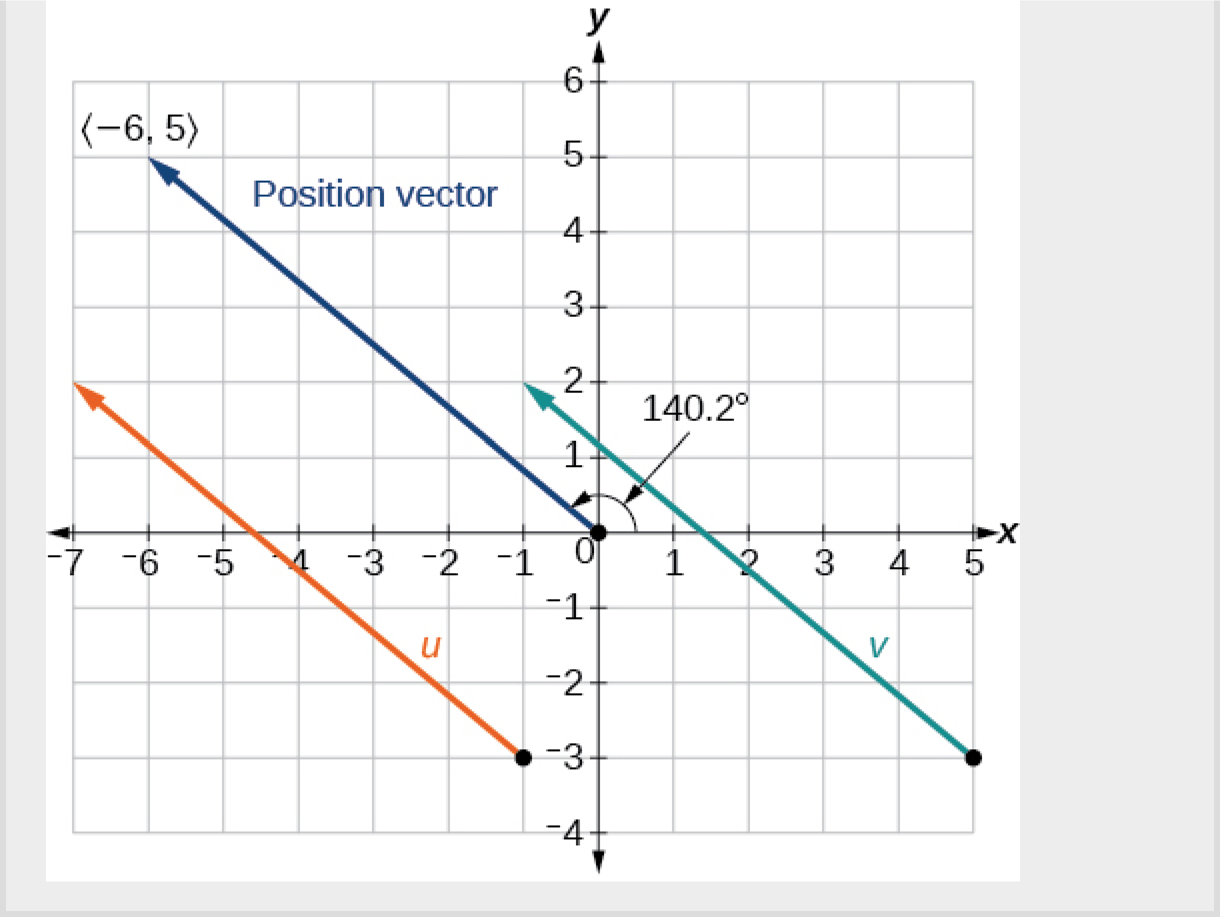
$$\begin{aligned} v &= \sqrt{(-1-5)^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \\ u &= \sqrt{(-7-(-1))^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \end{aligned}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

Equation:

$$\begin{aligned} &\tan\ \theta = -\tfrac{5}{6} \Rightarrow \theta = \tan^{-1}\left(-\tfrac{5}{6}\right) \\ &= -39.8^{\circ} \end{aligned}$$

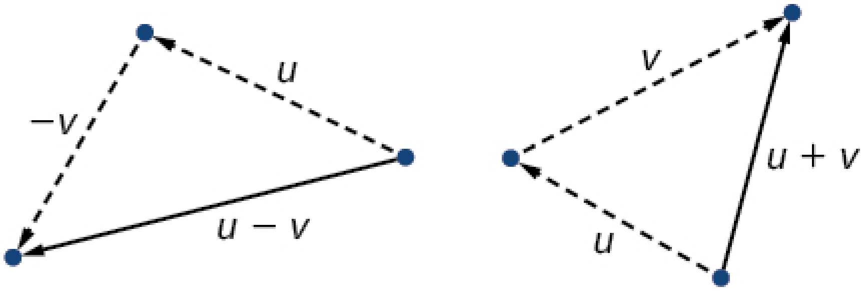
However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8^{\circ} + 180^{\circ} = 140.2^{\circ}$.



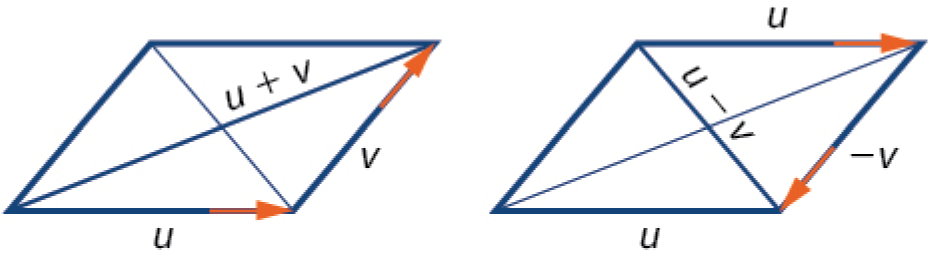
Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = \langle x, y \rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in the plane. The sum of two vectors ***u*** and ***v***, or **vector addition**, produces a third vector ***u***+ ***v***, the **resultant** vector.

To find ***u*** + ***v***, we first draw the vector ***u***, and from the terminal end of ***u***, we drawn the vector ***v***. In other words, we have the initial point of ***v*** meet the terminal end of ***u***. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum ***u*** + ***v*** is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of ***u*** to the end of ***v*** in a straight path, as shown in [link].



Vector subtraction is similar to vector addition. To find ***u*** − ***v***, view it as ***u*** + (−***v***). Adding −***v*** is reversing direction of ***v*** and adding it to the end of ***u***. The new vector begins at the start of ***u*** and stops at the end point of −***v***. See [link] for a visual that compares vector addition and vector subtraction using parallelograms.



Example:
Exercise:

Problem:
Adding and Subtracting Vectors

Given $u=\langle 3,-2\rangle$ and $v=\langle -1,4\rangle$, find two new vectors ***u*** + ***v***, and ***u*** − ***v***.

Solution:

To find the sum of two vectors, we add the components. Thus,
Equation:

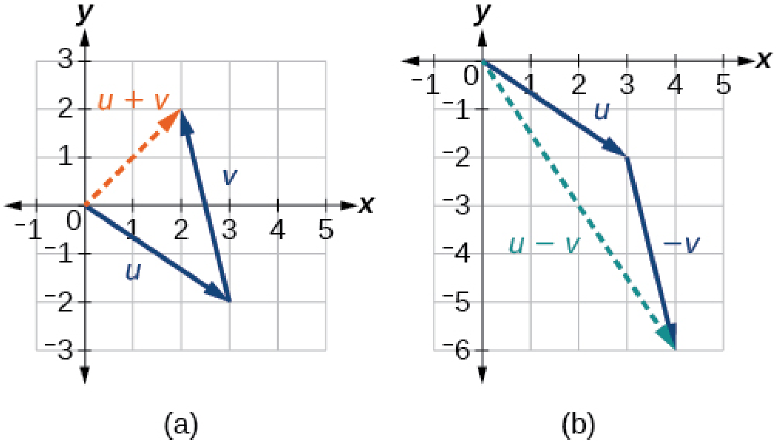
$$u+v=\langle 3,-2\rangle+\langle -1,4\rangle$$
$$=\langle 3+(-1),-2+4\rangle$$
$$=\langle 2,2\rangle$$

See [link]**(a)**.

To find the difference of two vectors, add the negative components of v to u. Thus,
Equation:

$$u+(-v)=\langle 3,-2\rangle+\langle 1,-4\rangle$$
$$=\langle 3+1,-2+(-4)\rangle$$
$$=\langle 4,-6\rangle$$

See [link]**(b).**


y
x
3
2
1
0
-1
-2
-3
-1
1
2
3
4
5
u + v
v
u
(a)
y
x
-1
0
1
2
3
4
5
-1
-2
-3
-4
-5
-6
u
u − v
−v
(b)

(a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

Note:
Scalar Multiplication
Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = \langle a, b\rangle$ by k, we have
Equation:

$$kv = \langle ka, kb\rangle$$

Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example:
Exercise:

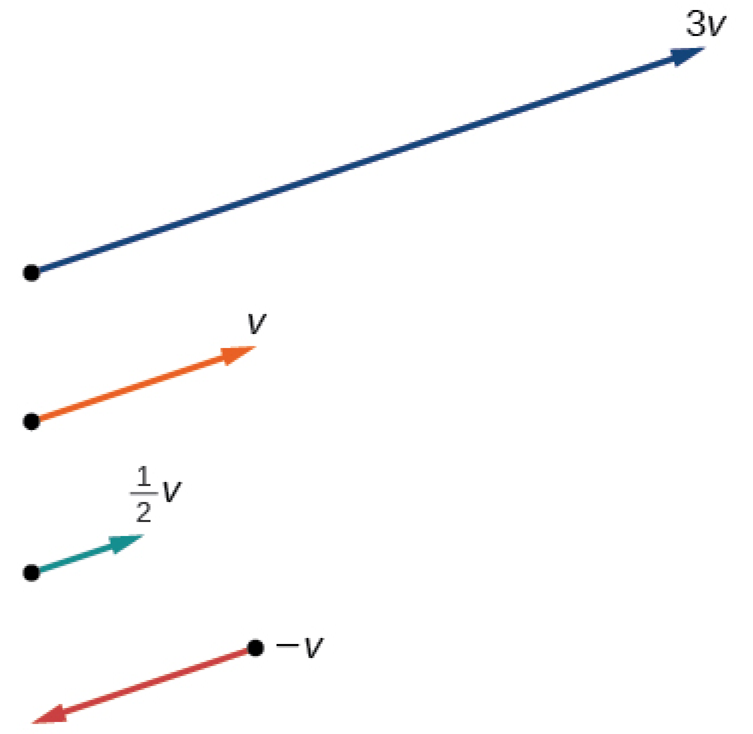
Problem:
Performing Scalar Multiplication

Given vector $v = \langle 3, 1\rangle$, find 3$\boldsymbol{v}$, $\frac{1}{2}$ v, and $-\boldsymbol{v}$.

Solution:

See [link] for a geometric interpretation. If $v = \langle 3, 1\rangle$, then
Equation:

$$\begin{aligned} 3v &= \langle 3\cdot 3, 3\cdot 1\rangle \\ &= \langle 9, 3\rangle \\ \tfrac{1}{2}v &= \left\langle \tfrac{1}{2}\cdot 3, \tfrac{1}{2}\cdot 1\right\rangle \\ &= \left\langle \tfrac{3}{2}, \tfrac{1}{2}\right\rangle \\ -v &= \langle -3, -1\rangle \end{aligned}$$



Analysis

Notice that the vector 3***v*** is three times the length of ***v***, $\frac{1}{2}$ v is half the length of ***v***, and –***v*** is the same length of ***v***, but in the opposite direction.

Note:
Exercise:

Problem: Find the scalar multiple 3 u given $u = \langle 5, 4\rangle$.

Solution:

$3u = \langle 15, 12 \rangle$

Example:
Exercise:

Problem:
Using Vector Addition and Scalar Multiplication to Find a New Vector

Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find a new vector ***w*** = 3***u*** + 2***v***.

Solution:

First, we must multiply each vector by the scalar.
Equation:

$$\begin{aligned} 3u &= 3\left\langle 3, -2 \right\rangle \\ &= \langle 9, -6 \rangle \\ 2v &= 2\left\langle -1, 4 \right\rangle \\ &= \langle -2, 8 \rangle \end{aligned}$$

Then, add the two together.
Equation:

$$\begin{aligned} w &= 3u + 2v \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 9 - 2, -6 + 8 \rangle \\ &= \langle 7, 2 \rangle \end{aligned}$$

So, $w = \langle 7, 2 \rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in [link] that the position vector $\langle 2,3\rangle$ comes from adding the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. We have $\boldsymbol{v}_1$ with initial point $(0,0)$ and terminal point $(2,0)$.

Equation:

$$\begin{array}{l}v_1=\langle 2-0,0-0\rangle \\ =\langle 2,0\rangle\end{array}$$

We also have $\boldsymbol{v}_2$ with initial point $(0,0)$ and terminal point $(0,\ 3)$.

Equation:

$$\begin{array}{l}v_2=\langle 0-0,3-0\rangle \\ =\langle 0,3\rangle\end{array}$$

Therefore, the position vector is

Equation:

$$\begin{array}{l}v=\langle 2+0,3+0\rangle \\ =\langle 2,3\rangle\end{array}$$

Using the Pythagorean Theorem, the magnitude of $\boldsymbol{v}_1$ is 2, and the magnitude of $\boldsymbol{v}_2$ is 3. To find the magnitude of $\boldsymbol{v}$, use the formula with the position vector.

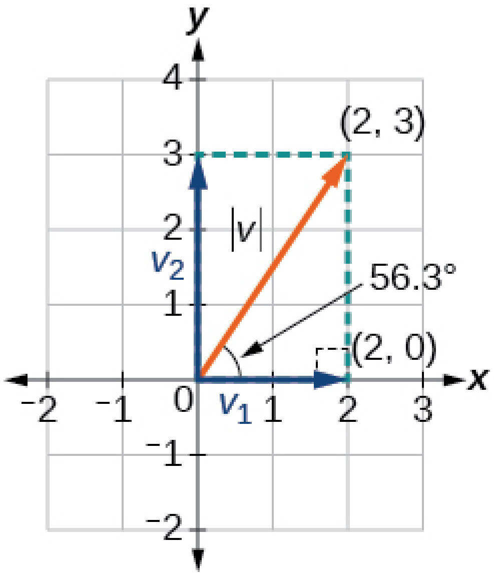
Equation:

$$\begin{aligned} v &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of ***v*** is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\ \theta = \frac{y}{x}$.

Equation:

$$\begin{aligned} &\tan\ \theta = \frac{v_2}{v_1} \\ &\tan\ \theta = \frac{3}{2} \\ &\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ \end{aligned}$$



Thus, the magnitude of v is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example:
Exercise:

Problem:
Finding the Components of the Vector

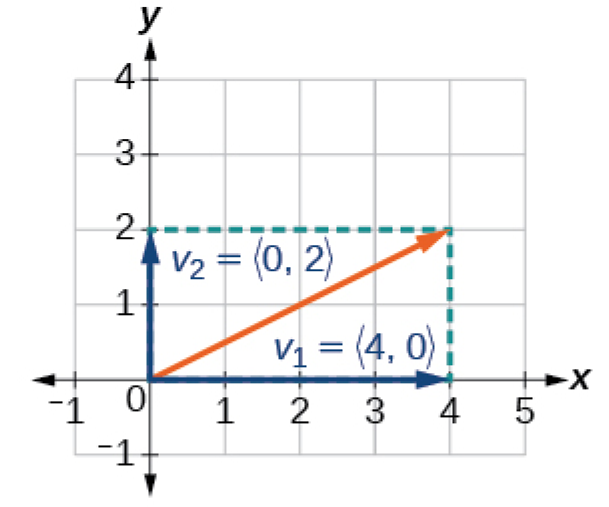
Find the components of the vector v with initial point $(3, 2)$ and terminal point $(7, 4)$.

Solution:

First find the standard position.
Equation:

$$\begin{aligned} v &= \langle 7-3, 4-2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration in [link].

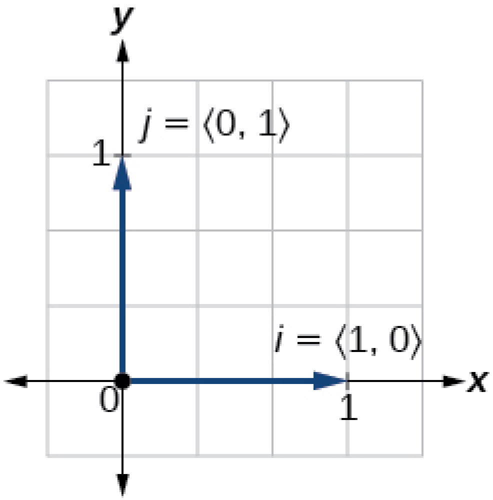


The horizontal component is $v_1 = \langle 4, 0 \rangle$ and the vertical component is $v_2 = \langle 0, 2 \ \rangle$.

Finding the Unit Vector in the Direction of *v*

In addition to finding a vector’s components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a **unit vector**. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $i = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $j = \langle 0, 1 \rangle$ and is directed along the positive vertical axis. See [link].



Note:
The Unit Vectors
If v is a nonzero vector, then $\frac{v}{|v|}$ is a unit vector in the direction of v. Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example:

Exercise:

Problem:
Finding the Unit Vector in the Direction of *v*

Find a unit vector in the same direction as $v = \langle -5, 12\, \rangle$.

Solution:

First, we will find the magnitude.
Equation:

$$\begin{aligned} v \ &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

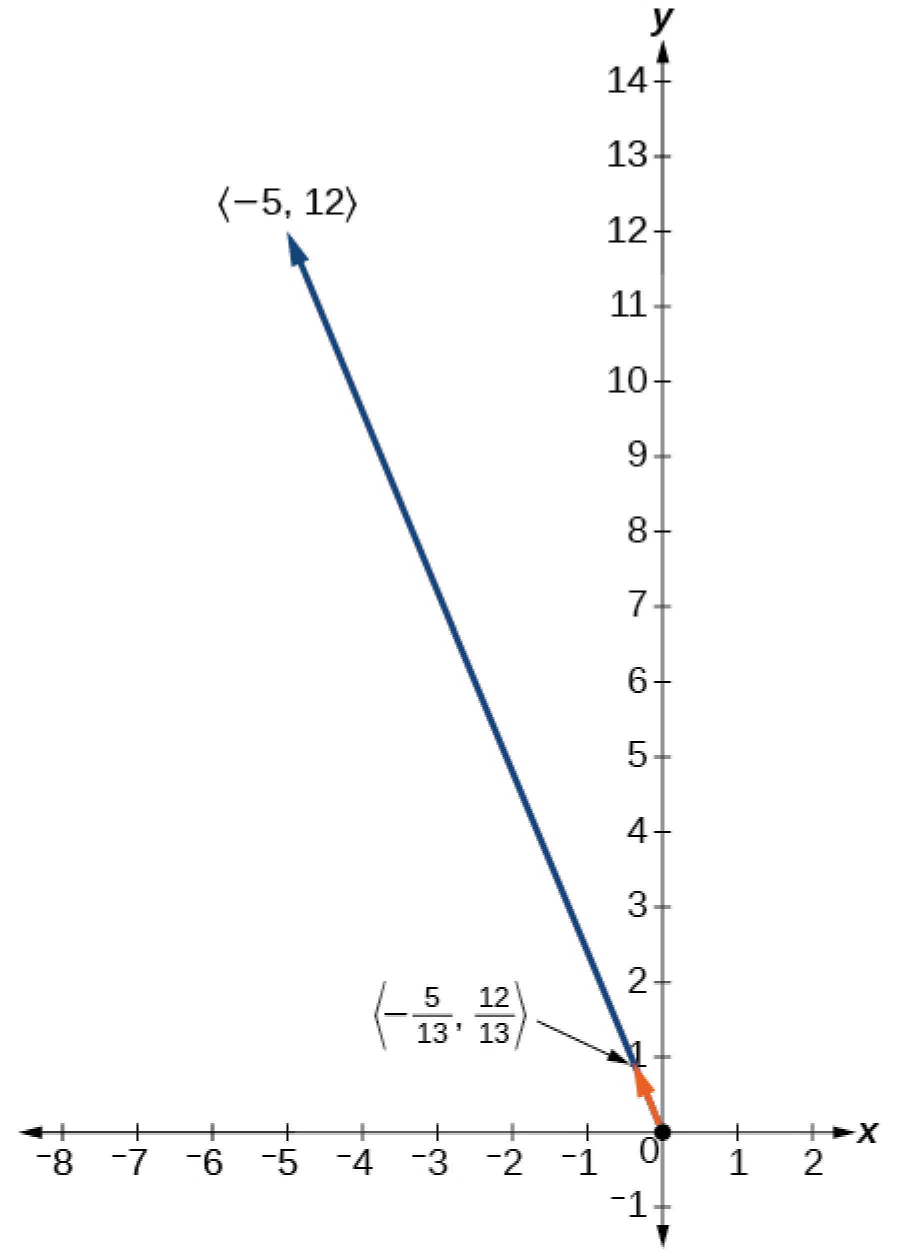
Then we divide each component by $|v\,|$, which gives a unit vector in the same direction as ***v***:
Equation:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form
Equation:

$$\frac{v}{|v|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$$

See [link].



Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i + \frac{12}{13}j$ is given as

Equation:

$$\begin{aligned} \sqrt{\left(-\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2} &= \sqrt{\frac{25}{169} + \frac{144}{169}} \\ &= \sqrt{\frac{169}{169}} = 1 \end{aligned}$$

The vector $\boldsymbol{u}=\frac{5}{13}\,\boldsymbol{i}+\frac{12}{13}\,\boldsymbol{j}$ is the unit vector in the same direction as $\boldsymbol{v}=\langle -5,12\rangle$.

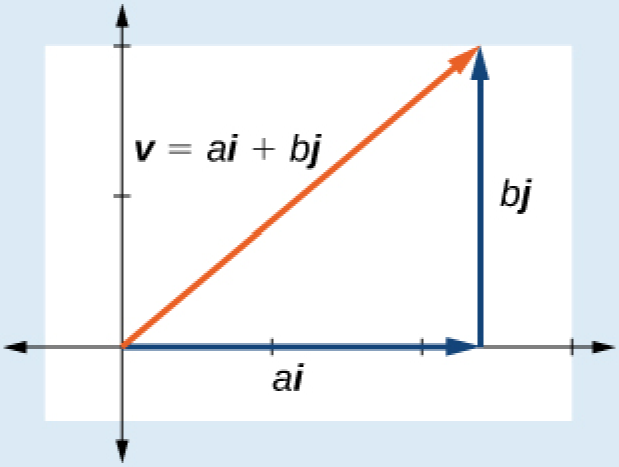
Performing Operations with Vectors in Terms of *i* and *j*

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of ***i*** and ***j***.

Note:
Vectors in the Rectangular Plane
Given a vector v with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2\ )$, ***v*** is written as
Equation:

$$v=(x_2-x_1)i+(y_2-y_1)j$$

The position vector from $(0,0)$ to (a,b), where $(x_2-x_1)=a$ and $(y_2-y_1)=b$, is written as ***v*** = *a**i*** + *b**j***. This vector sum is called a linear combination of the vectors ***i*** and ***j***.
The magnitude of ***v*** = *a**i*** + *b**j*** is given as $|v|=\sqrt{a^2+b^2}$. See [link].



Example:
Exercise:

Problem:
Writing a Vector in Terms of *i* and *j*

Given a vector v with initial point $P=(2,-6)$ and terminal point $Q=(-6,6)$, write the vector in terms of i and j.

Solution:

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

Equation:

$$\begin{aligned} v &= (x_2-x_1)i+(y_2-y_1)j \\ &= (-6-2)i+(6-(-6))j \\ &= -8i+12j \end{aligned}$$

Example:
Exercise:

Problem:
Writing a Vector in Terms of *i* and *j* Using Initial and Terminal Points

Given initial point $P_1 = (-1, 3)$ and terminal point $P_2 = (2, 7)$, write the vector v in terms of i and j.

Solution:

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

Equation:

$$\begin{aligned} v &= (x_2 - x_1)i + (y_2 - y_1)j \\ v &= (2 - (-1))i + (7 - 3)j \\ &= 3i + 4j \end{aligned}$$

Note:
Exercise:

Problem:

Write the vector u with initial point $P = (-1, 6)$ and terminal point $Q = (7, -5)$ in terms of i and j.

Solution:

$u = 8i - 11j$

Performing Operations on Vectors in Terms of *i* and *j*

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

Note:
Adding and Subtracting Vectors in Rectangular Coordinates
Given ***v*** = *a**i*** + *b**j*** and ***u*** = *c**i*** + *d**j***, then
Equation:

$$v + u = (a + c)i + (b + d)j$$
$$v - u = (a - c)i + (b - d)j$$

Example:
Exercise:

Problem:
Finding the Sum of the Vectors

Find the sum of $v_1 = 2i - 3j$ and $v_2 = 4i + 5j$.

Solution:

According to the formula, we have
Equation:

$$v_1 + v_2 = (2 + 4)i + (-3 + 5)j$$
$$= 6i + 2j$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using $i \text{ and } j$. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

Note:
Vector Components in Terms of Magnitude and Direction
Given a position vector $v = \langle x, y \rangle$ and a direction angle θ,
Equation:

$$\begin{array}{lcl} \cos\ \theta = \frac{x}{|v|} & \text{and} & \sin\ \theta = \frac{y}{|v|} \\ x = |v| \cos\ \theta & & y = |v| \sin\ \theta \end{array}$$

Thus, $v = xi + yj = |v| \cos\ \theta i + |v| \sin\ \theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Example:
Exercise:

Problem:
Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive *x*-axis in terms of magnitude and direction.

Solution:

Using the conversion formulas $x = |v| \cos\ \theta i$ and $y = |v| \sin\ \theta j$, we find that

Equation:

$$\begin{aligned} x &= 7\cos(135^\circ)i \\ &= -\tfrac{7\sqrt{2}}{2} \\ y &= 7\sin(135^\circ)j \\ &= \tfrac{7\sqrt{2}}{2} \end{aligned}$$

This vector can be written as $v = 7\cos(135^\circ)i + 7\sin(135^\circ)j$ or simplified as

Equation:

$$v = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

Note:

Exercise:

Problem:

A vector travels from the origin to the point $(3, 5)$. Write the vector in terms of magnitude and direction.

Solution:

$v = \sqrt{34}\cos(59^\circ)i + \sqrt{34}\sin(59^\circ)j$

Magnitude = $\sqrt{34}$

$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04^\circ$

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

Note:
Dot Product
The **dot product** of two vectors $v = \langle a, b \rangle$ and $u = \langle c, d \rangle$ is the sum of the product of the horizontal components and the product of the vertical components.
Equation:

$$v \cdot u = ac + bd$$

To find the angle between the two vectors, use the formula below.
Equation:

$$\cos\ \theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$$

Example:
Exercise:

Problem:
Finding the Dot Product of Two Vectors

Find the dot product of $v = \langle 5, 12 \rangle$ and $u = \langle -3, 4 \rangle$.

Solution:

Using the formula, we have
Equation:

$$\begin{aligned} v \cdot u &= \langle 5, 12\rangle \cdot \langle -3, 4\rangle \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33 \end{aligned}$$

Example:
Exercise:

Problem:
Finding the Dot Product of Two Vectors and the Angle between Them

Find the dot product of $\boldsymbol{v}_1 = 5\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{v}_2 = 3\boldsymbol{i} + 7\boldsymbol{j}$. Then, find the angle between the two vectors.

Solution:

Finding the dot product, we multiply corresponding components.
Equation:

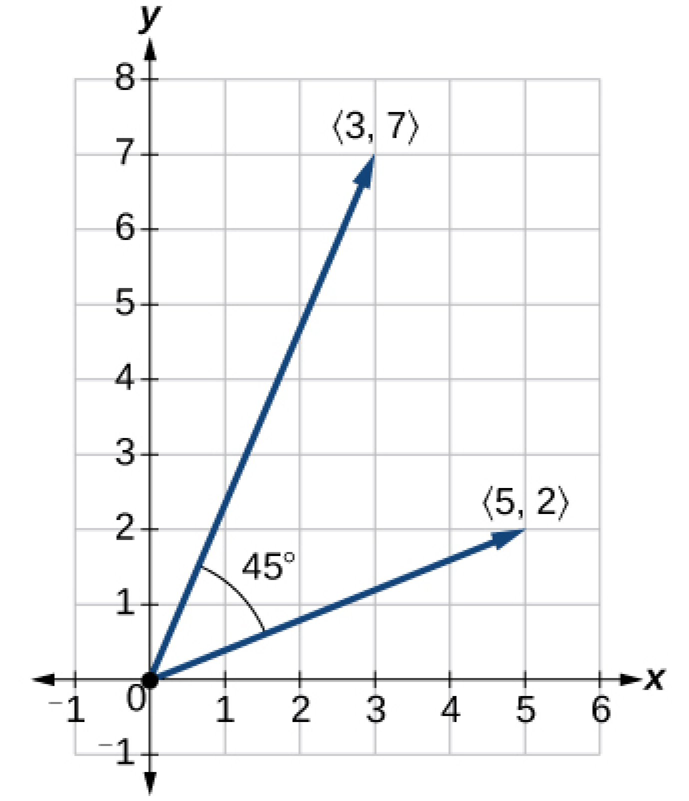
$$\begin{aligned} v_1 \cdot v_2 &= \langle 5, 2\rangle \cdot \langle 3, 7\rangle \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

To find the angle between them, we use the formula
$\cos\ \theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$.

Equation:

$$\begin{aligned}
&\frac{v}{|v|}\cdot\frac{u}{|u|}=\left\langle\frac{5}{\sqrt{29}}+\frac{2}{\sqrt{29}}\right\rangle\cdot\left\langle\frac{3}{\sqrt{58}}+\frac{7}{\sqrt{58}}\right\rangle\\
&=\frac{5}{\sqrt{29}}\cdot\frac{3}{\sqrt{58}}+\frac{2}{\sqrt{29}}\cdot\frac{7}{\sqrt{58}}\\
&=\frac{15}{\sqrt{1682}}+\frac{14}{\sqrt{1682}}=\frac{29}{\sqrt{1682}}\\
&=0.707107\\
&\cos^{-1}(0.707107)=45^{\circ}
\end{aligned}$$

See [link].



Example:
Exercise:

Problem:

Finding the Angle between Two Vectors

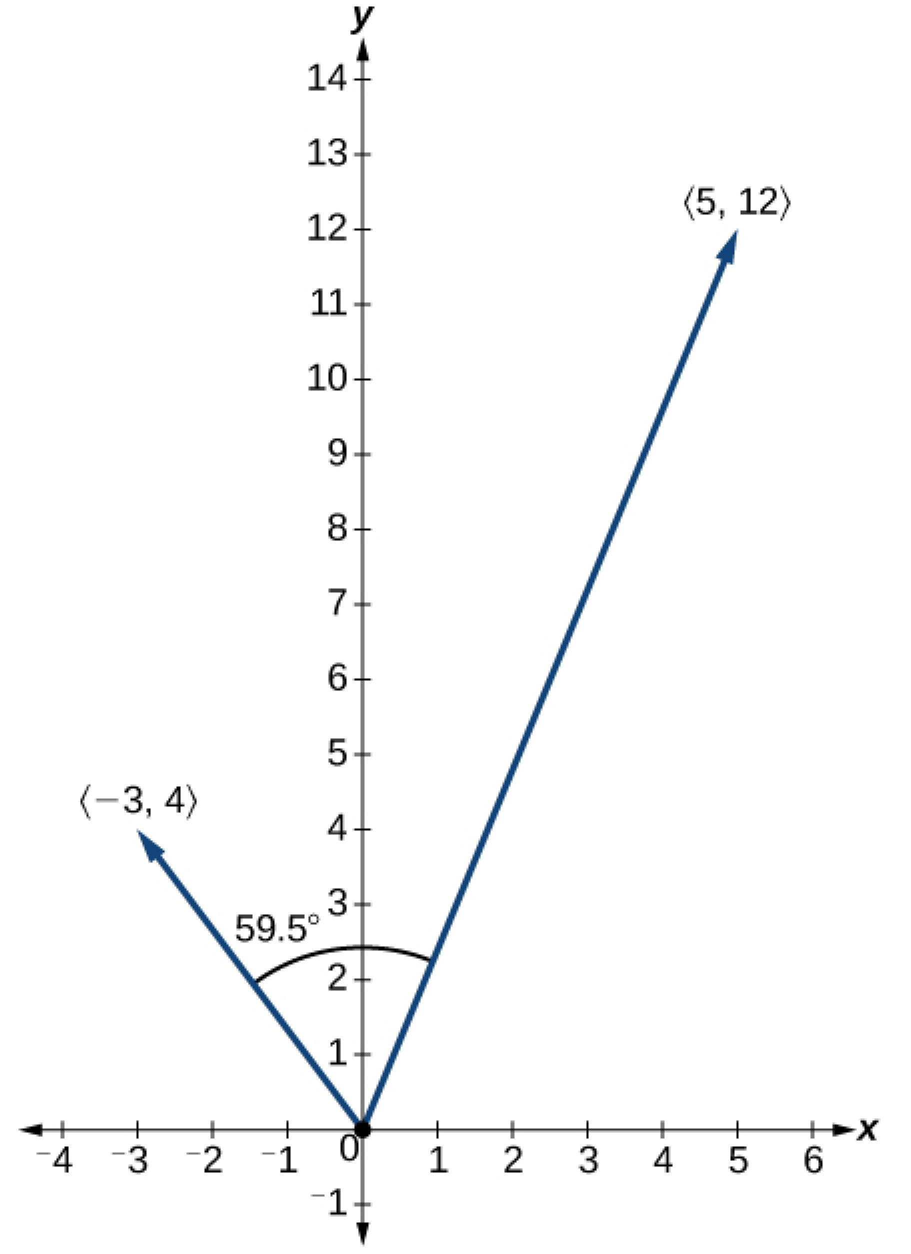
Find the angle between $u = \langle -3, 4 \rangle$ and $v = \langle 5, 12 \rangle$.

Solution:

Using the formula, we have
Equation:

$$\begin{aligned}
&\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) \\
&\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) = \frac{-3i+4j}{5} \cdot \frac{5i+12j}{13} \\
&= \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right) \\
&= -\frac{15}{65} + \frac{48}{65} \\
&= \frac{33}{65} \\
&\theta = \cos^{-1}\left(\frac{33}{65}\right) \\
&= 59.5^\circ
\end{aligned}$$

See [link].



Example:

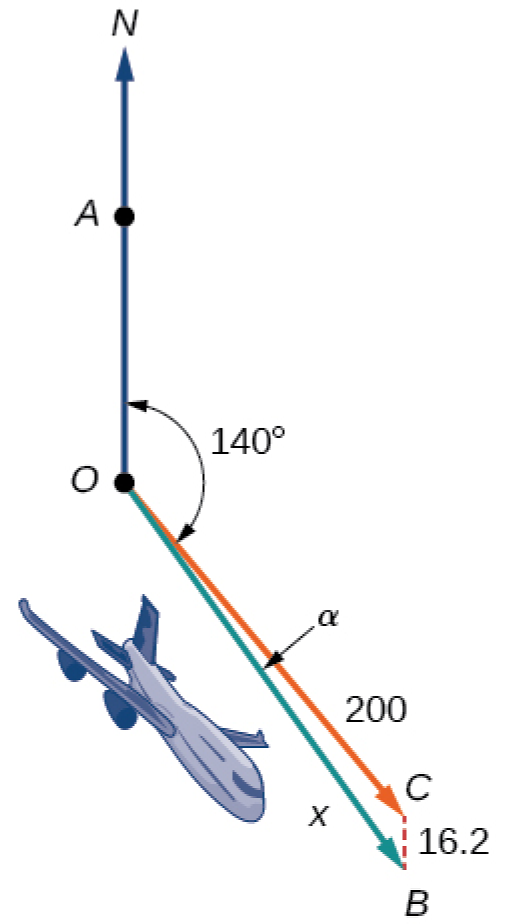
Exercise:

Problem:

Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See [link].



Solution:

The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140^\circ + \alpha$.

Notice in [link], that angle BCO must be equal to angle AOC by the rule of alternating interior angles, so angle BCO is 140°. We can find

x by the Law of Cosines:

Equation:

$$\begin{aligned} x^2 &= (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140^\circ) \\ x^2 &= 45,226.41 \\ x &= \sqrt{45,226.41} \\ x &= 212.7 \end{aligned}$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

Equation:

$$\begin{aligned} &\frac{\sin\ \alpha}{16.2} = \frac{\sin(140^\circ)}{212.7} \\ &\sin\ \alpha = \frac{16.2\sin(140^\circ)}{212.7} \\ &= 0.04896 \\ &\sin^{-1}(0.04896) = 2.8^\circ \end{aligned}$$

Therefore, the plane has a SE bearing of 140°+2.8°=142.8°. The ground speed is 212.7 miles per hour.

Note:
Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors
- Vector Operations
- The Unit Vector

Key Concepts

- The position vector has its initial point at the origin. See [link].
- If the position vector is the same for two vectors, they are equal. See [link].
- Vectors are defined by their magnitude and direction. See [link].
- If two vectors have the same magnitude and direction, they are equal. See [link].
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See [link].
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See [link] and [link].
- Vectors are comprised of two components: the horizontal component along the positive *x*-axis, and the vertical component along the positive *y*-axis. See [link].
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v|=\sqrt{a^2+b^2}$. See [link].
- In the rectangular coordinate system, unit vectors may be represented in terms of i and j where i represents the horizontal component and j represents the vertical component. Then, $\mathbf{v} = \mathrm{a}\boldsymbol{i} + \mathrm{b}\boldsymbol{j}$ is a scalar multiple of v by real numbers a and b. See [link] and [link].
- Adding and subtracting vectors in terms of *i* and *j* consists of adding or subtracting corresponding coefficients of *i* and corresponding coefficients of *j*. See [link].
- A vector $v = \boldsymbol{a}\boldsymbol{i} + \boldsymbol{b}\boldsymbol{j}$ is written in terms of magnitude and direction as $v=|v|\cos\ \theta i+|v|\sin\ \theta j$. See [link].
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See [link].
- We can use the dot product to find the angle between two vectors. [link] and [link].
- Dot products are useful for many types of physics applications. See [link].

Section Exercises

Verbal

Exercise:

Problem:

What are the characteristics of the letters that are commonly used to represent vectors?

Solution:

lowercase, bold letter, usually u, v, w

Exercise:

Problem: How is a vector more specific than a line segment?

Exercise:

Problem: What are i and j, and what do they represent?

Solution:

They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.

Exercise:

Problem: What is component form?

Exercise:

Problem:

When a unit vector is expressed as $\langle a, b \rangle$, which letter is the coefficient of the i and which the j?

Solution:

The first number always represents the coefficient of the i, and the second represents the j.

Algebraic

Exercise:

Problem:

Given a vector with initial point $(5,2)$ and terminal point $(-1,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

Exercise:

Problem:

Given a vector with initial point $(-4,2)$ and terminal point $(3,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

Solution:

$\langle 7,-5\rangle$

Exercise:

Problem:

Given a vector with initial point $(7,-1)$ and terminal point $(-1,-7)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

For the following exercises, determine whether the two vectors u and v are equal, where u has an initial point P_1 and a terminal point P_2 and v has an initial point P_3 and a terminal point P_4.

Exercise:

Problem:

$P_1 = (5,1), P_2 = (3,-2), P_3 = (-1,3)$, and $P_4 = (9,-4)$

Solution:

not equal

Exercise:

Problem:$P_1 = (2,-3), P_2 = (5,1), P_3 = (6,-1)$, and $P_4 = (9,3)$

Exercise:

Problem:

$P_1 = (-1,-1), P_2 = (-4,5), P_3 = (-10,6)$, and $P_4 = (-13,12)$

Solution:

equal

Exercise:

Problem:$P_1 = (3,7), P_2 = (2,1), P_3 = (1,2)$, and $P_4 = (-1,-4)$

Exercise:

Problem:$P_1 = (8,3), P_2 = (6,5), P_3 = (11,8)$, and $P_4 = (9,10)$

Solution:

equal

Exercise:

Problem:

Given initial point $P_1 = (-3,1)$ and terminal point $P_2 = (5,2)$, write the vector v in terms of i and j.

Exercise:

Problem:

Given initial point $P_1 = (6, 0)$ and terminal point $P_2 = (-1, -3)$, write the vector v in terms of i and j.

Solution:

$7i - 3j$

For the following exercises, use the vectors ***u*** = ***i*** + 5***j***, ***v*** = −2***i***− 3***j***, and ***w*** = 4***i*** − ***j***.

Exercise:

Problem: Find ***u*** + (***v*** − ***w***)

Exercise:

Problem: Find 4***v*** + 2***u***

Solution:

$-6i - 2j$

For the following exercises, use the given vectors to compute ***u*** + ***v***, ***u*** − ***v***, and 2***u*** − 3***v***.

Exercise:

Problem:$u = \langle 2, -3 \rangle, v = \langle 1, 5 \rangle$

Exercise:

Problem:$u = \langle -3, 4 \rangle, v = \langle -2, 1 \rangle$

Solution:

$u+v=\langle -5,5\rangle, u-v=\langle -1,3\rangle, 2u-3v=\langle 0,5\rangle$

Exercise:

Problem:

Let ***v*** = −4***i*** + 3***j***. Find a vector that is half the length and points in the same direction as v.

Exercise:

Problem:

Let ***v*** = 5***i*** + 2***j***. Find a vector that is twice the length and points in the opposite direction as v.

Solution:

$-10i–4j$

For the following exercises, find a unit vector in the same direction as the given vector.

Exercise:

Problem: ***a*** = 3***i*** + 4***j***

Exercise:

Problem: ***b*** = −2***i*** + 5***j***

Solution:

$-\frac{2\sqrt{29}}{29}i+\frac{5\sqrt{29}}{29}j$

Exercise:

Problem: ***c*** = 10***i*** – ***j***

Exercise:

Problem: $d = -\frac{1}{3}i + \frac{5}{2}j$

Solution:

$-\frac{2\sqrt{229}}{229}i + \frac{15\sqrt{229}}{229}j$

Exercise:

Problem: ***u*** = 100***i*** + 200***j***

Exercise:

Problem: ***u*** = −14***i*** + 2***j***

Solution:

$-\frac{7\sqrt{2}}{10}i + \frac{\sqrt{2}}{10}j$

For the following exercises, find the magnitude and direction of the vector, $0 \le \theta < 2\pi$.

Exercise:

Problem: $\langle 0, 4\rangle$

Exercise:

Problem: $\langle 6, 5\rangle$

Solution:

$|v| = 7.810, \theta = 39.806^{\circ}$

Exercise:

Problem: $\langle 2, -5\rangle$

Exercise:

Problem:$\langle -4, -6\rangle$

Solution:

$|v| = 7.211, \theta = 236.310^{\circ}$

Exercise:

Problem: Given ***u*** = 3***i*** − 4***j*** and ***v*** = −2***i*** + 3***j***, calculate $u \cdot v$.

Exercise:

Problem: Given ***u*** = −***i*** − ***j*** and ***v*** = ***i*** + 5***j***, calculate $u \cdot v$.

Solution:

-6

Exercise:

Problem: Given $u = \langle -2, 4\rangle$ and $v = \langle -3, 1\rangle$, calculate $u \cdot v$.

Exercise:

Problem: Given $\boldsymbol{u} = \langle -1, 6\rangle$ and $\boldsymbol{v} = \langle 6, -1\rangle$, calculate $u \cdot v$.

Solution:

-12

Graphical

For the following exercises, given v, draw v, 3***v*** and $\frac{1}{2}v$.

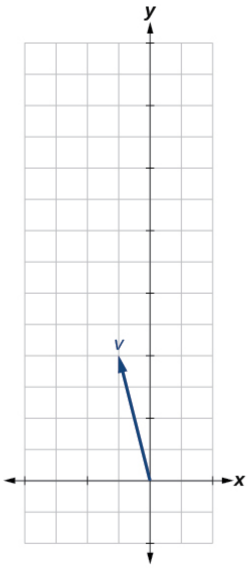
Exercise:

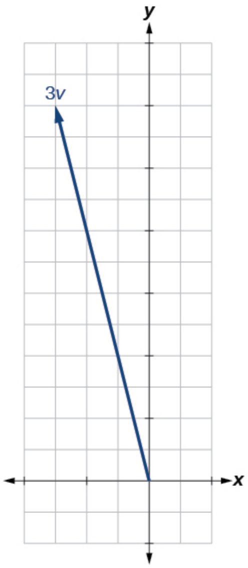
Problem: $\langle 2, -1 \rangle$

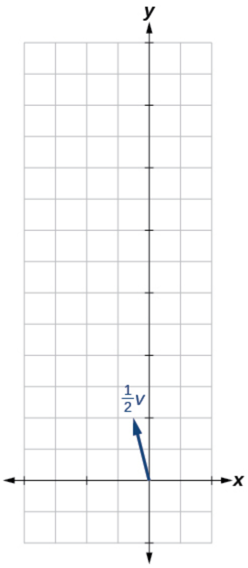
Exercise:

Problem: $\langle -1, 4 \rangle$

Solution:


y
v
x


y
3v
x


y
$\frac{1}{2}v$
x

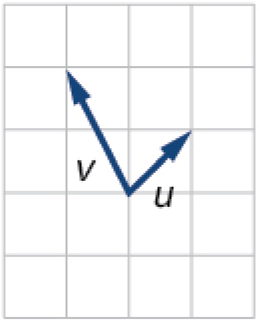
Exercise:

Problem: $\langle -3, -2 \rangle$

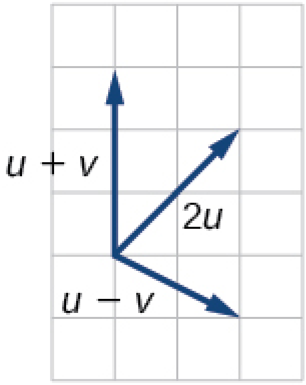
For the following exercises, use the vectors shown to sketch $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{u} - \boldsymbol{v}$, and $2\boldsymbol{u}$.

Exercise:

Problem:

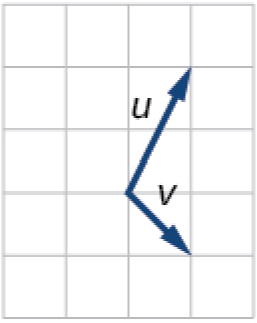


Solution:



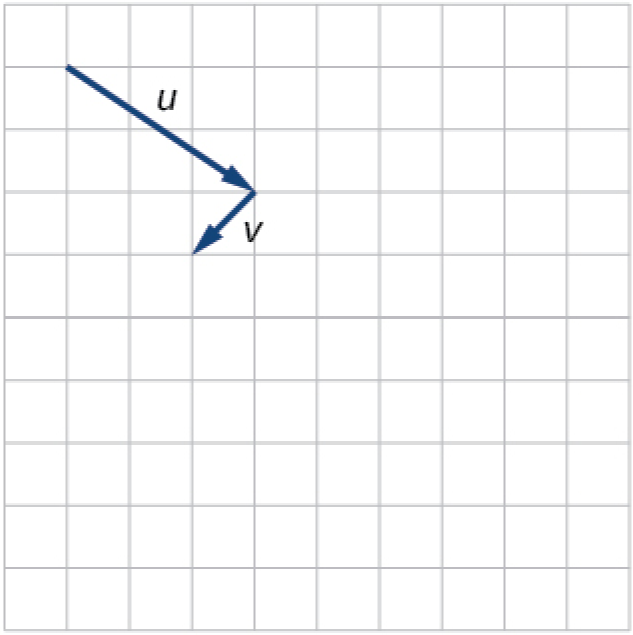
Exercise:

Problem:

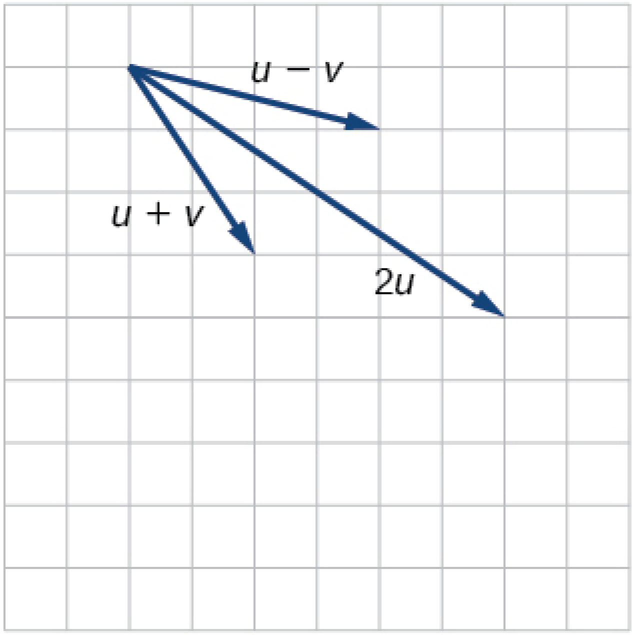


Exercise:

Problem:



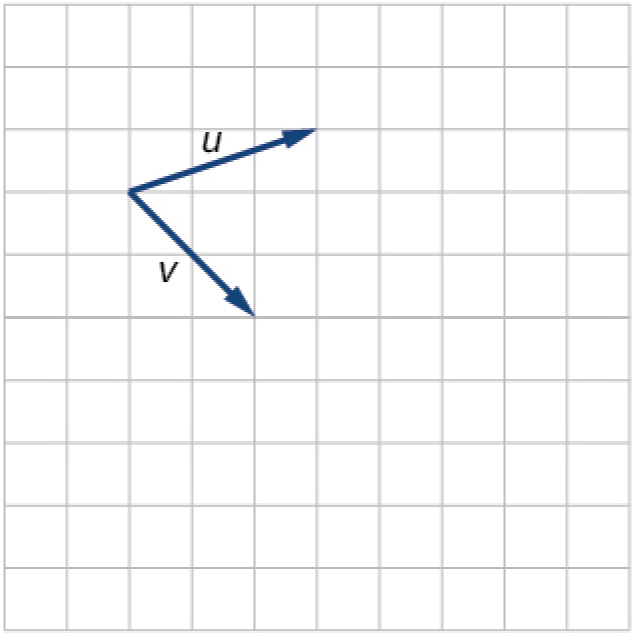
Solution:



For the following exercises, use the vectors shown to sketch 2***u*** + ***v***.

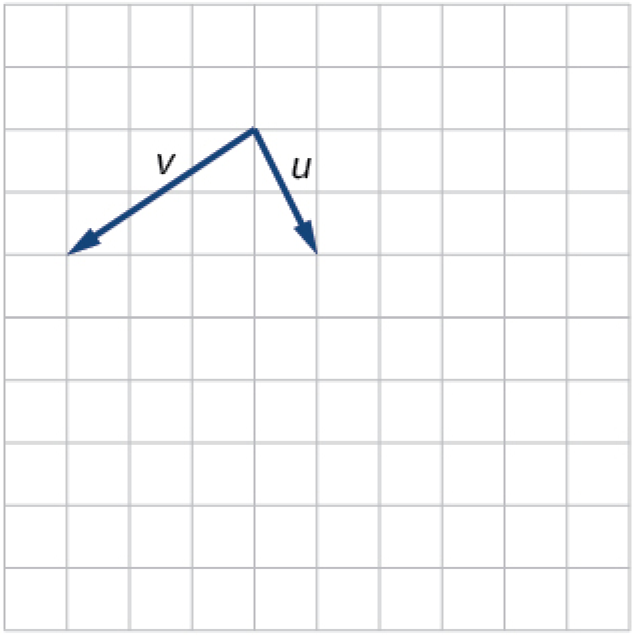
Exercise:

Problem:

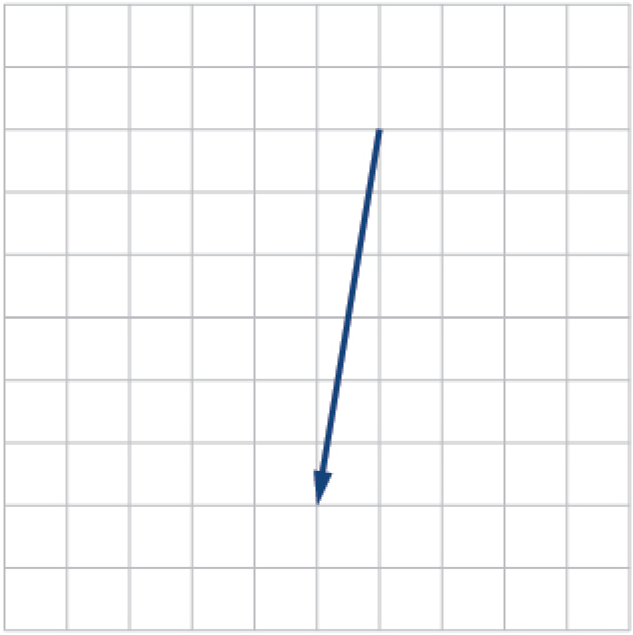


Exercise:

Problem:



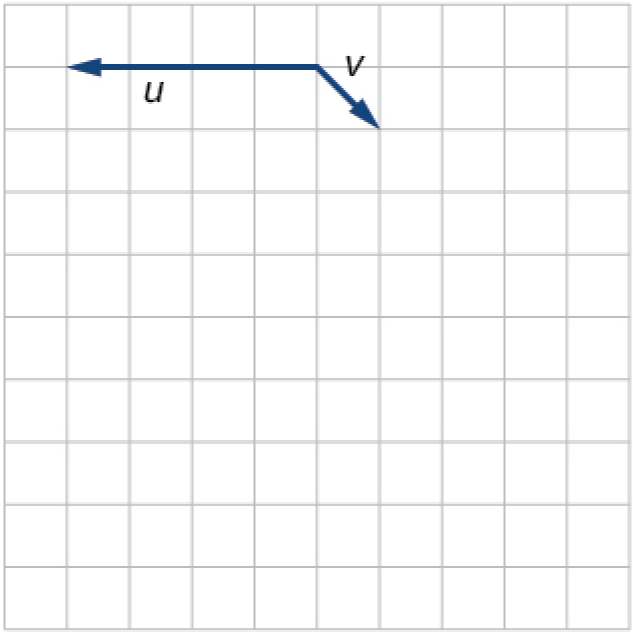
Solution:



For the following exercises, use the vectors shown to sketch $\boldsymbol{u} - 3\boldsymbol{v}$.

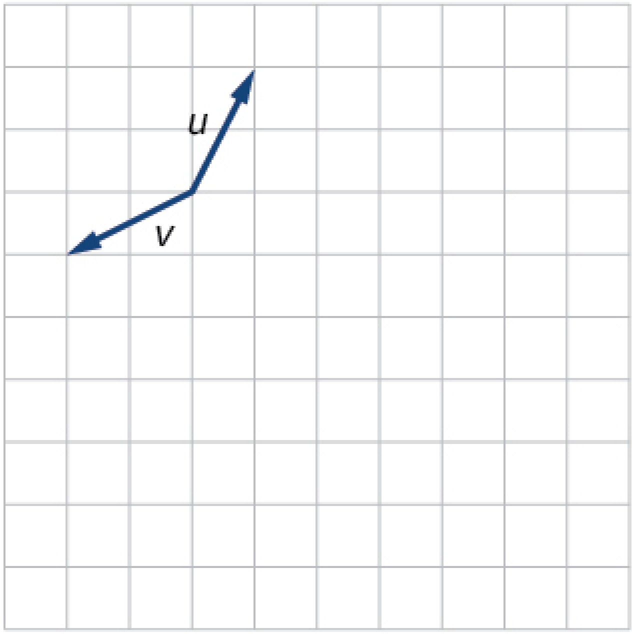
Exercise:

Problem:

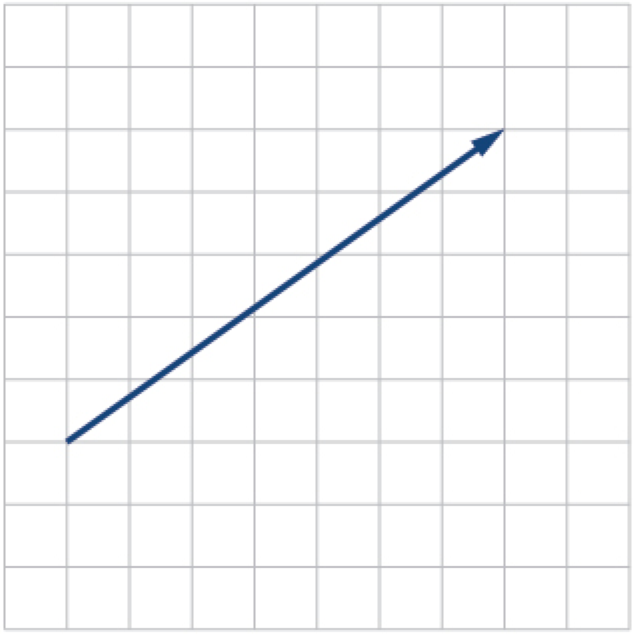


Exercise:

Problem:



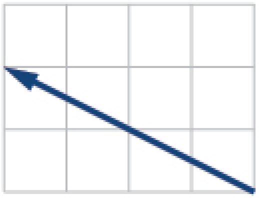
Solution:



For the following exercises, write the vector shown in component form.

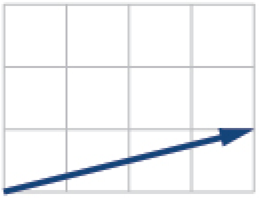
Exercise:

Problem:



Exercise:

Problem:



Solution:

$\langle 4, 1 \rangle$

Exercise:

Problem:

Given initial point $P_1 = (2, 1)$ and terminal point $P_2 = (-1, 2)$, write the vector v in terms of i and j, then draw the vector on the graph.

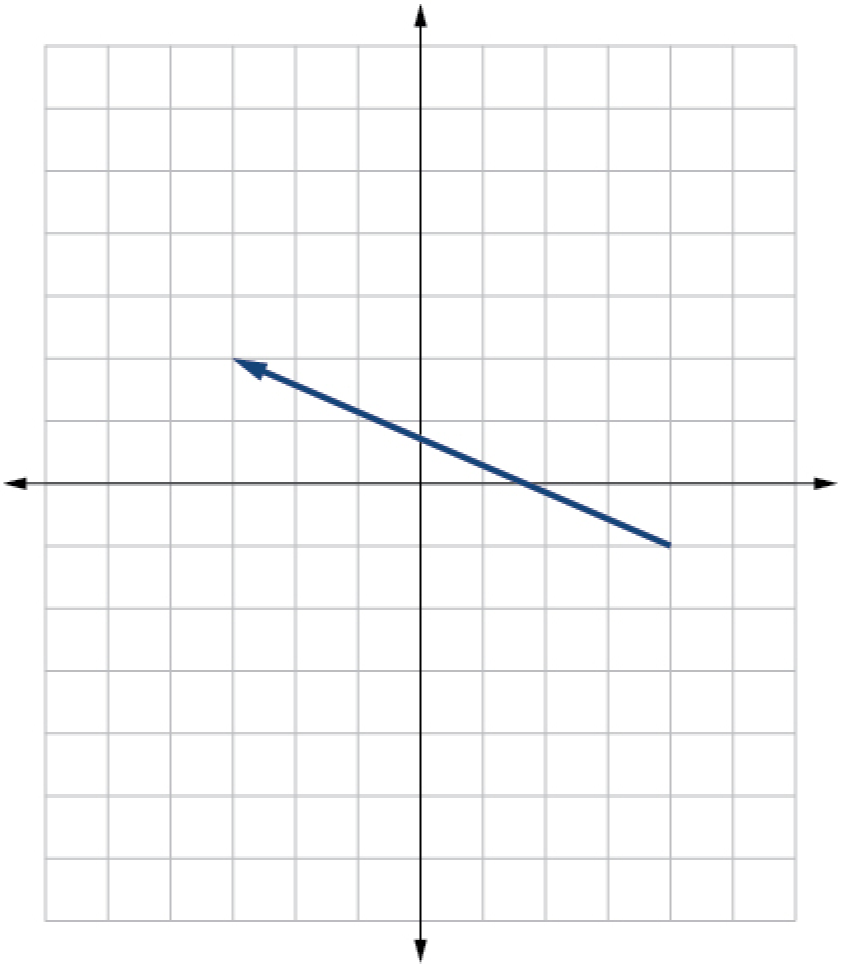
Exercise:

Problem:

Given initial point $P_1 = (4, -1)$ and terminal point $P_2 = (-3, 2)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Solution:

$v = -7i + 3j$



Exercise:

Problem:

Given initial point $P_1 = (3, 3)$ and terminal point $P_2 = (-3, 3)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Extensions

For the following exercises, use the given magnitude and direction in standard position, write the vector in component form.

Exercise:

Problem:$|v| = 6, \theta = 45^{\circ}$

Solution:

$3\sqrt{2}i + 3\sqrt{2}j$

Exercise:

Problem:$|v| = 8, \theta = 220^{\circ}$

Exercise:

Problem:$|v| = 2, \theta = 300^{\circ}$

Solution:

$i - \sqrt{3}j$

Exercise:

Problem:$|v| = 5, \theta = 135^{\circ}$

Exercise:

Problem:

A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

Solution:

a. 58.7; b. 12.5

Exercise:

Problem:

A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

Exercise:

Problem:

Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

Solution:

$x = 7.13$ pounds, $y = 3.63$ pounds

Exercise:

Problem:

Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

Exercise:

Problem:

Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

Solution:

$x = 2.87$ pounds, $y = 4.10$ pounds

Exercise:

Problem:

Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

Real-World Applications

Exercise:

Problem:

A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

Solution:

4.635 miles, 17.764° N of E

Exercise:

Problem:

A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

Exercise:

Problem:

A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

Solution:

17 miles. 10.318 miles

Exercise:

Problem:

A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

Exercise:

Problem:

A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

Solution:

Distance: 2.868. Direction: 86.474° North of West, or 3.526° West of North

Exercise:

Problem:

A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

Exercise:

Problem:

An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane’s speed relative to the ground?

Solution:

4.924°. 659 km/hr

Exercise:

Problem:

An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane’s speed relative to the ground?

Exercise:

Problem:

An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

Solution:

4.424°

Exercise:

Problem:

An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

Exercise:

Problem:

As part of a video game, the point $(5, 7)$ is rotated counterclockwise about the origin through an angle of 35°. Find the new coordinates of this point.

Solution:

$(0.081, 8.602)$

Exercise:

Problem:

As part of a video game, the point $(7, 3)$ is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

Exercise:

Problem:

Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

Solution:

21.801°, relative to the car’s forward direction

Exercise:

Problem:

Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

Exercise:

Problem:

A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

Solution:

parallel: 16.28, perpendicular: 47.28 pounds

Exercise:

Problem:

Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

Exercise:

Problem:

Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it −135° from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

Solution:

19.35 pounds, 231.54° from the horizontal

Exercise:

Problem:

The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

Exercise:

Problem:

Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it 30° from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

Solution:

5.1583 pounds, 75.8° from the horizontal

Chapter Review Exercises

Non-right Triangles: Law of Sines

For the following exercises, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

Exercise:

Problem:$\beta = 50^\circ, a = 105, b = 45$

Solution:

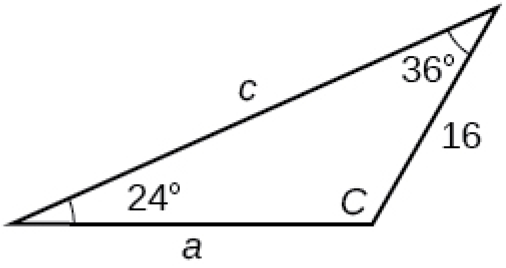
Not possible

Exercise:

Problem:$\alpha = 43.1^\circ, a = 184.2, b = 242.8$

Exercise:

Problem: Solve the triangle.

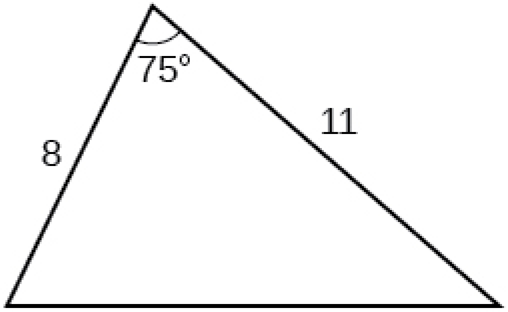


Solution:

$C = 120^\circ, a = 23.1, c = 34.1$

Exercise:

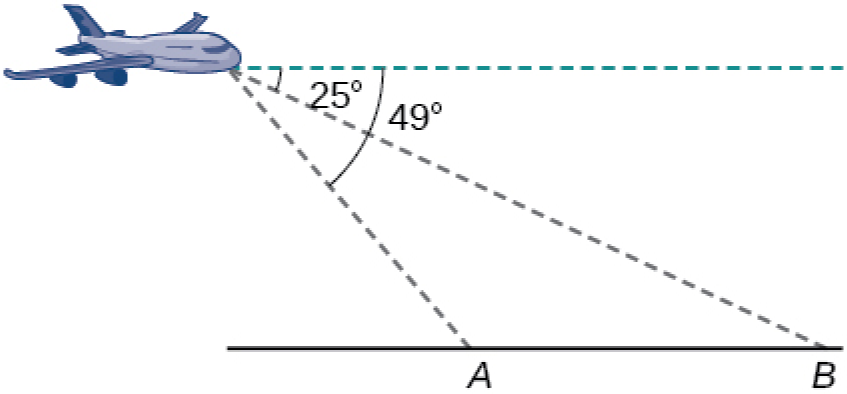
Problem: Find the area of the triangle.



Exercise:

Problem:

A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 2.1 km apart, to be 25° and 49°, as shown in [link]. Find the distance of the plane from point A and the elevation of the plane.



Solution:

distance of the plane from point A : 2.2 km, elevation of the plane: 1.6 km

Non-right Triangles: Law of Cosines

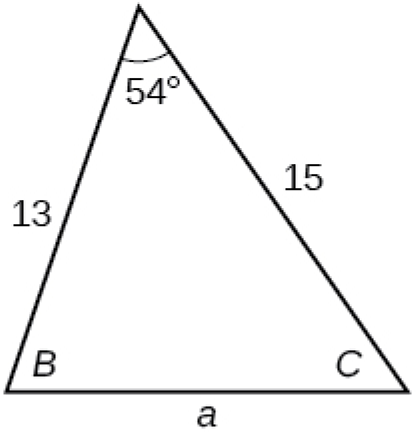
Exercise:

Problem:

Solve the triangle, rounding to the nearest tenth, assuming α is opposite side a, β is opposite side b, and γ s opposite side $c: \quad a=4, \quad b=6, c=8$.

Exercise:

Problem: Solve the triangle in [link], rounding to the nearest tenth.



Solution:

$B = 71.0^\circ, C = 55.0^\circ, a = 12.8$

Exercise:

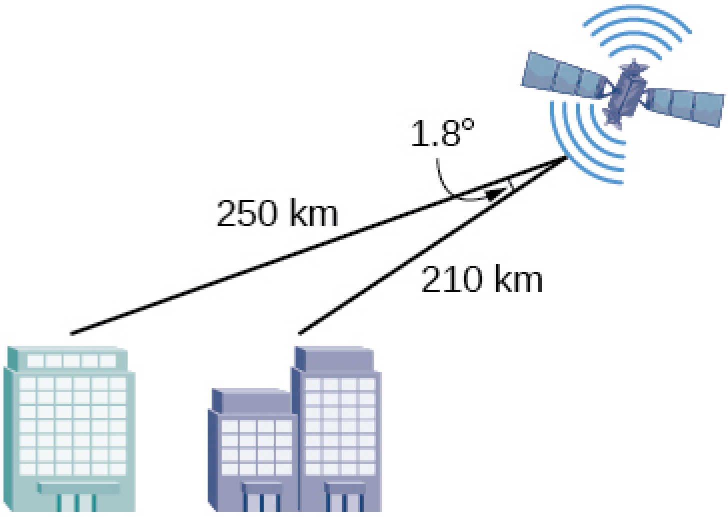
Problem:

Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.

Exercise:

Problem:

To find the distance between two cities, a satellite calculates the distances and angle shown in [link] (not to scale). Find the distance between the cities. Round answers to the nearest tenth.



Solution:

40.6 km

Polar Coordinates

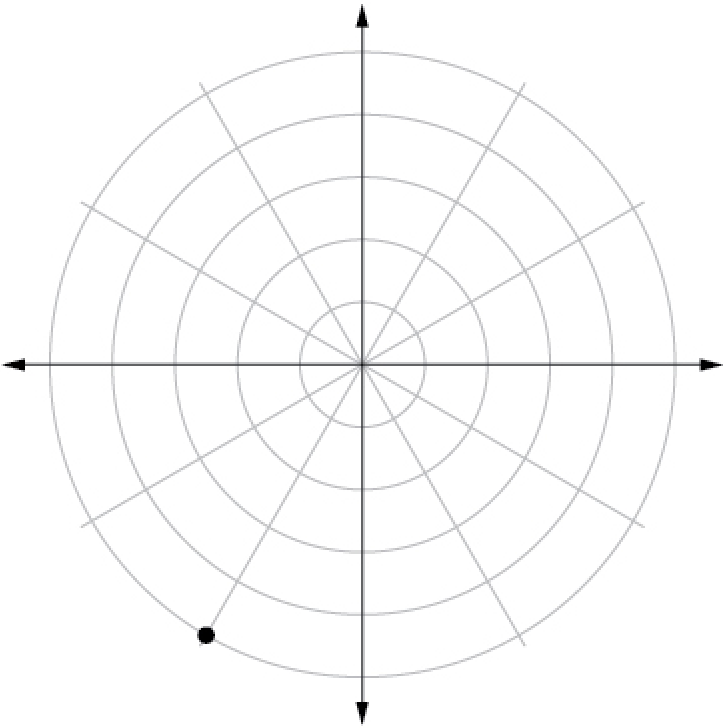
Exercise:

Problem: Plot the point with polar coordinates $\left(3, \frac{\pi}{6}\right)$.

Exercise:

Problem: Plot the point with polar coordinates $\left(5, -\frac{2\pi}{3}\right)$

Solution:



Exercise:

Problem: Convert $\left(6,-\frac{3\pi}{4}\right)$ to rectangular coordinates.

Exercise:

Problem: Convert $\left(-2,\frac{3\pi}{2}\right)$ to rectangular coordinates.

Solution:

$(0,2)$

Exercise:

Problem: Convert $(7,-2)$ to polar coordinates.

Exercise:

Problem: Convert $(-9,-4)$ to polar coordinates.

Solution:

$(9.8489,203.96^{\circ})$

For the following exercises, convert the given Cartesian equation to a polar equation.

Exercise:

Problem:$x=-2$

Exercise:

Problem:$x^2+y^2=64$

Solution:

$r=8$

Exercise:

Problem:$x^2 + y^2 = -2y$

For the following exercises, convert the given polar equation to a Cartesian equation.

Exercise:

Problem:$r = 7\cos\,\theta$

Solution:

$x^2 + y^2 = 7x$

Exercise:

Problem:$r = \frac{-2}{4\cos\,\theta + \sin\,\theta}$

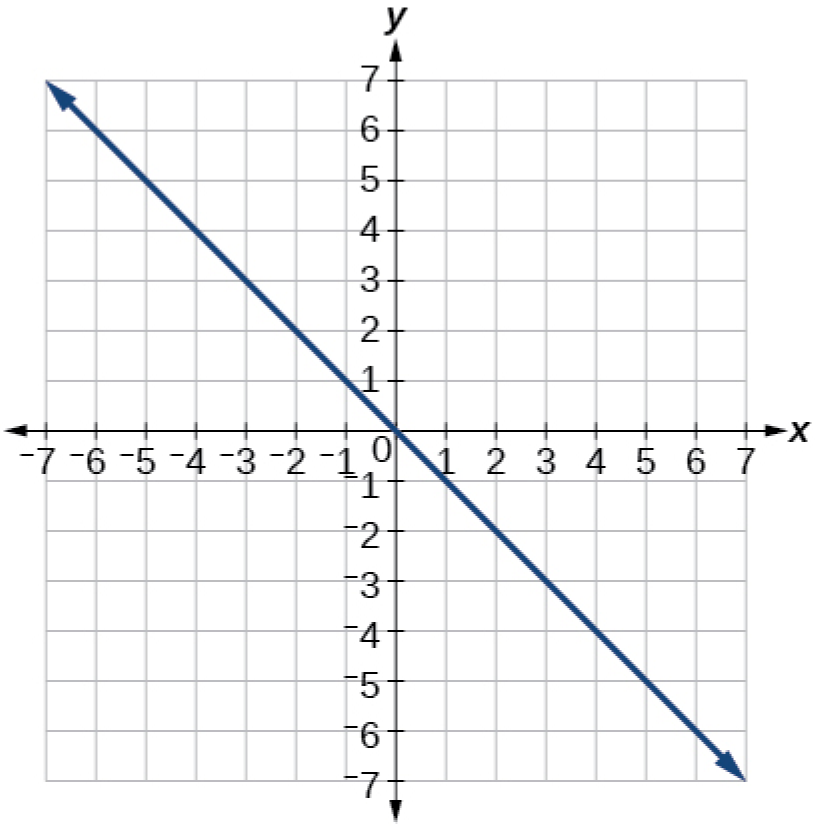
For the following exercises, convert to rectangular form and graph.

Exercise:

Problem:$\theta = \frac{3\pi}{4}$

Solution:

$y = -x$



Exercise:

Problem:$r=5\sec\ \theta$

Polar Coordinates: Graphs

For the following exercises, test each equation for symmetry.

Exercise:

Problem:$r=4+4\sin\ \theta$

Solution:

symmetric with respect to the line $\theta=\frac{\pi}{2}$

Exercise:

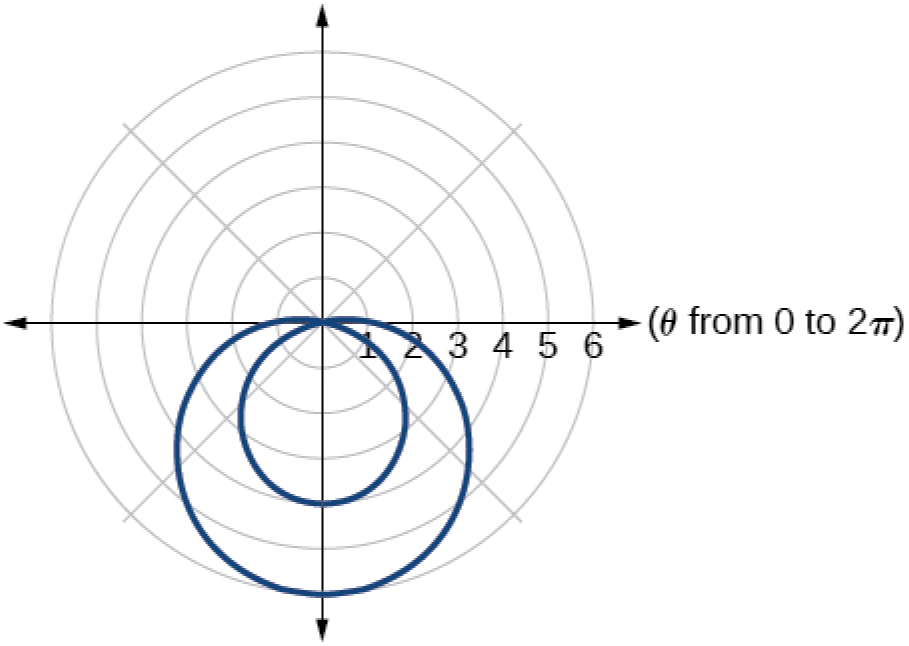
Problem:$r=7$

Exercise:

Problem:

Sketch a graph of the polar equation $r=1-5\sin\ \theta$. Label the axis intercepts.

Solution:



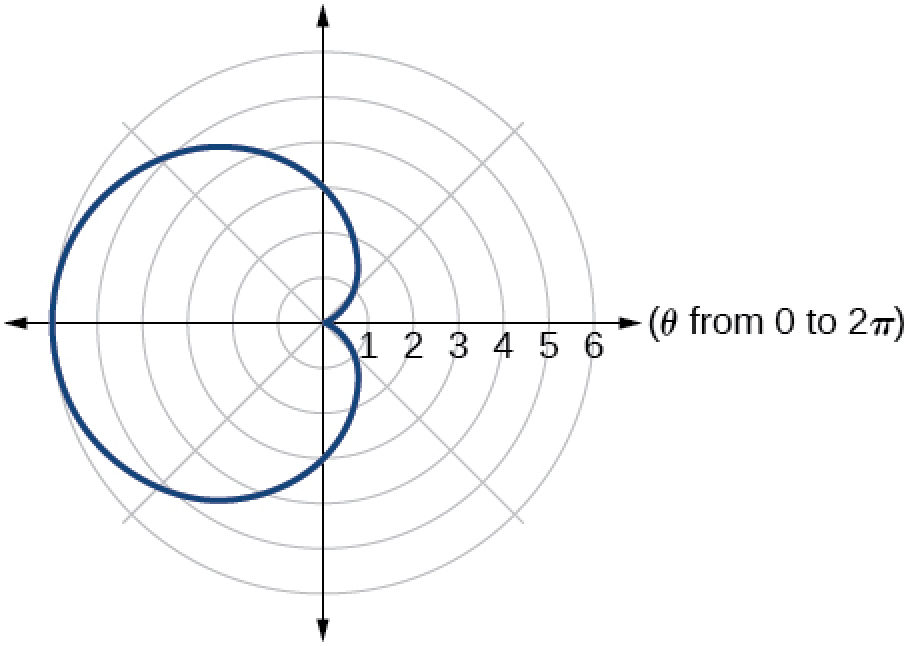
Exercise:

Problem: Sketch a graph of the polar equation $r=5\sin\left(7\theta\right)$.

Exercise:

Problem: Sketch a graph of the polar equation $r=3-3\cos\ \theta$

Solution:



Polar Form of Complex Numbers

For the following exercises, find the absolute value of each complex number.

Exercise:

Problem: $-2+6i$

Exercise:

Problem: $4-3i$

Solution:

5

Write the complex number in polar form.

Exercise:

Problem:$5+9i$

Exercise:

Problem:$\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Solution:

$\text{cis}\left(-\frac{\pi}{3}\right)$

For the following exercises, convert the complex number from polar to rectangular form.

Exercise:

Problem:$z=5\text{ cis}\left(\frac{5\pi}{6}\right)$

Exercise:

Problem:$z=3\text{ cis}\left(40^{\circ}\right)$

Solution:

$2.3+1.9i$

For the following exercises, find the product z_1z_2 in polar form.

Exercise:

Problem: $z_1=2\text{ cis}\left(89^{\circ}\right)$

$z_2=5\text{ cis}\left(23^{\circ}\right)$

Exercise:

Problem: $z_1=10\text{ cis}\left(\frac{\pi}{6}\right)$

$z_2 = 6 \operatorname{cis}\left(\frac{\pi}{3}\right)$

Solution:

$60 \operatorname{cis}\left(\frac{\pi}{2}\right)$

For the following exercises, find the quotient $\frac{z_1}{z_2}$ in polar form.

Exercise:

Problem: $z_1 = 12 \operatorname{cis}(55^\circ)$

$z_2 = 3 \operatorname{cis}(18^\circ)$

Exercise:

Problem: $z_1 = 27 \operatorname{cis}\left(\frac{5\pi}{3}\right)$

$z_2 = 9 \operatorname{cis}\left(\frac{\pi}{3}\right)$

Solution:

$3 \operatorname{cis}\left(\frac{4\pi}{3}\right)$

For the following exercises, find the powers of each complex number in polar form.

Exercise:

Problem: Find z^4 when $z = 2 \operatorname{cis}(70^\circ)$

Exercise:

Problem: Find z^2 when $z = 5 \operatorname{cis}\left(\frac{3\pi}{4}\right)$

Solution:

$25 \operatorname{cis}\left(\frac{3\pi}{2}\right)$

For the following exercises, evaluate each root.

Exercise:

Problem: Evaluate the cube root of z when $z = 64 \operatorname{cis}(210^\circ)$.

Exercise:

Problem: Evaluate the square root of z when $z = 25 \operatorname{cis}\left(\frac{3\pi}{2}\right)$.

Solution:

$5 \operatorname{cis}\left(\frac{3\pi}{4}\right), 5 \operatorname{cis}\left(\frac{7\pi}{4}\right)$

For the following exercises, plot the complex number in the complex plane.

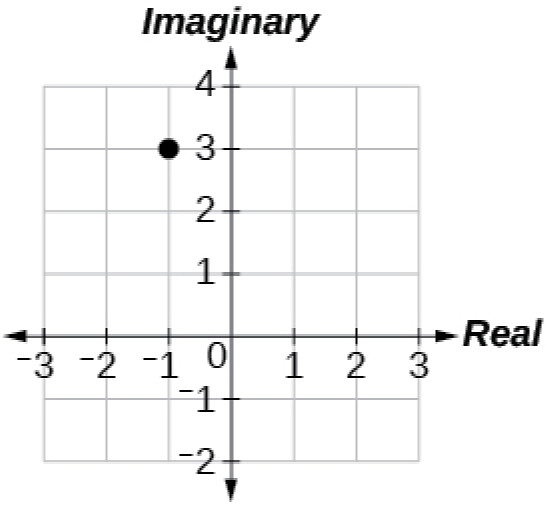
Exercise:

Problem:$6 - 2i$

Exercise:

Problem:$-1 + 3i$

Solution:



Parametric Equations

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

Exercise:

Problem: $\begin{cases} x\left(t\right) = 3t - 1 \\ y\left(t\right) = \sqrt{t} \end{cases}$

Exercise:

Problem: $\begin{cases} x(t) = -\cos\ t \\ y(t) = 2\sin^2 t \end{cases}$

Solution:

$x^2 + \frac{1}{2}y = 1$

Exercise:

Problem:

Parameterize (write a parametric equation for) each Cartesian equation by using $x\left(t\right)=a\cos\ t$ and $y(t)=b\sin\ t$ for $\frac{x^2}{25}+\frac{y^2}{16}=1.$

Exercise:

Problem:

Parameterize the line from $\left(-2,3\right)$ to $\left(4,7\right)$ so that the line is at $\left(-2,3\right)$ at $t=0$ and $\left(4,7\right)$ at $t=1.$

Solution:

$$\begin{cases} x\left(t\right)=-2+6t \\ y\left(t\right)=3+4t \end{cases}$$

Parametric Equations: Graphs

For the following exercises, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

Exercise:

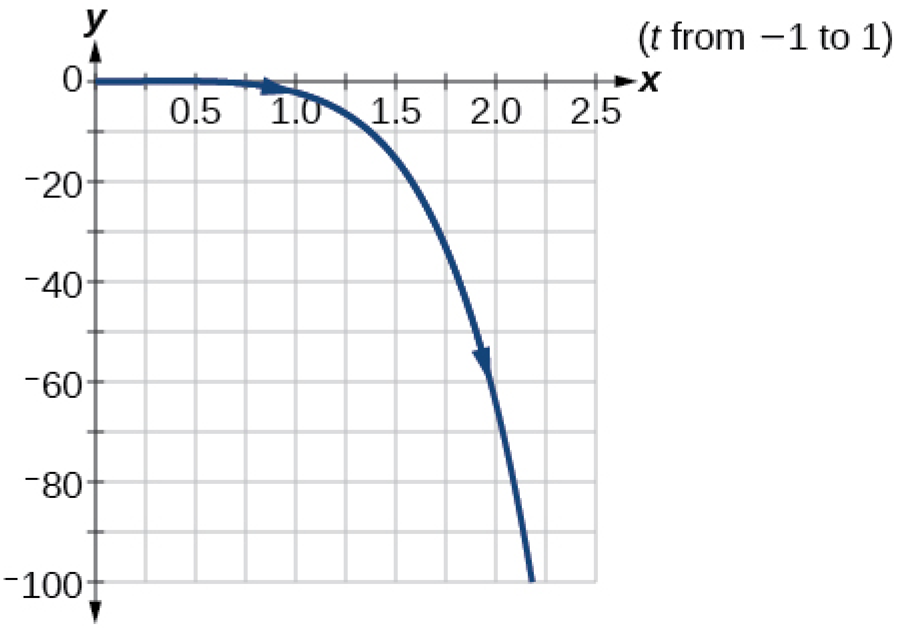
Problem: $\begin{cases} x\left(t\right)=3t^2 \\ y\left(t\right)=2t-1 \end{cases}$

Exercise:

Problem: $\begin{cases} x(t)=e^t \\ y(t)=-2e^{5\ t} \end{cases}$

Solution:

$y=-2x^5$



Exercise:

Problem: $\begin{cases} x(t) = 3\cos\ t \\ y(t) = 2\sin\ t \end{cases}$

Exercise:

Problem:

A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.

a. Find the parametric equations to model the path of the ball.
b. Where is the ball after 3 seconds?
c. How long is the ball in the air?

Solution:

a. $\begin{cases} x\,(t) = (80\cos\,(40^\circ))t \\ y\,(t) = -16t^2 + (80\sin\,(40^\circ))t + 4 \end{cases}$
b. The ball is 14 feet high and 184 feet from where it was launched.

c. 3.3 seconds

Vectors

For the following exercises, determine whether the two vectors, u and v, are equal, where u has an initial point P_1 and a terminal point P_2, and v has an initial point P_3 and a terminal point P_4.

Exercise:

Problem:$P_1 = (-1, 4), P_2 = (3, 1), P_3 = (5, 5)$ and $P_4 = (9, 2)$

Exercise:

Problem:

$P_1 = (6, 11), P_2 = (-2, 8), P_3 = (0, -1)$ and $P_4 = (-8, 2)$

Solution:

not equal

For the following exercises, use the vectors $u = 2i - j$,$v = 4i - 3j$, and $w = -2i + 5j$ to evaluate the expression.

Exercise:

Problem:$\boldsymbol{u} - \boldsymbol{v}$

Exercise:

Problem:$2\boldsymbol{v} - \boldsymbol{u} + \boldsymbol{w}$

Solution:

$4\boldsymbol{i}$

For the following exercises, find a unit vector in the same direction as the given vector.
Exercise:

Problem: $\boldsymbol{a} = 8\boldsymbol{i} - 6\boldsymbol{j}$

Exercise:

Problem: $\boldsymbol{b} = -3\boldsymbol{i} - \boldsymbol{j}$

Solution:

$-\frac{3\sqrt{10}}{10}\boldsymbol{i} - \frac{\sqrt{10}}{10}\boldsymbol{j}$

For the following exercises, find the magnitude and direction of the vector.
Exercise:

Problem: $\langle 6, -2 \rangle$

Exercise:

Problem: $\langle -3, -3 \rangle$

Solution:

Magnitude: $3\sqrt{2}$, Direction: 225°

For the following exercises, calculate $u \cdot v$.
Exercise:

Problem: $\boldsymbol{u} = -2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{v} = 3\boldsymbol{i} + 7\boldsymbol{j}$

Exercise:

Problem: $\boldsymbol{u} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{v} = 4\boldsymbol{i} + 3\boldsymbol{j}$

Solution:

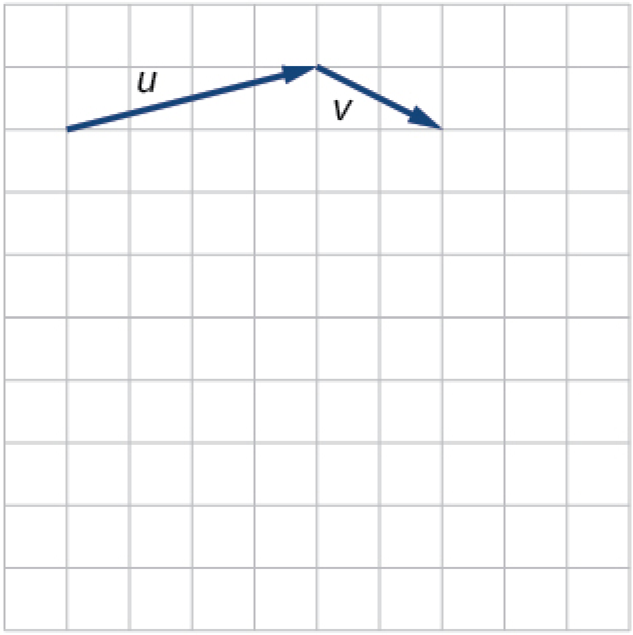
16

Exercise:

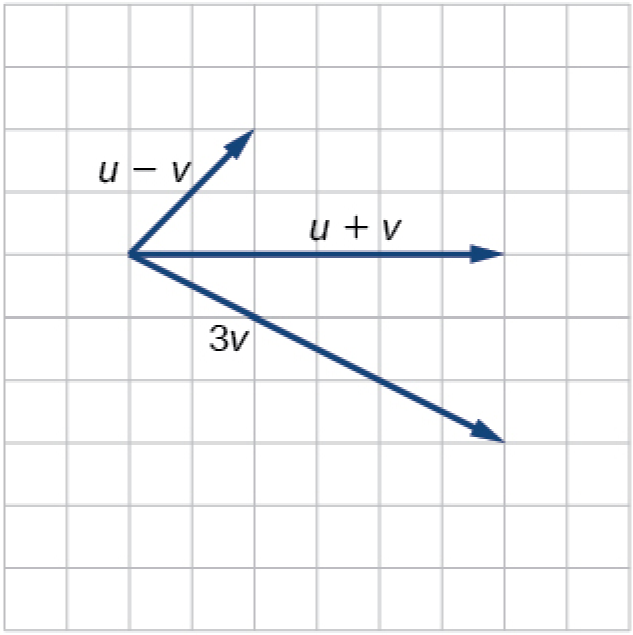
Problem: Given $\boldsymbol{v} = \langle -3, 4 \rangle$ draw $\boldsymbol{v}$, $2\boldsymbol{v}$, and $\frac{1}{2}\ \boldsymbol{v}$.

Exercise:

Problem: Given the vectors shown in [link], sketch $\boldsymbol{u} + \boldsymbol{v}$, $\boldsymbol{u} - \boldsymbol{v}$ and $3\boldsymbol{v}$.



Solution:



Exercise:

Problem:

Given initial point $P_1 = (3, 2)$ and terminal point $P_2 = (-5, -1)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Practice Test

Exercise:

Problem:

Assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve the triangle, if possible, and round each answer to the nearest tenth, given $\beta = 68^{\circ}, b = 21, c = 16$.

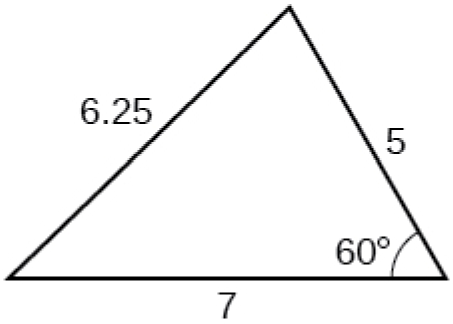
Solution:

$\alpha = 67.1^{\circ}, \gamma = 44.9^{\circ}, a = 20.9$

Exercise:

Problem:

Find the area of the triangle in [link]. Round each answer to the nearest tenth.



Exercise:

Problem:

A pilot flies in a straight path for 2 hours. He then makes a course correction, heading 15° to the right of his original course, and flies 1 hour in the new direction. If he maintains a constant speed of 575 miles per hour, how far is he from his starting position?

Solution:

1712 miles

Exercise:

Problem: Convert $(2,2)$ to polar coordinates, and then plot the point.

Exercise:

Problem: Convert $\left(2,\frac{\pi}{3}\right)$ to rectangular coordinates.

Solution:

$\left(1,\sqrt{3}\right)$

Exercise:

Problem:

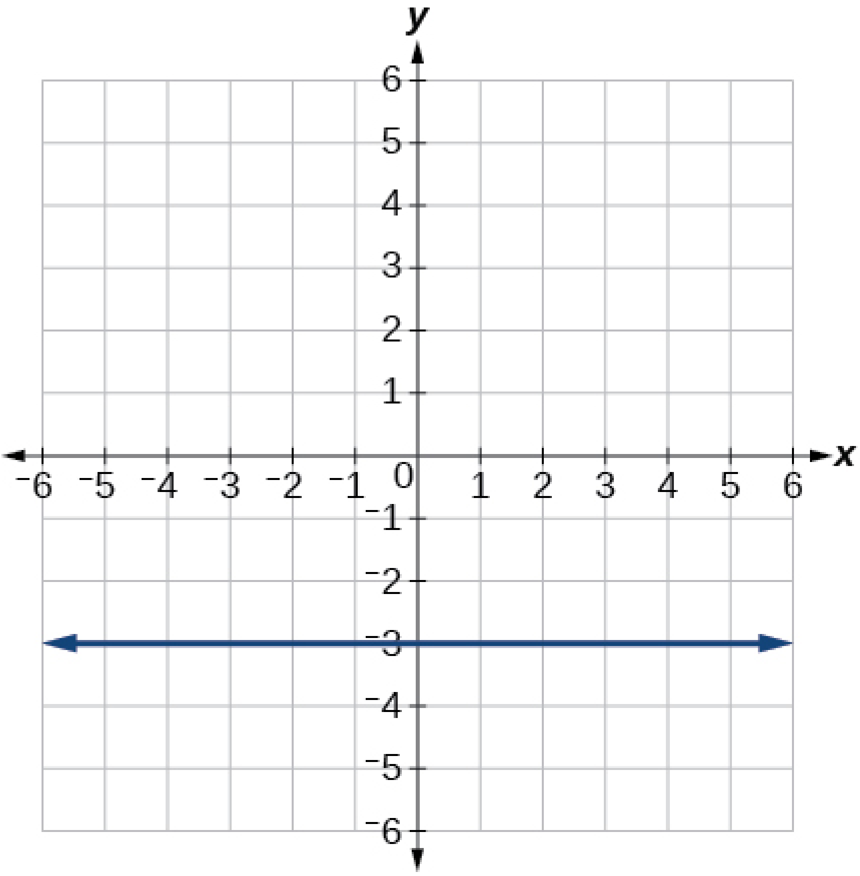
Convert the polar equation to a Cartesian equation: $x^2 + y^2 = 5$y.

Exercise:

Problem: Convert to rectangular form and graph: $r = -3\csc\ \theta$.

Solution:

$y = -3$



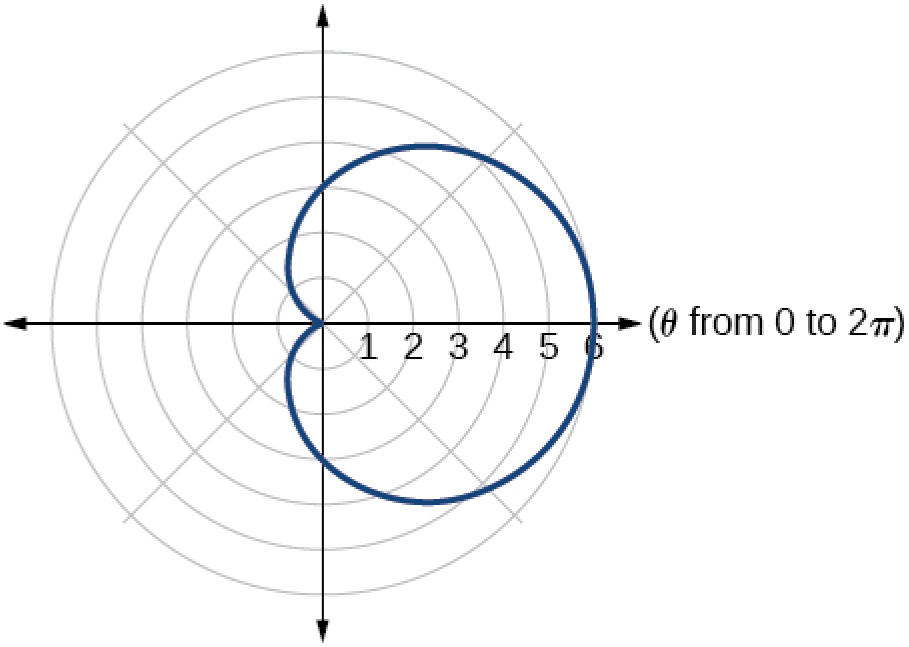
Exercise:

Problem: Test the equation for symmetry: $r = -4\sin(2\theta\ )$.

Exercise:

Problem: Graph $r=3+3\cos\ \theta$.

Solution:



Exercise:

Problem: Graph $r=3-5\sin\ \theta$.

Exercise:

Problem: Find the absolute value of the complex number $5-9i$.

Solution:

$\sqrt{106}$

Exercise:

Problem: Write the complex number in polar form: $4+i$.

Exercise:

Problem:

Convert the complex number from polar to rectangular form:
$z = 5\text{cis}\left(\frac{2\pi}{3}\right)$.

Solution:

$\frac{-5}{2} + i\frac{5\sqrt{3}}{2}$

Given $z_1 = 8\,\text{cis}\,(36^\circ)$ and $z_2 = 2\,\text{cis}\,(15^\circ)$, evaluate each expression.

Exercise:

Problem: $z_1 z_2$

Exercise:

Problem: $\frac{z_1}{z_2}$

Solution:

$4\,\text{cis}\,(21^\circ)$

Exercise:

Problem: $(z_2)^3$

Exercise:

Problem: $\sqrt{z_1}$

Solution:

$2\sqrt{2}\,\text{cis}\,(18^\circ), 2\sqrt{2}\,\text{cis}\,(198^\circ)$

Exercise:

Problem: Plot the complex number $-5 - i$ in the complex plane.

Exercise:

Problem:

Eliminate the parameter t to rewrite the following parametric equations as a Cartesian equation: $\begin{cases} x(t) = t + 1 \\ y(t) = 2t^2 \end{cases}$.

Solution:

$y = 2(x - 1)^2$

Exercise:

Problem:

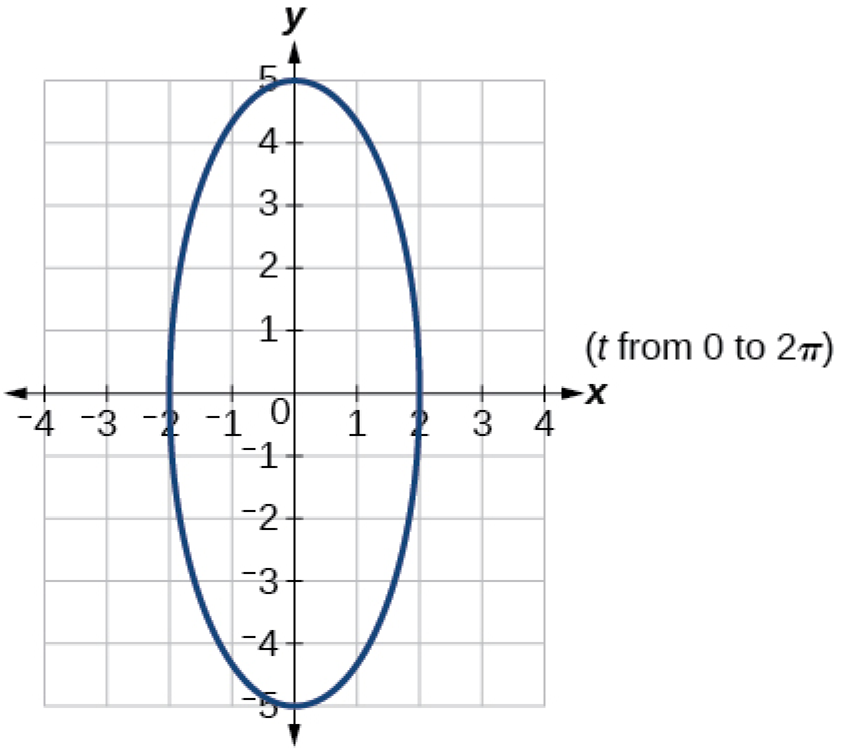
Parameterize (write a parametric equation for) the following Cartesian equation by using $x\,(t) = a\cos\ t$ and $y(t) = b\sin\ t$:
$\frac{x^2}{36} + \frac{y^2}{100} = 1.$

Exercise:

Problem:

Graph the set of parametric equations and find the Cartesian equation:
$\begin{cases} x(t) = -2\sin\ t \\ y(t) = 5\cos\ t \end{cases}$.

Solution:



Exercise:

Problem:

A ball is launched with an initial velocity of 95 feet per second at an angle of 52° to the horizontal. The ball is released at a height of 3.5 feet above the ground.

a. Find the parametric equations to model the path of the ball.
b. Where is the ball after 2 seconds?
c. How long is the ball in the air?

For the following exercises, use the vectors $\boldsymbol{u} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} + 3\boldsymbol{j}$.

Exercise:

Problem: Find $2\boldsymbol{u} - 3\boldsymbol{v}$.

Solution:

$-4\boldsymbol{i} - 15\boldsymbol{j}$

Exercise:

Problem: Calculate $u \cdot v$.

Exercise:

Problem: Find a unit vector in the same direction as v.

Solution:

$\frac{2\sqrt{13}}{13}i + \frac{3\sqrt{13}}{13}j$

Exercise:

Problem:

Given vector v has an initial point $P_1 = (2, 2)$ and terminal point $P_2 = (-1, 0)$, write the vector v in terms of i and j. On the graph, draw v, and $-v$.

Glossary

dot product	given two vectors, the sum of the product of the horizontal components and the product of the vertical components
initial point	the origin of a vector
magnitude	the length of a vector; may represent a quantity such as speed, and is calculated using the Pythagorean Theorem
resultant	a vector that results from addition or subtraction of two vectors, or from scalar multiplication

scalar

a quantity associated with magnitude but not direction; a constant

scalar multiplication

the product of a constant and each component of a vector

standard position

the placement of a vector with the initial point at $(0, 0)$ and the terminal point (a, b), represented by the change in the *x*-coordinates and the change in the *y*-coordinates of the original vector

terminal point

the end point of a vector, usually represented by an arrow indicating its direction

unit vector

a vector that begins at the origin and has magnitude of 1; the horizontal unit vector runs along the *x*-axis and is defined as $v_1 = \langle 1, 0 \rangle$ the vertical unit vector runs along the *y*-axis and is defined as $v_2 = \langle 0, 1 \rangle$.

vector

a quantity associated with both magnitude and direction, represented as a directed line segment with a starting point (initial point) and an end point (terminal point)

vector addition

the sum of two vectors, found by adding corresponding components

Products of Vectors

By the end of this section, you will be able to:

- Explain the difference between the scalar product and the vector product of two vectors.
- Determine the scalar product of two vectors.
- Determine the vector product of two vectors.
- Describe how the products of vectors are used in physics.

A vector can be multiplied by another vector but may not be divided by another vector. There are two kinds of products of vectors used broadly in physics and engineering. One kind of multiplication is a *scalar multiplication of two vectors*. Taking a scalar product of two vectors results in a number (a scalar), as its name indicates. Scalar products are used to define work and energy relations. For example, the work that a force (a vector) performs on an object while causing its displacement (a vector) is defined as a scalar product of the force vector with the displacement vector. A quite different kind of multiplication is a *vector multiplication of vectors*. Taking a vector product of two vectors returns as a result a vector, as its name suggests. Vector products are used to define other derived vector quantities. For example, in describing rotations, a vector quantity called *torque* is defined as a vector product of an applied force (a vector) and its distance from pivot to force (a vector). It is important to distinguish between these two kinds of vector multiplications because the scalar product is a scalar quantity and a vector product is a vector quantity.

The Scalar Product of Two Vectors (the Dot Product)

Scalar multiplication of two vectors yields a scalar product.

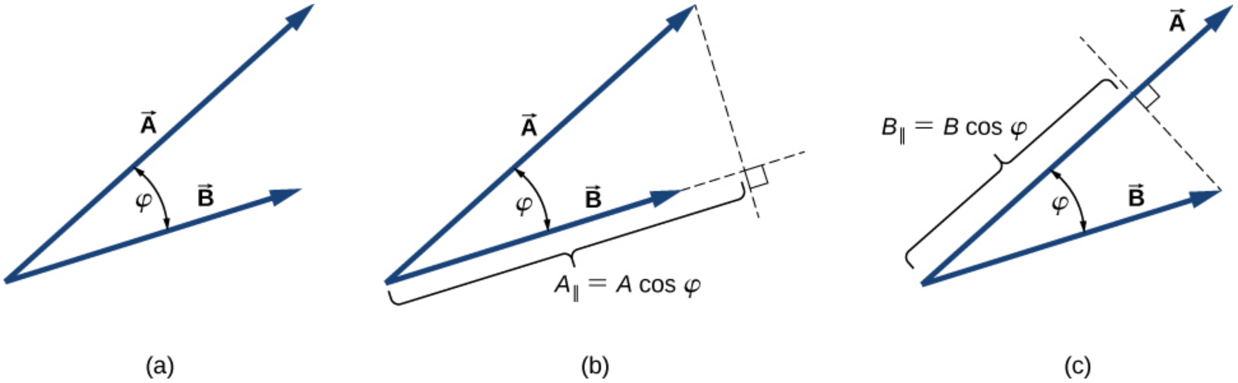
Note:
Scalar Product (Dot Product)
The **scalar product** $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is a number defined by the equation
Equation:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\varphi,$$

where φ is the angle between the vectors (shown in [link]). The scalar product is also called the **dot product** because of the dot notation that indicates it.

In the definition of the dot product, the direction of angle φ does not matter, and φ can be measured from either of the two vectors to the other because $\cos\varphi = \cos(-\varphi) = \cos(2\pi - \varphi)$. The dot product is a negative number when $90^\circ < \varphi \le 180^\circ$ and is a positive number when $0^\circ \le \varphi < 90^\circ$. Moreover, the dot product of two parallel vectors is $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos 0^\circ = AB$, and the dot product of two antiparallel vectors is $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos 180^\circ = -AB$. The scalar product of two *orthogonal vectors* vanishes: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos 90^\circ = 0$. The scalar product of a vector with itself is the square of its magnitude:
Equation:

$$\vec{\mathbf{A}}^2 \equiv \vec{\mathbf{A}} \cdot \vec{\mathbf{A}} = AA\cos 0^\circ = A^2.$$


$\vec{\mathbf{A}}$
φ
$\vec{\mathbf{B}}$
$A_{\parallel} = A\cos\varphi$
$B_{\parallel} = B\cos\varphi$
(a)
(b)
(c)

The scalar product of two vectors. (a) The angle between the two vectors. (b) The orthogonal projection $A_{\perp}$ of vector $\vec{\mathbf{A}}$ onto the direction of vector $\vec{\mathbf{B}}$. (c) The orthogonal projection $B_{\perp}$ of vector $\vec{\mathbf{B}}$ onto the direction of vector $\vec{\mathbf{A}}$.

Example:
The Scalar Product

For the vectors shown in [link], find the scalar product $\vec{\mathbf{A}} \cdot \vec{\mathbf{F}}$.

Strategy

From [link], the magnitudes of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{F}}$ are $A = 10.0$ and $F = 20.0$. Angle θ, between them, is the difference: $\theta = \varphi - \alpha = 110^\circ - 35^\circ = 75^\circ$. Substituting these values into [link] gives the scalar product.

Solution

A straightforward calculation gives us

Equation:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{F}} = AF\cos\theta = (10.0)(20.0)\cos 75^\circ = 51.76.$$

Note:
Exercise:

Problem:

Check Your Understanding For the vectors given in [link], find the scalar products $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ and $\vec{\mathbf{F}} \cdot \vec{\mathbf{C}}$.

Solution:

$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = -57.3, \vec{\mathbf{F}} \cdot \vec{\mathbf{C}} = 27.8$

In the Cartesian coordinate system, scalar products of the unit vector of an axis with other unit vectors of axes always vanish because these unit vectors are orthogonal:

Equation:

$$\begin{aligned}\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} &= \ \hat{\mathbf{i}} \ \ \hat{\mathbf{j}} \ \cos 90^\circ = (1)(1)(0) = 0, \\ \hat{\mathbf{i}} \cdot \widehat{\mathbf{k}} &= \ \hat{\mathbf{i}} \ \ \widehat{\mathbf{k}} \ \cos 90^\circ = (1)(1)(0) = 0, \\ \widehat{\mathbf{k}} \cdot \hat{\mathbf{j}} &= \ \widehat{\mathbf{k}} \ \ \hat{\mathbf{j}} \ \cos 90^\circ = (1)(1)(0) = 0.\end{aligned}$$

In these equations, we use the fact that the magnitudes of all unit vectors are one: $\ \hat{\mathbf{i}} \ = \ \hat{\mathbf{j}} \ = \ \widehat{\mathbf{k}} \ = 1$. For unit vectors of the axes, [link] gives the following identities:

Equation:

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = i^2 = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = j^2 = \widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}} = k^2 = 1.$$

The scalar product $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$ can also be interpreted as either the product of *B* with the projection $A_{\parallel}$ of vector $\vec{\mathbf{A}}$ onto the direction of vector $\vec{\mathbf{B}}$ ([link](b)) or the product of *A* with the projection $B_{\parallel}$ of vector $\vec{\mathbf{B}}$ onto the direction of vector $\vec{\mathbf{A}}$ ([link](c)):

Equation:

$$\begin{aligned}\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} \ &= AB \cos \varphi \\ &= B(A \cos \varphi) = BA_{\parallel} \\ &= A(B \cos \varphi) = AB_{\parallel}.\end{aligned}$$

For example, in the rectangular coordinate system in a plane, the scalar *x*-component of a vector is its dot product with the unit vector $\hat{\mathbf{i}}$, and the scalar *y*-component of a vector is its dot product with the unit vector $\hat{\mathbf{j}}$
:

Equation:

$$\begin{aligned}\vec{\mathbf{A}} \cdot \hat{\mathbf{i}} &= \ \vec{\mathbf{A}} \ \ \hat{\mathbf{i}} \ \cos \theta_A = A \cos \theta_A = A_x \\ \vec{\mathbf{A}} \cdot \hat{\mathbf{j}} &= \ \vec{\mathbf{A}} \ \ \hat{\mathbf{j}} \ \cos(90^\circ - \theta_A) = A \sin \theta_A = A_y\end{aligned}.$$

Scalar multiplication of vectors is commutative,

Note:
Equation:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}},$$

and obeys the distributive law:

Note:

Equation:

$$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{C}}.$$

We can use the commutative and distributive laws to derive various relations for vectors, such as expressing the dot product of two vectors in terms of their scalar components.

Note:
Exercise:

Problem:

Check Your Understanding For vector $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\widehat{\mathbf{k}}$ in a rectangular coordinate system, use [link] through [link] to show that $\vec{\mathbf{A}} \cdot \hat{\mathbf{i}} = A_x$ $\vec{\mathbf{A}} \cdot \hat{\mathbf{j}} = A_y$ and $\vec{\mathbf{A}} \cdot \widehat{\mathbf{k}} = A_z$.

When the vectors in [link] are given in their vector component forms,
Equation:

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\widehat{\mathbf{k}} \text{ and } \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\widehat{\mathbf{k}},$$

we can compute their scalar product as follows:
Equation:

$$\begin{array}{ccc}\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} & = & (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\widehat{\mathbf{k}}) \cdot (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\widehat{\mathbf{k}}) \\ & = & \ A_xB_x\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + A_xB_y\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + A_xB_z\hat{\mathbf{i}} \cdot \widehat{\mathbf{k}} \\ & & +A_yB_x\hat{\mathbf{j}} \cdot \hat{\mathbf{i}} + A_yB_y\hat{\mathbf{j}} \cdot \hat{\mathbf{j}} + A_yB_z\hat{\mathbf{j}} \cdot \widehat{\mathbf{k}} \\ & & +A_zB_x\widehat{\mathbf{k}} \cdot \hat{\mathbf{i}} + A_zB_y\widehat{\mathbf{k}} \cdot \hat{\mathbf{j}} + A_zB_z\widehat{\mathbf{k}} \cdot \widehat{\mathbf{k}}.\end{array}$$

Since scalar products of two different unit vectors of axes give zero, and scalar products of unit vectors with themselves give one (see [link] and [link]), there are only three nonzero terms in this expression. Thus, the scalar product simplifies to

Note:
Equation:

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z.$$

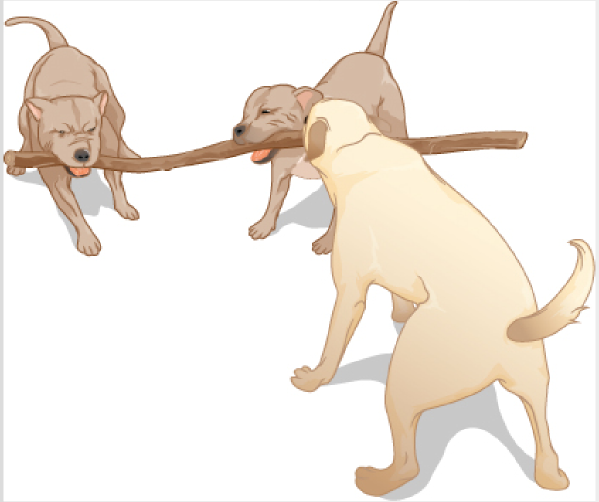
We can use [link] for the scalar product in terms of scalar components of vectors to find the angle between two vectors. When we divide [link] by *AB*, we obtain the equation for cos φ, into which we substitute [link]:

Note:
Equation:

$$\cos\varphi = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{AB} = \frac{A_x B_x + A_y B_y + A_z B_z}{AB}.$$

Angle φ between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is obtained by taking the inverse cosine of the expression in [link].

Example:
Angle between Two Forces
Three dogs are pulling on a stick in different directions, as shown in [link]. The first dog pulls with force $\vec{\mathbf{F}}_1 = (10.0\hat{\mathbf{i}} - 20.4\hat{\mathbf{j}} + 2.0\widehat{\mathbf{k}})\text{N}$, the second dog pulls with force $\vec{\mathbf{F}}_2 = (-15.0\hat{\mathbf{i}} - 6.2\widehat{\mathbf{k}})\text{N}$, and the third dog pulls with force $\vec{\mathbf{F}}_3 = (5.0\hat{\mathbf{i}} + 12.5\hat{\mathbf{j}})\text{N}$. What is the angle between forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$?



Three dogs are playing with a stick.

Strategy
The components of force vector $\vec{\mathbf{F}}_1$ are $F_{1x} = 10.0\,\text{N}$, $F_{1y} = -20.4\,\text{N}$, and $F_{1z} = 2.0\,\text{N}$, whereas those of force vector $\vec{\mathbf{F}}_2$ are $F_{2x} = -15.0\,\text{N}$, $F_{2y} = 0.0\,\text{N}$, and $F_{2z} = -6.2\,\text{N}$. Computing the scalar product of these vectors and their magnitudes, and substituting into [link] gives the angle of interest.
Solution
The magnitudes of forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ are
Equation:

$$F_1 = \sqrt{F_{1x}^2 + F_{1y}^2 + F_{1z}^2} = \sqrt{10.0^2 + 20.4^2 + 2.0^2}\,\text{N} = 22.8\,\text{N}$$

and

Equation:

$$F_2 = \sqrt{F_{2x}^2 + F_{2y}^2 + F_{2z}^2} = \sqrt{15.0^2 + 6.2^2}\,\text{N} = 16.2\,\text{N}.$$

Substituting the scalar components into [link] yields the scalar product

Equation:

$$\begin{aligned}\vec{\mathbf{F}}_1 \cdot \vec{\mathbf{F}}_2 &= F_{1x}F_{2x} + F_{1y}F_{2y} + F_{1z}F_{2z} \\ &= (10.0\,\text{N})(-15.0\,\text{N}) + (-20.4\,\text{N})(0.0\,\text{N}) + (2.0\,\text{N})(-6.2\,\text{N}) \\ &= -162.4\,\text{N}^2.\end{aligned}$$

Finally, substituting everything into [link] gives the angle

Equation:

$$\cos\varphi = \frac{\vec{\mathbf{F}}_1 \cdot \vec{\mathbf{F}}_2}{F_1 F_2} = \frac{-162.4\text{N}^2}{(22.8\,\text{N})(16.2\,\text{N})} = -0.439 \Rightarrow \quad \varphi = \cos^{-1}(-0.439) = 116.0^\circ.$$

Significance

Notice that when vectors are given in terms of the unit vectors of axes, we can find the angle between them without knowing the specifics about the geographic directions the unit vectors represent. Here, for example, the +*x*-direction might be to the east and the +*y*-direction might be to the north. But, the angle between the forces in the problem is the same if the +*x*-direction is to the west and the +*y*-direction is to the south.

Note:

Exercise:

Problem: Check Your Understanding Find the angle between forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_3$ in [link].

Solution:

131.9°

Example:

The Work of a Force

When force $\vec{\mathbf{F}}$ pulls on an object and when it causes its displacement $\vec{\mathbf{D}}$, we say the force performs work. The amount of work the force does is the scalar product $\vec{\mathbf{F}} \cdot \vec{\mathbf{D}}$. If the stick in [link] moves momentarily and gets displaced by vector $\vec{\mathbf{D}} = (-7.9\hat{\mathbf{j}} - 4.2\hat{\mathbf{k}})$ cm, how much work is done by the third dog in [link]?

Strategy

We compute the scalar product of displacement vector $\vec{\mathbf{D}}$ with force vector $\vec{\mathbf{F}}_3 = (5.0\hat{\mathbf{i}} + 12.5\hat{\mathbf{j}})\text{N}$, which is the pull from the third dog. Let's use W_3 to denote the work done by force $\vec{\mathbf{F}}_3$ on displacement $\vec{\mathbf{D}}$.

Solution

Calculating the work is a straightforward application of the dot product:
Equation:

$$\begin{aligned} W_3 &= \vec{\mathbf{F}}_3 \cdot \vec{\mathbf{D}} = F_{3x}D_x + F_{3y}D_y + F_{3z}D_z \\ &= (5.0\,\text{N})(0.0\,\text{cm}) + (12.5\,\text{N})(-7.9\,\text{cm}) + (0.0\,\text{N})(-4.2\,\text{cm}) \\ &= -98.7\,\text{N} \cdot \text{cm}. \end{aligned}$$

Significance
The SI unit of work is called the joule (J), where 1 J = 1 N · m. The unit cm · N can be written as $10^{-2}\text{m} \cdot \text{N} = 10^{-2}\text{J}$, so the answer can be expressed as $W_3 = -0.9875\,\text{J} \approx -1.0\,\text{J}$.

Note:
Exercise:

Problem:

Check Your Understanding How much work is done by the first dog and by the second dog in [link] on the displacement in [link]?

Solution:

$W_1 = 1.5\,\text{J}$, $W_2 = 0.3\,\text{J}$

The Vector Product of Two Vectors (the Cross Product)

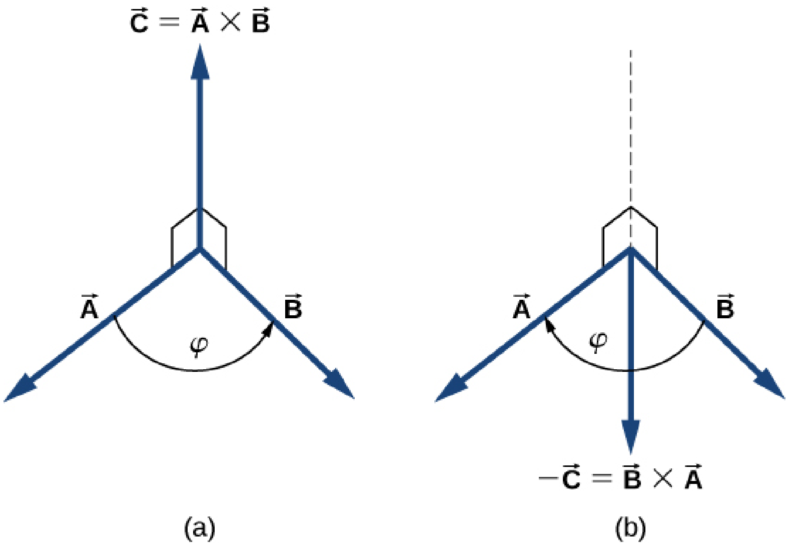
Vector multiplication of two vectors yields a vector product.

Note:
Vector Product (Cross Product)
The **vector product** of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is denoted by $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ and is often referred to as a **cross product**. The vector product is a vector that has its direction perpendicular to both vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. In other words, vector $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is perpendicular to the plane that contains vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, as shown in [link]. The magnitude of the vector product is defined as
Equation:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = AB \sin \varphi,$$

where angle φ, between the two vectors, is measured from vector $\vec{\mathbf{A}}$ (first vector in the product) to vector $\vec{\mathbf{B}}$ (second vector in the product), as indicated in [link], and is between 0° and 180°.

According to [link], the vector product vanishes for pairs of vectors that are either parallel $(\varphi = 0^\circ)$ or antiparallel $(\varphi = 180^\circ)$ because $\sin 0^\circ = \sin 180^\circ = 0$.



$\vec{C} = \vec{A} \times \vec{B}$
$\vec{A}$
$\vec{B}$
φ
$\vec{A}$
$\vec{B}$
φ
$-\vec{C} = \vec{B} \times \vec{A}$
(a)
(b)

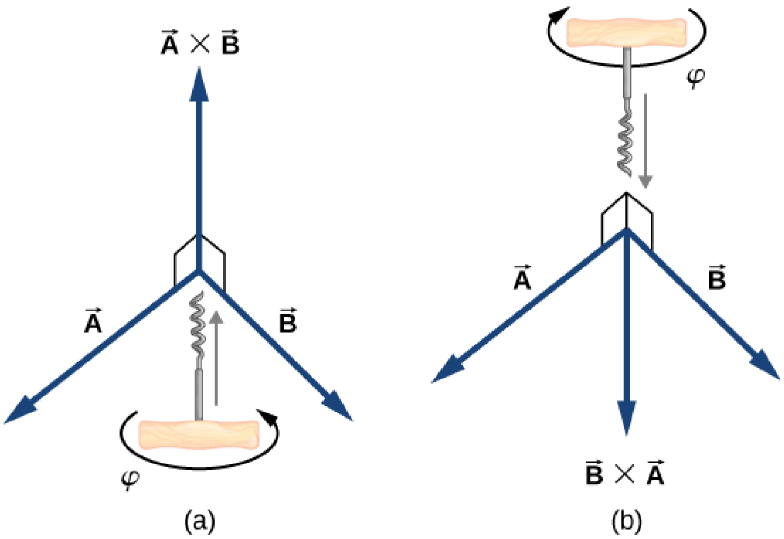
The vector product of two vectors is drawn in three-dimensional space. (a) The vector product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is a vector perpendicular to the plane that contains vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. Small squares drawn in perspective mark right angles between $\vec{\mathbf{A}}$ and $\vec{\mathbf{C}}$, and between $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$ so that if $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ lie on the floor, vector $\vec{\mathbf{C}}$ points vertically upward to the ceiling. (b) The vector product $\vec{\mathbf{B}} \times \vec{\mathbf{A}}$ is a vector antiparallel to vector $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$.

On the line perpendicular to the plane that contains vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ there are two alternative directions—either up or down, as shown in [link]—and the direction of the vector product may be either one of them. In the standard right-handed orientation, where the angle between vectors is measured counterclockwise from the first vector, vector $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ points *upward*, as seen in [link](a). If we reverse the order of multiplication, so that now $\vec{\mathbf{B}}$ comes first in the product, then vector $\vec{\mathbf{B}} \times \vec{\mathbf{A}}$ must point *downward*, as seen in [link](b). This means that vectors $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ and $\vec{\mathbf{B}} \times \vec{\mathbf{A}}$ are *antiparallel* to each other and that vector multiplication is *not* commutative but *anticommutative*. The **anticommutative property** means the vector product reverses the sign when the order of multiplication is reversed:

Note:
Equation:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}.$$

The **corkscrew right-hand rule** is a common mnemonic used to determine the direction of the vector product. As shown in [link], a corkscrew is placed in a direction perpendicular to the plane that contains vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, and its handle is turned in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew.



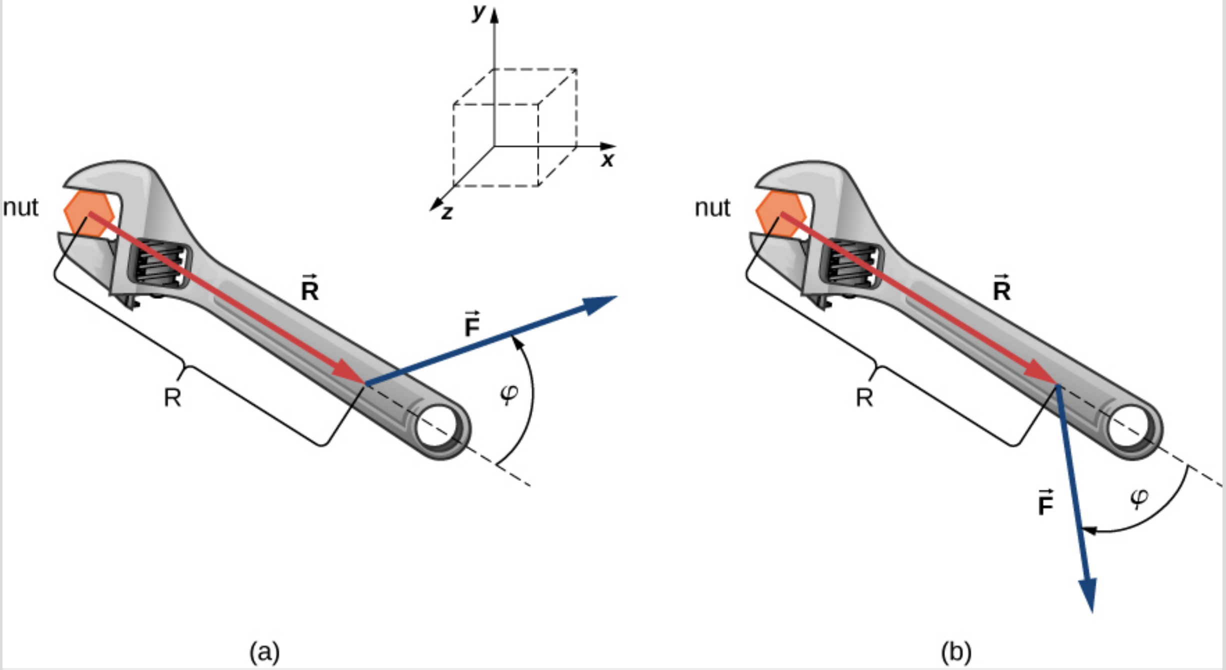
The corkscrew right-hand rule can be used to determine the direction of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$. Place a corkscrew in the direction perpendicular to the plane that contains vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, and turn it in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew. (a) Upward movement means the cross-product vector points up. (b) Downward movement means the cross-product vector points downward.

Example:

The Torque of a Force

The mechanical advantage that a familiar tool called a *wrench* provides ([link]) depends on magnitude F of the applied force, on its direction with respect to the wrench handle, and on how far from the nut this force is applied. The distance R from the nut to the point where force vector $\vec{\mathbf{F}}$ is attached is represented by the radial vector $\vec{\mathbf{R}}$. The physical vector quantity that makes the nut turn is called *torque* (denoted by $\vec{\boldsymbol{\tau}}$), and it is the vector product of the distance between the pivot to force with the force: $\vec{\boldsymbol{\tau}} = \vec{\mathbf{R}} \times \vec{\mathbf{F}}$.

To loosen a rusty nut, a 20.00-N force is applied to the wrench handle at angle $\varphi = 40^\circ$ and at a distance of 0.25 m from the nut, as shown in [link](a). Find the magnitude and direction of the torque applied to the nut. What would the magnitude and direction of the torque be if the force were applied at angle $\varphi = 45^\circ$, as shown in [link](b)? For what value of angle φ does the torque have the largest magnitude?



(a) (b)

A wrench provides grip and mechanical advantage in applying torque to turn a nut. (a) Turn counterclockwise to loosen the nut. (b) Turn clockwise to tighten the nut.

Strategy

We adopt the frame of reference shown in [link], where vectors $\vec{\mathbf{R}}$ and $\vec{\mathbf{F}}$ lie in the *xy*-plane and the origin is at the position of the nut. The radial direction along vector $\vec{\mathbf{R}}$ (pointing away from the origin) is the reference direction for measuring the angle φ because $\vec{\mathbf{R}}$ is the first vector in the vector product $\vec{\boldsymbol{\tau}} = \vec{\mathbf{R}} \times \vec{\mathbf{F}}$. Vector $\vec{\boldsymbol{\tau}}$ must lie along the *z*-axis because this is the axis that is perpendicular to the *xy*-plane, where both $\vec{\mathbf{R}}$ and $\vec{\mathbf{F}}$ lie. To compute the magnitude τ, we use [link]. To find the direction of $\vec{\boldsymbol{\tau}}$, we use the corkscrew right-hand rule ([link]).

Solution

For the situation in (a), the corkscrew rule gives the direction of $\vec{\mathbf{R}} \times \vec{\mathbf{F}}$ in the positive direction of the *z*-axis. Physically, it means the torque vector $\vec{\boldsymbol{\tau}}$ points out of the page, perpendicular to the wrench handle. We identify *F* = 20.00 N and *R* = 0.25 m, and compute the magnitude using [link]:

Equation:

$$\tau = \left| \vec{\mathbf{R}} \times \vec{\mathbf{F}} \right| = RF \sin \varphi = (0.25\ \mathrm{m})(20.00\ \mathrm{N}) \sin 40^\circ = 3.21\ \mathrm{N} \cdot \mathrm{m}.$$

For the situation in (b), the corkscrew rule gives the direction of $\vec{\mathbf{R}} \times \vec{\mathbf{F}}$ in the negative direction of the *z*-axis. Physically, it means the vector $\vec{\boldsymbol{\tau}}$ points into the page, perpendicular to the wrench handle. The magnitude of this torque is

Equation:

$$\tau = \left| \vec{\mathbf{R}} \times \vec{\mathbf{F}} \right| = RF \sin \varphi = (0.25\ \mathrm{m})(20.00\ \mathrm{N}) \sin 45^\circ = 3.53\ \mathrm{N} \cdot \mathrm{m}.$$

The torque has the largest value when $\sin \varphi = 1$, which happens when $\varphi = 90^\circ$. Physically, it means the wrench is most effective—giving us the best mechanical advantage—when we apply the force perpendicular to the wrench handle. For the situation in this example, this best-torque value is $\tau_{\text{best}} = RF = (0.25\,\text{m})(20.00\,\text{N}) = 5.00\,\text{N} \cdot \text{m}$.

Significance

When solving mechanics problems, we often do not need to use the corkscrew rule at all, as we'll see now in the following equivalent solution. Notice that once we have identified that vector $\vec{\mathbf{R}} \times \vec{\mathbf{F}}$ lies along the *z*-axis, we can write this vector in terms of the unit vector $\widehat{\mathbf{k}}$ of the *z*-axis:

Equation:

$$\vec{\mathbf{R}} \times \vec{\mathbf{F}} = RF \sin \varphi \widehat{\mathbf{k}}.$$

In this equation, the number that multiplies $\widehat{\mathbf{k}}$ is the scalar *z*-component of the vector $\vec{\mathbf{R}} \times \vec{\mathbf{F}}$. In the computation of this component, care must be taken that the angle φ is measured *counterclockwise* from $\vec{\mathbf{R}}$ (first vector) to $\vec{\mathbf{F}}$ (second vector). Following this principle for the angles, we obtain $RF \sin(+40^\circ) = +3.2\,\text{N} \cdot \text{m}$ for the situation in (a), and we obtain $RF \sin(-45^\circ) = -3.5\,\text{N} \cdot \text{m}$ for the situation in (b). In the latter case, the angle is negative because the graph in [link] indicates the angle is measured clockwise; but, the same result is obtained when this angle is measured counterclockwise because $+(360^\circ - 45^\circ) = +315^\circ$ and $\sin(+315^\circ) = \sin(-45^\circ)$. In this way, we obtain the solution without reference to the corkscrew rule. For the situation in (a), the solution is $\vec{\mathbf{R}} \times \vec{\mathbf{F}} = +3.2\,\text{N} \cdot \text{m}\widehat{\mathbf{k}}$; for the situation in (b), the solution is $\vec{\mathbf{R}} \times \vec{\mathbf{F}} = -3.5\,\text{N} \cdot \text{m}\widehat{\mathbf{k}}$.

Note:

Exercise:

Problem:

Check Your Understanding For the vectors given in [link], find the vector products $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ and $\vec{\mathbf{C}} \times \vec{\mathbf{F}}$.

Solution:

$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -40.1\widehat{\mathbf{k}}$ or, equivalently, $\left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = 40.1$, and the direction is into the page;

$\vec{\mathbf{C}} \times \vec{\mathbf{F}} = +157.6\widehat{\mathbf{k}}$ or, equivalently, $\left|\vec{\mathbf{C}} \times \vec{\mathbf{F}}\right| = 157.6$, and the direction is out of the page.

Similar to the dot product ([link]), the cross product has the following distributive property:

Note:

Equation:

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}.$$

The distributive property is applied frequently when vectors are expressed in their component forms, in terms of unit vectors of Cartesian axes.

When we apply the definition of the cross product, [link], to unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$ that define the positive *x*-, *y*-, and *z*-directions in space, we find that
Equation:

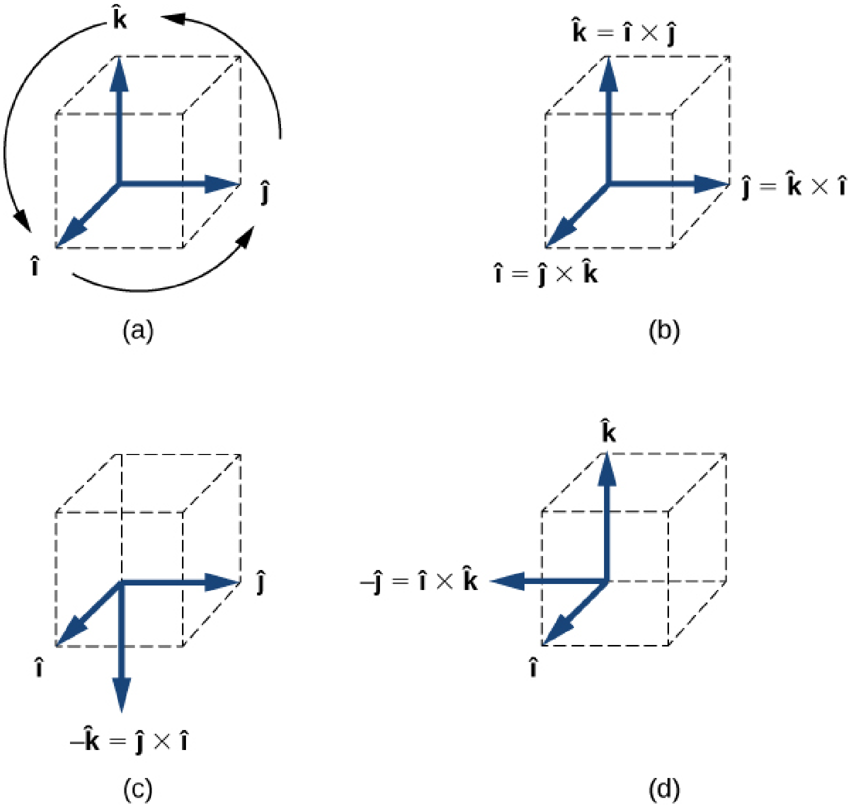
$$\hat{\mathbf{i}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \widehat{\mathbf{k}} \times \widehat{\mathbf{k}} = 0.$$

All other cross products of these three unit vectors must be vectors of unit magnitudes because $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$ are orthogonal. For example, for the pair $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, the magnitude is $\hat{\mathbf{i}} \times \hat{\mathbf{j}} = ij \sin 90^\circ = (1)(1)(1) = 1$. The direction of the vector product $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ must be orthogonal to the *xy*-plane, which means it must be along the *z*-axis. The only unit vectors along the *z*-axis are $-\widehat{\mathbf{k}}$ or $+\widehat{\mathbf{k}}$. By the corkscrew rule, the direction of vector $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ must be parallel to the positive *z*-axis. Therefore, the result of the multiplication $\hat{\mathbf{i}} \times \hat{\mathbf{j}}$ is identical to $+\widehat{\mathbf{k}}$. We can repeat similar reasoning for the remaining pairs of unit vectors. The results of these multiplications are

Note:
Equation:

$$\begin{aligned} \hat{\mathbf{i}} \times \hat{\mathbf{j}} &= +\widehat{\mathbf{k}}, \\ \hat{\mathbf{j}} \times \widehat{\mathbf{k}} &= +\hat{\mathbf{i}}, \\ \widehat{\mathbf{k}} \times \hat{\mathbf{i}} &= +\hat{\mathbf{j}}. \end{aligned}$$

Notice that in [link], the three unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\widehat{\mathbf{k}}$ appear in the *cyclic order* shown in a diagram in [link](a). The cyclic order means that in the product formula, $\hat{\mathbf{i}}$ follows $\widehat{\mathbf{k}}$ and comes before $\hat{\mathbf{j}}$, or $\widehat{\mathbf{k}}$ follows $\hat{\mathbf{j}}$ and comes before $\hat{\mathbf{i}}$, or $\hat{\mathbf{j}}$ follows $\hat{\mathbf{i}}$ and comes before $\widehat{\mathbf{k}}$. The cross product of two different unit vectors is always a third unit vector. When two unit vectors in the cross product appear in the cyclic order, the result of such a multiplication is the remaining unit vector, as illustrated in [link](b). When unit vectors in the cross product appear in a different order, the result is a unit vector that is antiparallel to the remaining unit vector (i.e., the result is with the minus sign, as shown by the examples in [link](c) and [link](d). In practice, when the task is to find cross products of vectors that are given in vector component form, this rule for the cross-multiplication of unit vectors is very useful.


$\hat{\mathbf{k}}$
$\hat{\mathbf{j}}$
$\hat{\mathbf{i}}$
(a)
$\hat{\mathbf{k}} = \hat{\mathbf{i}} \times \hat{\mathbf{j}}$
$\hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{i}}$
$\hat{\mathbf{i}} = \hat{\mathbf{j}} \times \hat{\mathbf{k}}$
(b)
$\hat{\mathbf{j}}$
$\hat{\mathbf{i}}$
$-\hat{\mathbf{k}} = \hat{\mathbf{j}} \times \hat{\mathbf{i}}$
(c)
$\hat{\mathbf{k}}$
$-\hat{\mathbf{j}} = \hat{\mathbf{i}} \times \hat{\mathbf{k}}$
$\hat{\mathbf{i}}$
(d)

(a) The diagram of the cyclic order of the unit vectors of the axes. (b) The only cross products where the unit vectors appear in the cyclic order. These products have the positive sign. (c, d) Two examples of cross products where the unit vectors do not appear in the cyclic order. These products have the negative sign.

Suppose we want to find the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ for vectors $\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}$ and $\vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$. We can use the distributive property ([link]), the anticommutative property ([link]), and the results in [link] and [link] for unit vectors to perform the following algebra:

Equation:

$$\begin{aligned}
\vec{\mathbf{A}} \times \vec{\mathbf{B}} &= (A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}) \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) \\
&= A_x\hat{\mathbf{i}} \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) + A_y\hat{\mathbf{j}} \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) + A_z\hat{\mathbf{k}} \times (B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}) \\
&= \quad A_xB_x\hat{\mathbf{i}} \times \hat{\mathbf{i}} + A_xB_y\hat{\mathbf{i}} \times \hat{\mathbf{j}} + A_xB_z\hat{\mathbf{i}} \times \hat{\mathbf{k}} \\
&\quad + A_yB_x\hat{\mathbf{j}} \times \hat{\mathbf{i}} + A_yB_y\hat{\mathbf{j}} \times \hat{\mathbf{j}} + A_yB_z\hat{\mathbf{j}} \times \hat{\mathbf{k}} \\
&\quad + A_zB_x\hat{\mathbf{k}} \times \hat{\mathbf{i}} + A_zB_y\hat{\mathbf{k}} \times \hat{\mathbf{j}} + A_zB_z\hat{\mathbf{k}} \times \hat{\mathbf{k}} \\
&= \quad A_xB_x(0) + A_xB_y(+\hat{\mathbf{k}}) + A_xB_z(-\hat{\mathbf{j}}) \\
&\quad + A_yB_x(-\hat{\mathbf{k}}) + A_yB_y(0) + A_yB_z(+\hat{\mathbf{i}}) \\
&\quad + A_zB_x(+\hat{\mathbf{j}}) + A_zB_y(-\hat{\mathbf{i}}) + A_zB_z(0).
\end{aligned}$$

When performing algebraic operations involving the cross product, be very careful about keeping the correct order of multiplication because the cross product is anticommutative. The last two steps that we still have to do to complete our task are, first, grouping the terms that contain a common unit vector and, second, factoring. In this way we obtain the following very useful expression for the computation of the cross product:

Note:
Equation:

$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\widehat{\mathbf{k}}.$$

In this expression, the scalar components of the cross-product vector are
Equation:

$$\begin{aligned} C_x &= A_y B_z - A_z B_y, \\ C_y &= A_z B_x - A_x B_z, \\ C_z &= A_x B_y - A_y B_x. \end{aligned}$$

When finding the cross product, in practice, we can use either [link] or [link], depending on which one of them seems to be less complex computationally. They both lead to the same final result. One way to make sure if the final result is correct is to use them both.

Example:
A Particle in a Magnetic Field
When moving in a magnetic field, some particles may experience a magnetic force. Without going into details—a detailed study of magnetic phenomena comes in later chapters—let’s acknowledge that the magnetic field $\vec{\mathbf{B}}$ is a vector, the magnetic force $\vec{\mathbf{F}}$ is a vector, and the velocity $\vec{\mathbf{u}}$ of the particle is a vector. The magnetic force vector is proportional to the vector product of the velocity vector with the magnetic field vector, which we express as $\vec{\mathbf{F}} = \zeta \vec{\mathbf{u}} \times \vec{\mathbf{B}}$. In this equation, a constant ζ takes care of the consistency in physical units, so we can omit physical units on vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{B}}$. In this example, let’s assume the constant ζ is positive.

A particle moving in space with velocity vector $\vec{\mathbf{u}} = -5.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}} + 3.5\widehat{\mathbf{k}}$ enters a region with a magnetic field and experiences a magnetic force. Find the magnetic force $\vec{\mathbf{F}}$ on this particle at the entry point to the region where the magnetic field vector is (a) $\vec{\mathbf{B}} = 7.2\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2.4\widehat{\mathbf{k}}$ and (b) $\vec{\mathbf{B}} = 4.5\widehat{\mathbf{k}}$. In each case, find magnitude F of the magnetic force and angle θ the force vector $\vec{\mathbf{F}}$ makes with the given magnetic field vector $\vec{\mathbf{B}}$.

Strategy
First, we want to find the vector product $\vec{\mathbf{u}} \times \vec{\mathbf{B}}$, because then we can determine the magnetic force using $\vec{\mathbf{F}} = \zeta \vec{\mathbf{u}} \times \vec{\mathbf{B}}$. Magnitude F can be found either by using components, $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$, or by computing the magnitude $\left| \vec{\mathbf{u}} \times \vec{\mathbf{B}} \right|$ directly using [link]. In the latter approach, we would have to find the angle between vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{B}}$. When we have $\vec{\mathbf{F}}$, the general method for finding the direction angle θ

involves the computation of the scalar product $\vec{\mathbf{F}} \cdot \vec{\mathbf{B}}$ and substitution into [link]. To compute the vector product we can either use [link] or compute the product directly, whichever way is simpler.

Solution

The components of the velocity vector are $u_x = -5.0$, $u_y = -2.0$, and $u_z = 3.5$.

(a) The components of the magnetic field vector are $B_x = 7.2$, $B_y = -1.0$, and $B_z = -2.4$. Substituting them into [link] gives the scalar components of vector $\vec{\mathbf{F}} = \zeta \vec{\mathbf{u}} \times \vec{\mathbf{B}}$:

Equation:

$$\begin{aligned} F_x &= \zeta(u_y B_z - u_z B_y) = \zeta[(-2.0)(-2.4) - (3.5)(-1.0)] = 8.3\zeta \\ F_y &= \zeta(u_z B_x - u_x B_z) = \zeta[(3.5)(7.2) - (-5.0)(-2.4)] = 13.2\zeta \\ F_z &= \zeta(u_x B_y - u_y B_x) = \zeta[(-5.0)(-1.0) - (-2.0)(7.2)] = 19.4\zeta \end{aligned}.$$

Thus, the magnetic force is $\vec{\mathbf{F}} = \zeta(8.3\hat{\mathbf{i}} + 13.2\hat{\mathbf{j}} + 19.4\widehat{\mathbf{k}})$ and its magnitude is

Equation:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta\sqrt{(8.3)^2 + (13.2)^2 + (19.4)^2} = 24.9\zeta.$$

To compute angle θ, we may need to find the magnitude of the magnetic field vector,

Equation:

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(7.2)^2 + (-1.0)^2 + (-2.4)^2} = 7.6,$$

and the scalar product $\vec{\mathbf{F}} \cdot \vec{\mathbf{B}}$:

Equation:

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{B}} = F_x B_x + F_y B_y + F_z B_z = (8.3\zeta)(7.2) + (13.2\zeta)(-1.0) + (19.4\zeta)(-2.4) = 0.$$

Now, substituting into [link] gives angle θ:

Equation:

$$\cos\theta = \frac{\vec{\mathbf{F}} \cdot \vec{\mathbf{B}}}{FB} = \frac{0}{(18.2\zeta)(7.6)} = 0 \;\Rightarrow\; \theta = 90^\circ.$$

Hence, the magnetic force vector is perpendicular to the magnetic field vector. (We could have saved some time if we had computed the scalar product earlier.)

(b) Because vector $\vec{\mathbf{B}} = 4.5\widehat{\mathbf{k}}$ has only one component, we can perform the algebra quickly and find the vector product directly:

Equation:

$$\begin{aligned} \vec{\mathbf{F}} &= \zeta \vec{\mathbf{u}} \times \vec{\mathbf{B}} = \zeta(-5.0\hat{\mathbf{i}} - 2.0\hat{\mathbf{j}} + 3.5\widehat{\mathbf{k}}) \times (4.5\widehat{\mathbf{k}}) \\ &= \zeta[(-5.0)(4.5)\hat{\mathbf{i}} \times \widehat{\mathbf{k}} + (-2.0)(4.5)\hat{\mathbf{j}} \times \widehat{\mathbf{k}} + (3.5)(4.5)\widehat{\mathbf{k}} \times \widehat{\mathbf{k}}] \\ &= \zeta[-22.5(-\hat{\mathbf{j}}) - 9.0(+\hat{\mathbf{i}}) + 0] = \zeta(-9.0\hat{\mathbf{i}} + 22.5\hat{\mathbf{j}}). \end{aligned}$$

The magnitude of the magnetic force is

Equation:

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta\sqrt{(-9.0)^2 + (22.5)^2 + (0.0)^2} = 24.2\zeta.$$

Because the scalar product is

Equation:

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{B}} = F_x B_x + F_y B_y + F_z B_z = (-9.0\zeta)(0) + (22.5\zeta)(0) + (0)(4.5) = 0,$$

the magnetic force vector $\vec{\mathbf{F}}$ is perpendicular to the magnetic field vector $\vec{\mathbf{B}}$.

Significance

Even without actually computing the scalar product, we can predict that the magnetic force vector must always be perpendicular to the magnetic field vector because of the way this vector is constructed. Namely, the magnetic force vector is the vector product $\vec{\mathbf{F}} = \zeta\vec{\mathbf{u}} \times \vec{\mathbf{B}}$ and, by the definition of the vector product (see [link]), vector $\vec{\mathbf{F}}$ must be perpendicular to both vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{B}}$.

Note:

Exercise:

Problem:

Check Your Understanding Given two vectors $\vec{\mathbf{A}} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\vec{\mathbf{B}} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}}$, find (a) $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$, (b) $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$, (c) the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, and (d) the angle between $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ and vector $\vec{\mathbf{C}} = \hat{\mathbf{i}} + \widehat{\mathbf{k}}$.

Solution:

a. $-2\widehat{\mathbf{k}}$, b. 2, c. 153.4°, d. 135°

In conclusion to this section, we want to stress that “dot product” and “cross product” are entirely different mathematical objects that have different meanings. The dot product is a scalar; the cross product is a vector. Later chapters use the terms *dot product* and *scalar product* interchangeably. Similarly, the terms *cross product* and *vector product* are used interchangeably.

Summary

- There are two kinds of multiplication for vectors. One kind of multiplication is the scalar product, also known as the dot product. The other kind of multiplication is the vector product, also known as the cross product. The scalar product of vectors is a number (scalar). The vector product of vectors is a vector.
- Both kinds of multiplication have the distributive property, but only the scalar product has the commutative property. The vector product has the anticommutative property, which means that when we change the order in which two vectors are multiplied, the result acquires a minus sign.
- The scalar product of two vectors is obtained by multiplying their magnitudes with the cosine of the angle between them. The scalar product of orthogonal vectors vanishes; the scalar product of antiparallel vectors is negative.
- The vector product of two vectors is a vector perpendicular to both of them. Its magnitude is obtained by multiplying their magnitudes by the sine of the angle between them. The direction of the vector product can be determined by the corkscrew right-hand rule. The vector product of two either parallel or antiparallel vectors vanishes. The magnitude of the vector product is largest for orthogonal vectors.
- The scalar product of vectors is used to find angles between vectors and in the definitions of derived scalar physical quantities such as work or energy.

- The cross product of vectors is used in definitions of derived vector physical quantities such as torque or magnetic force, and in describing rotations.

Key Equations

Multiplication by a scalar (vector equation)	$\vec{\mathbf{B}} = \alpha\vec{\mathbf{A}}$
Multiplication by a scalar (scalar equation for magnitudes)	$B = \lvert\alpha\rvert A$
Resultant of two vectors	$\vec{\mathbf{D}}_{AD} = \vec{\mathbf{D}}_{AC} + \vec{\mathbf{D}}_{CD}$
Commutative law	$\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$
Associative law	$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$
Distributive law	$\alpha_1\vec{\mathbf{A}} + \alpha_2\vec{\mathbf{A}} = (\alpha_1 + \alpha_2)\vec{\mathbf{A}}$
The component form of a vector in two dimensions	$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$
Scalar components of a vector in two dimensions	$\begin{cases} A_x = x_e - x_b \\ A_y = y_e - y_b \end{cases}$
Magnitude of a vector in a plane	$A = \sqrt{A_x^2 + A_y^2}$
The direction angle of a vector in a plane	$\theta_A = \tan^{-1}\left(\frac{A_y}{A_x}\right)$
Scalar components of a vector in a plane	$\begin{cases} A_x = A\cos\theta_A \\ A_y = A\sin\theta_A \end{cases}$
Polar coordinates in a plane	$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$
The component form of a vector in three dimensions	$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\widehat{\mathbf{k}}$

The scalar *z*-component of a vector in three dimensions	$A_z = z_e - z_b$
Magnitude of a vector in three dimensions	$A = \sqrt{A_x^2 + A_y^2 + A_z^2}$
Distributive property	$\alpha(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = \alpha\vec{\mathbf{A}} + \alpha\vec{\mathbf{B}}$
Antiparallel vector to $\vec{\mathbf{A}}$	$-\vec{\mathbf{A}} = -A_x\hat{\mathbf{i}} - A_y\hat{\mathbf{j}} - A_z\hat{\mathbf{k}}$
Equal vectors	$\vec{\mathbf{A}} = \vec{\mathbf{B}} \;\Leftrightarrow\; \begin{array}{l} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{array}$
Components of the resultant of *N* vectors	$F_{Rx} = \sum_{k=1}^{N} F_{kx} = F_{1x} + F_{2x} + \ldots + F_{Nx}$ $F_{Ry} = \sum_{k=1}^{N} F_{ky} = F_{1y} + F_{2y} + \ldots + F_{Ny}$ $F_{Rz} = \sum_{k=1}^{N} F_{kz} = F_{1z} + F_{2z} + \ldots + F_{Nz}$
General unit vector	$\hat{\mathbf{V}} = \frac{\vec{\mathbf{V}}}{V}$
Definition of the scalar product	$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = AB\cos\varphi$
Commutative property of the scalar product	$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{A}}$
Distributive property of the scalar product	$\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}} + \vec{\mathbf{A}} \cdot \vec{\mathbf{C}}$
Scalar product in terms of scalar components of vectors	$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_xB_x + A_yB_y + A_zB_z$
Cosine of the angle between two vectors	$\cos\varphi = \frac{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}}{AB}$
Dot products of unit vectors	$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{i}} = 0$
Magnitude of the vector product (definition)	$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = AB\sin\varphi$

Anticommutative property of the vector product	$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$
Distributive property of the vector product	$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}}$
Cross products of unit vectors	$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = +\widehat{\mathbf{k}}$, $\hat{\mathbf{j}} \times \widehat{\mathbf{k}} = +\hat{\mathbf{i}}$, $\widehat{\mathbf{k}} \times \hat{\mathbf{i}} = +\hat{\mathbf{j}}$.
The cross product in terms of scalar components of vectors	$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\widehat{\mathbf{k}}$

Conceptual Questions

Exercise:

Problem:

What is wrong with the following expressions? How can you correct them? (a) $C = \vec{\mathbf{A}}\vec{\mathbf{B}}$, (b) $\vec{\mathbf{C}} = \vec{\mathbf{A}}\vec{\mathbf{B}}$, (c) $C = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$, (d) $C = A\vec{\mathbf{B}}$, (e) $C + 2\vec{\mathbf{A}} = B$, (f) $\vec{\mathbf{C}} = A \times \vec{\mathbf{B}}$, (g) $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$, (h) $\vec{\mathbf{C}} = 2\vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$, (i) $C = \vec{\mathbf{A}}/\vec{\mathbf{B}}$, and (j) $C = \vec{\mathbf{A}}/B$.

Solution:

a. $C = \vec{\mathbf{A}} \cdot \vec{\mathbf{B}}$, b. $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ or $\vec{\mathbf{C}} = \vec{\mathbf{A}} - \vec{\mathbf{B}}$, c. $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$, d. $\vec{\mathbf{C}} = A\vec{\mathbf{B}}$, e. $\vec{\mathbf{C}} + 2\vec{\mathbf{A}} = \vec{\mathbf{B}}$, f. $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$, g. left side is a scalar and right side is a vector, h. $\vec{\mathbf{C}} = 2\vec{\mathbf{A}} \times \vec{\mathbf{B}}$, i. $\vec{\mathbf{C}} = \vec{\mathbf{A}}/B$, j. $\vec{\mathbf{C}} = \vec{\mathbf{A}}/B$

Exercise:

Problem: If the cross product of two vectors vanishes, what can you say about their directions?

Exercise:

Problem: If the dot product of two vectors vanishes, what can you say about their directions?

Solution:

They are orthogonal.

Exercise:

Problem:

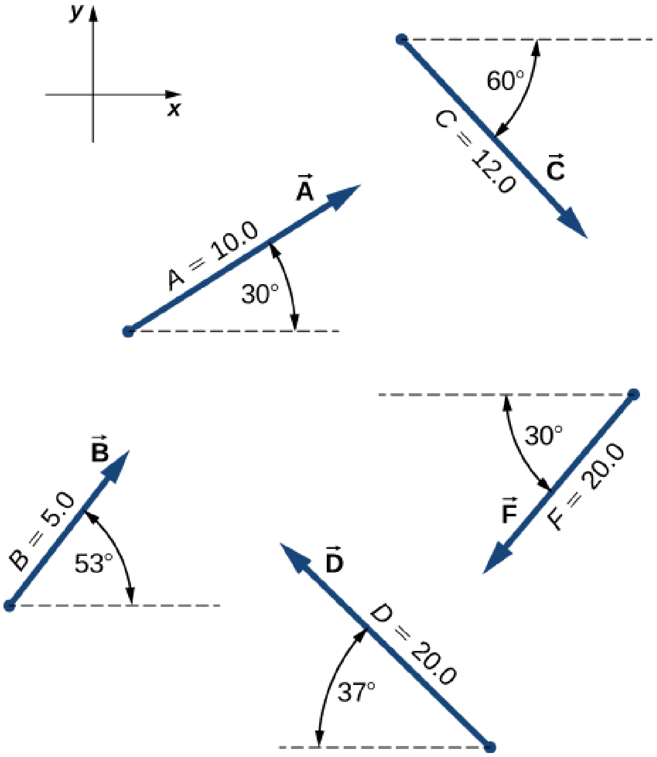
What is the dot product of a vector with the cross product that this vector has with another vector?

Problems

Exercise:

Problem:

Assuming the +*x*-axis is horizontal to the right for the vectors in the following figure, find the following scalar products: (a) $\vec{\mathbf{A}} \cdot \vec{\mathbf{C}}$, (b) $\vec{\mathbf{A}} \cdot \vec{\mathbf{F}}$, (c) $\vec{\mathbf{D}} \cdot \vec{\mathbf{C}}$, (d) $\vec{\mathbf{A}} \cdot (\vec{\mathbf{F}} + 2\vec{\mathbf{C}})$, (e) $\hat{\mathbf{i}} \cdot \vec{\mathbf{B}}$, (f) $\hat{\mathbf{j}} \cdot \vec{\mathbf{B}}$, (g) $(3\hat{\mathbf{i}} - \hat{\mathbf{j}}) \cdot \vec{\mathbf{B}}$, and (h) $\widehat{\boldsymbol{B}} \cdot \boldsymbol{B}$.



Exercise:

Problem:

Assuming the +*x*-axis is horizontal to the right for the vectors in the preceding figure, find (a) the component of vector $\vec{\mathbf{A}}$ along vector $\vec{\mathbf{C}}$, (b) the component of vector $\vec{\mathbf{C}}$ along vector $\vec{\mathbf{A}}$, (c) the component of vector $\hat{\mathbf{i}}$ along vector $\vec{\mathbf{F}}$, and (d) the component of vector $\vec{\mathbf{F}}$ along vector $\hat{\mathbf{i}}$.

Solution:

a. 0, b. 0, c. 0.866, d. 17.32

Exercise:

Problem:

Find the angle between vectors for (a) $\vec{\mathbf{D}} = (-3.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}})\text{m}$ and $\vec{\mathbf{A}} = (-3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}})\text{m}$ and (b) $\vec{\mathbf{D}} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}} + \widehat{\mathbf{k}})\text{m}$ and $\vec{\mathbf{B}} = (-2.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}} + 2.0\widehat{\mathbf{k}})\text{m}$.

Exercise:

Problem: Find the angles that vector $\vec{\mathbf{D}} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}} + \widehat{\mathbf{k}})\mathrm{m}$ makes with the *x*-, *y*-, and *z*- axes.

Solution:

$\theta_i = 64.12^\circ, \theta_j = 150.79^\circ, \theta_k = 77.39^\circ$

Exercise:

Problem:

Show that the force vector $\vec{\mathbf{D}} = (2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}} + \widehat{\mathbf{k}})\mathrm{N}$ is orthogonal to the force vector $\vec{\mathbf{G}} = (3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}} + 10.0\widehat{\mathbf{k}})\mathrm{N}$.

Exercise:

Problem:

Assuming the +*x*-axis is horizontal to the right for the vectors in the previous figure, find the following vector products: (a) $\vec{\mathbf{A}} \times \vec{\mathbf{C}}$, (b) $\vec{\mathbf{A}} \times \vec{\mathbf{F}}$, (c) $\vec{\mathbf{D}} \times \vec{\mathbf{C}}$, (d) $\vec{\mathbf{A}} \times (\vec{\mathbf{F}} + 2\vec{\mathbf{C}})$, (e) $\hat{\mathbf{i}} \times \vec{\mathbf{B}}$, (f) $\hat{\mathbf{j}} \times \vec{\mathbf{B}}$, (g) $(3\hat{\mathbf{i}} - \hat{\mathbf{j}}) \times \vec{\mathbf{B}}$, and (h) $\widehat{\mathbf{B}} \times \vec{\mathbf{B}}$.

Solution:

a. $-120\widehat{\mathbf{k}}$, b. $0\widehat{\mathbf{k}}$, c. $-94\widehat{\mathbf{k}}$, d. –240$\widehat{\mathbf{k}}$, e. $4.0\widehat{\mathbf{k}}$, f. $-3.0\widehat{\mathbf{k}}$, g. $15\widehat{\mathbf{k}}$, h. 0

Exercise:

Problem:

Find the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{C}}$ for (a) $\vec{\mathbf{A}} = 2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}} + \widehat{\mathbf{k}}$ and $\vec{\mathbf{C}} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}} + 10.0\widehat{\mathbf{k}}$, (b) $\vec{\mathbf{A}} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}} + 10.0\widehat{\mathbf{k}}$ and $\vec{\mathbf{C}} = 2.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}} + \widehat{\mathbf{k}}$, (c) $\vec{\mathbf{A}} = -3.0\hat{\mathbf{i}} - 4.0\hat{\mathbf{j}}$ and $\vec{\mathbf{C}} = -3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$, and (d) $\vec{\mathbf{C}} = -2.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}} + 2.0\widehat{\mathbf{k}}$ and $\vec{\mathbf{A}} = -9.0\hat{\mathbf{j}}$.

Exercise:

Problem:

For the vectors in the earlier figure, find (a) $(\vec{\mathbf{A}} \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{D}}$, (b) $(\vec{\mathbf{A}} \times \vec{\mathbf{F}}) \cdot (\vec{\mathbf{D}} \times \vec{\mathbf{B}})$, and (c) $(\vec{\mathbf{A}} \cdot \vec{\mathbf{F}})(\vec{\mathbf{D}} \times \vec{\mathbf{B}})$.

Solution:

a. 0, b. 0, c. $+-20{,}000\widehat{\mathbf{k}}$

Exercise:

Problem:

(a) If $\vec{\mathbf{A}} \times \vec{\mathbf{F}} = \vec{\mathbf{B}} \times \vec{\mathbf{F}}$, can we conclude $\vec{\mathbf{A}} = \vec{\mathbf{B}}$? (b) If $\vec{\mathbf{A}} \cdot \vec{\mathbf{F}} = \vec{\mathbf{B}} \cdot \vec{\mathbf{F}}$, can we conclude $\vec{\mathbf{A}} = \vec{\mathbf{B}}$? (c) If $F\vec{\mathbf{A}} = \vec{\mathbf{B}}F$, can we conclude $\vec{\mathbf{A}} = \vec{\mathbf{B}}$? Why or why not?

Additional Problems

Exercise:

Problem:

You fly $32.0\,\text{km}$ in a straight line in still air in the direction 35.0° south of west. (a) Find the distances you would have to fly due south and then due west to arrive at the same point. (b) Find the distances you would have to fly first in a direction 45.0° south of west and then in a direction 45.0° west of north. Note these are the components of the displacement along a different set of axes—namely, the one rotated by 45° with respect to the axes in (a).

Solution:

a. 18.4 km and 26.2 km, b. 31.5 km and 5.56 km

Exercise:

Problem:

Rectangular coordinates of a point are given by (2, y) and its polar coordinates are given by $(r, \pi/6)$. Find y and r.

Exercise:

Problem:

If the polar coordinates of a point are (r, φ) and its rectangular coordinates are (x, y), determine the polar coordinates of the following points: (a) $(-x, y)$, (b) $(-2x, -2y)$, and (c) $(3x, -3y)$.

Solution:

a. $(r, \pi - \varphi)$, b. $(2r, \varphi + 2\pi)$, (c) $(3r, -\varphi)$

Exercise:

Problem:

Vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ have identical magnitudes of 5.0 units. Find the angle between them if $\vec{\mathbf{A}} + \vec{\mathbf{B}} = 5\sqrt{2}\hat{\mathbf{j}}$.

Exercise:

Problem:

Starting at the island of Moi in an unknown archipelago, a fishing boat makes a round trip with two stops at the islands of Noi and Poi. It sails from Moi for 4.76 nautical miles (nmi) in a direction 37° north of east to Noi. From Noi, it sails 69° west of north to Poi. On its return leg from Poi, it sails 28° east of south. What distance does the boat sail between Noi and Poi? What distance does it sail between Moi and Poi? Express your answer both in nautical miles and in kilometers. Note: 1 nmi = 1852 m.

Solution:

$d_{\text{PM}} = 6.2\,\text{nmi} = 11.4\,\text{km},\;\; d_{\text{NP}} = 7.2\,\text{nmi} = 13.3\,\text{km}$

Exercise:

Problem:

An air traffic controller notices two signals from two planes on the radar monitor. One plane is at altitude 800 m and in a 19.2-km horizontal distance to the tower in a direction 25° south of west. The second plane is at altitude 1100 m and its horizontal distance is 17.6 km and 20° south of west. What is the distance between these planes?

Exercise:

Problem:

Show that when $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{C}}$, then $C^2 = A^2 + B^2 + 2AB \cos \varphi$, where φ is the angle between vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

Solution:

proof

Exercise:

Problem:

Four force vectors each have the same magnitude *f*. What is the largest magnitude the resultant force vector may have when these forces are added? What is the smallest magnitude of the resultant? Make a graph of both situations.

Exercise:

Problem:

A skater glides along a circular path of radius 5.00 m in clockwise direction. When he coasts around one-half of the circle, starting from the west point, find (a) the magnitude of his displacement vector and (b) how far he actually skated. (c) What is the magnitude of his displacement vector when he skates all the way around the circle and comes back to the west point?

Solution:

a. 10.00 m, b. 5π m, c. 0

Exercise:

Problem:

A stubborn dog is being walked on a leash by its owner. At one point, the dog encounters an interesting scent at some spot on the ground and wants to explore it in detail, but the owner gets impatient and pulls on the leash with force $\vec{\mathbf{F}} = (98.0\hat{\mathbf{i}} + 132.0\hat{\mathbf{j}} + 32.0\widehat{\mathbf{k}})\text{N}$ along the leash. (a) What is the magnitude of the pulling force? (b) What angle does the leash make with the vertical?

Exercise:

Problem:

If the velocity vector of a polar bear is $\vec{\mathbf{u}} = (-18.0\hat{\mathbf{i}} - 13.0\hat{\mathbf{j}})\text{km/h}$, how fast and in what geographic direction is it heading? Here, $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are directions to geographic east and north, respectively.

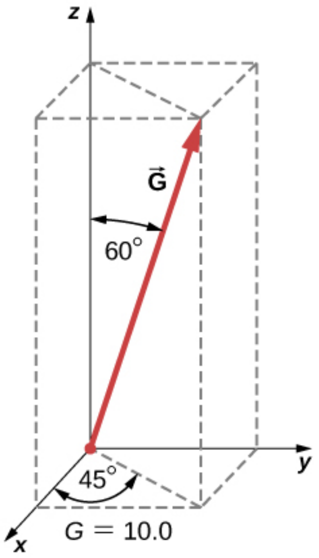
Solution:

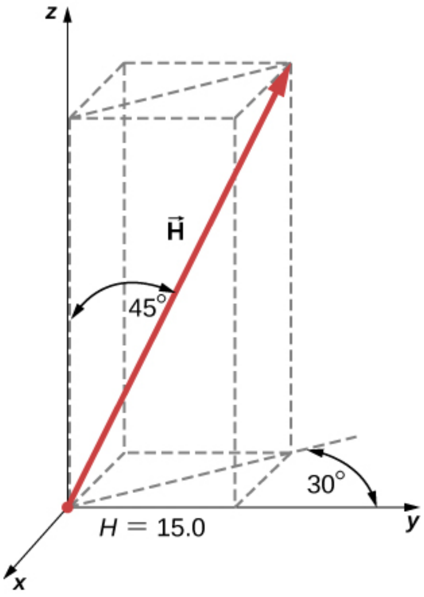
22.2 km/h, 35.8° south of west

Exercise:

Problem:

Find the scalar components of three-dimensional vectors $\vec{\mathbf{G}}$ and $\vec{\mathbf{H}}$ in the following figure and write the vectors in vector component form in terms of the unit vectors of the axes.


z
$\vec{\mathbf{G}}$
60°
y
45°
$G = 10.0$
x


z
$\vec{\mathbf{H}}$
45°
30°
$H = 15.0$
y
x

Exercise:

Problem:

A diver explores a shallow reef off the coast of Belize. She initially swims 90.0 m north, makes a turn to the east and continues for 200.0 m, then follows a big grouper for 80.0 m in the direction 30° north of east. In the meantime, a local current displaces her by 150.0 m south. Assuming the current is no longer present, in what direction and how far should she now swim to come back to the point where she started?

Solution:

270 m, 4.2° north of west

Exercise:

Problem:

A force vector $\vec{\mathbf{A}}$ has *x*- and *y*-components, respectively, of −8.80 units of force and 15.00 units of force. The *x*- and *y*-components of force vector $\vec{\mathbf{B}}$ are, respectively, 13.20 units of force and −6.60 units of force. Find the components of force vector $\vec{\mathbf{C}}$ that satisfies the vector equation $\vec{\mathbf{A}} - \vec{\mathbf{B}} + 3\vec{\mathbf{C}} = 0$.

Exercise:

Problem:

Vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are two orthogonal vectors in the *xy*-plane and they have identical magnitudes. If $\vec{\mathbf{A}} = 3.0\hat{\mathbf{i}} + 4.0\hat{\mathbf{j}}$, find $\vec{\mathbf{B}}$.

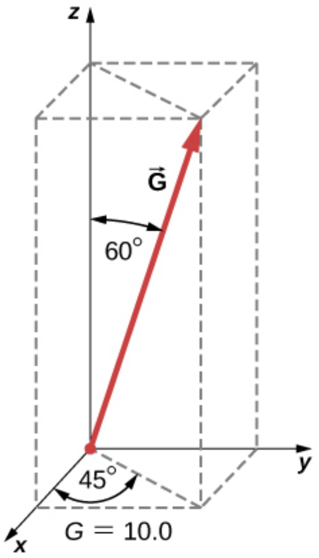
Solution:

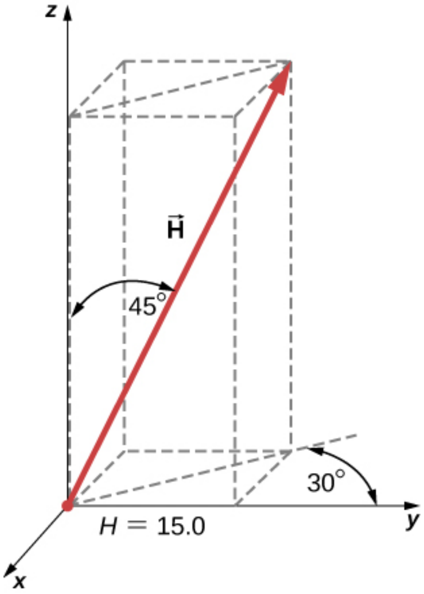
$\vec{\mathbf{B}} = -4.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}}$ or $\vec{\mathbf{B}} = 4.0\hat{\mathbf{i}} - 3.0\hat{\mathbf{j}}$

Exercise:

Problem:

For the three-dimensional vectors in the following figure, find (a) $\vec{\mathbf{G}} \times \vec{\mathbf{H}}$, (b) $\vec{\mathbf{G}} \times \vec{\mathbf{H}}$, and (c) $\vec{\mathbf{G}} \cdot \vec{\mathbf{H}}$.

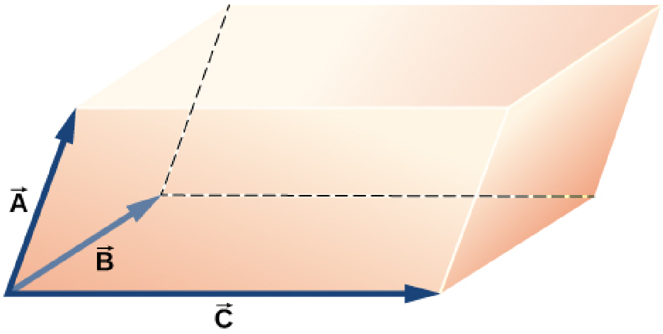

z
$\vec{G}$
60°
y
45°
$G = 10.0$
x


z
$\vec{H}$
45°
30°
$H = 15.0$
y
x

Exercise:

Problem:

Show that $(\vec{\mathbf{B}} \times \vec{\mathbf{C}}) \cdot \vec{\mathbf{A}}$ is the volume of the parallelepiped, with edges formed by the three vectors in the following figure.



Solution:

proof

Challenge Problems

Exercise:

Problem:

Vector $\vec{\mathbf{B}}$ is 5.0 cm long and vector $\vec{\mathbf{A}}$ is 4.0 cm long. Find the angle between these two vectors when $\left|\vec{\mathbf{A}}+\vec{\mathbf{B}}\right| = 3.0\text{ cm}$ and $\left|\vec{\mathbf{A}}-\vec{\mathbf{B}}\right| = 3.0\text{ cm}$.

Exercise:

Problem:

What is the component of the force vector $\vec{\mathbf{G}}=(3.0\hat{\mathbf{i}}+4.0\hat{\mathbf{j}}+10.0\widehat{\mathbf{k}})\text{N}$ along the force vector $\vec{\mathbf{H}}=(1.0\hat{\mathbf{i}}+4.0\hat{\mathbf{j}})\text{N}$?

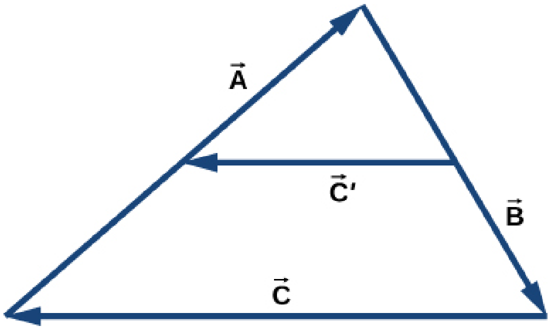
Solution:

$G_{\text{H}}=19\text{ N}/\sqrt{17}\approx 4.6\text{ N}$

Exercise:

Problem:

The following figure shows a triangle formed by the three vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{C}}$. If vector $\vec{\mathbf{C}}'$ is drawn between the midpoints of vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, show that $\vec{\mathbf{C}}'=\vec{\mathbf{C}}/2$.



Exercise:

Problem:

Distances between points in a plane do not change when a coordinate system is rotated. In other words, the magnitude of a vector is *invariant* under rotations of the coordinate system. Suppose a coordinate system S is rotated about its origin by angle φ to become a new coordinate system S', as shown in the following figure. A point in a plane has coordinates (*x*, *y*) in S and coordinates (x', y') in S'.

(a) Show that, during the transformation of rotation, the coordinates in S' are expressed in terms of the coordinates in S by the following relations:

Equation:

$$\begin{cases} x' = x\cos\varphi + y\sin\varphi \\ y' = -x\sin\varphi + y\cos\varphi \end{cases}.$$

(b) Show that the distance of point *P* to the origin is invariant under rotations of the coordinate system. Here, you have to show that

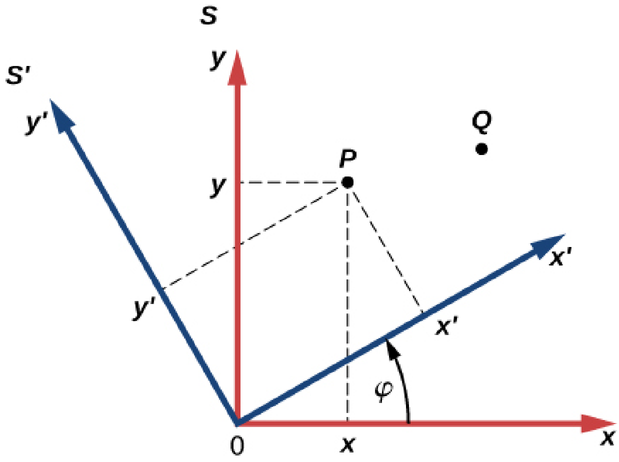
Equation:

$$\sqrt{x^2 + y^2} = \sqrt{x'^2 + y'^2}.$$

(c) Show that the distance between points *P* and *Q* is invariant under rotations of the coordinate system. Here, you have to show that

Equation:

$$\sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2} = \sqrt{(x'_P - x'_Q)^2 + (y'_P - y'_Q)^2}.$$



Solution:

proof

Glossary

anticommutative property
change in the order of operation introduces the minus sign

corkscrew right-hand rule
a rule used to determine the direction of the vector product

cross product
the result of the vector multiplication of vectors is a vector called a cross product; also called a vector product

dot product
the result of the scalar multiplication of two vectors is a scalar called a dot product; also called a scalar product

scalar product
the result of the scalar multiplication of two vectors is a scalar called a scalar product; also called a dot product

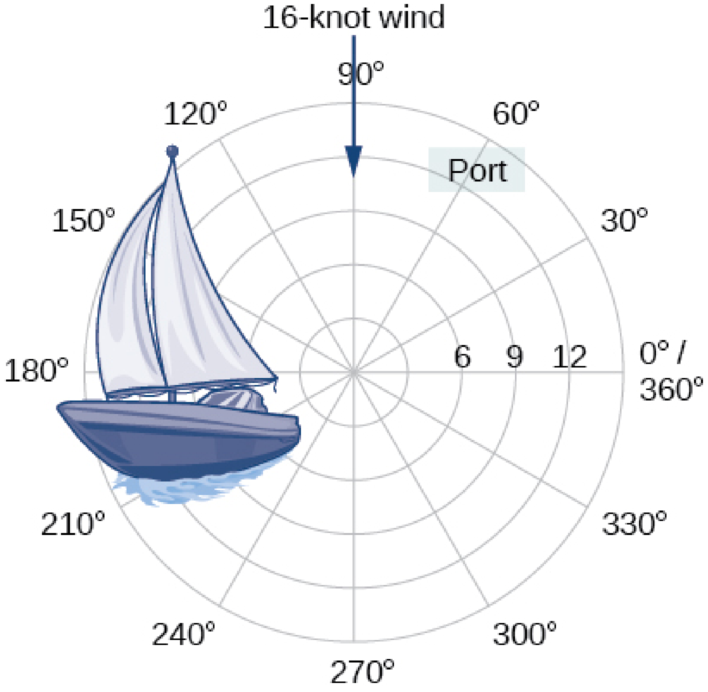
vector product
the result of the vector multiplication of vectors is a vector called a vector product; also called a cross product

Polar Coordinates

In this section, you will:

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

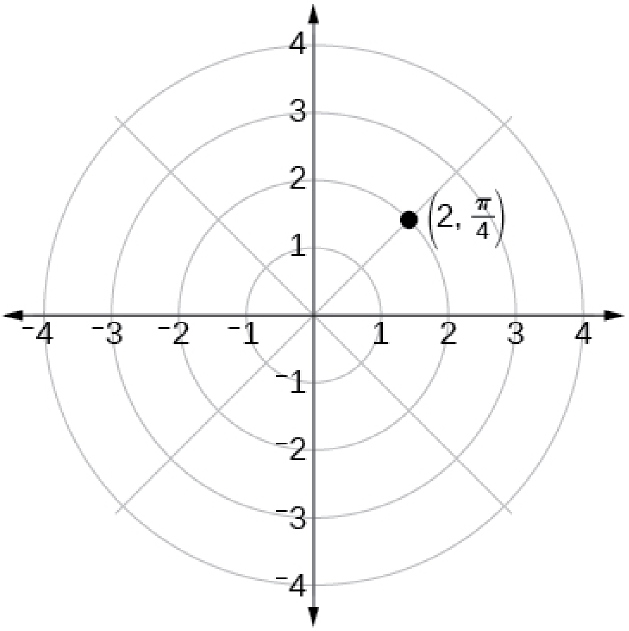
Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see [link]). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.



Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of rectangular coordinates (x, y) in the Cartesian coordinate plane. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to **polar coordinates**, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the **pole**, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive *x*-axis now viewed as the **polar axis** and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ, and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$, we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in [link].



Polar Grid

Example:
Exercise:

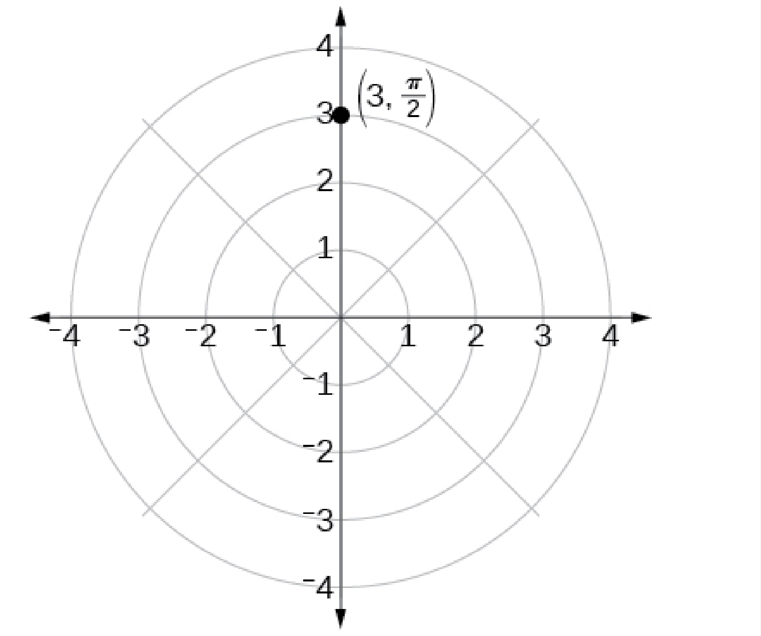
Problem:
Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution:

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$

direction, as shown in [link].

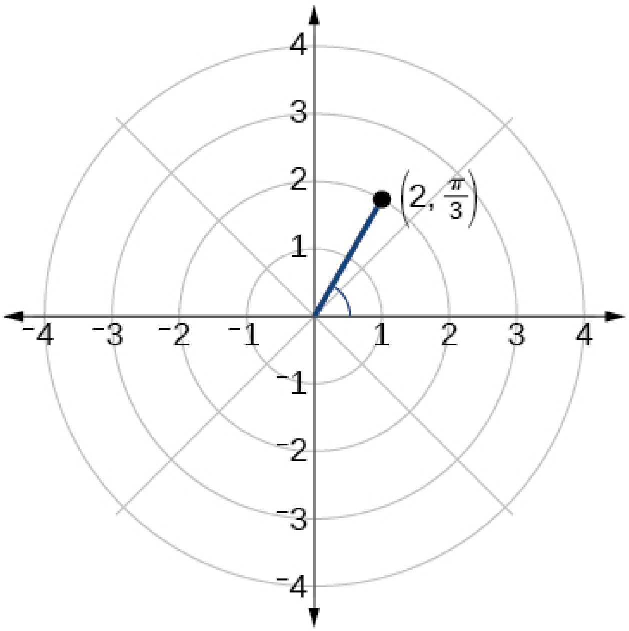


4
3
2
1
$\left(3, \frac{\pi}{2}\right)$
−4
−3
−2
−1
1
2
3
4
−1
−2
−3
−4

Note:
Exercise:

Problem: Plot the point $\left(2,\ \frac{\pi}{3}\right)$ in the polar grid.

Solution:



Example:
Exercise:

Problem:
Plotting a Point in the Polar Coordinate System with a Negative Component

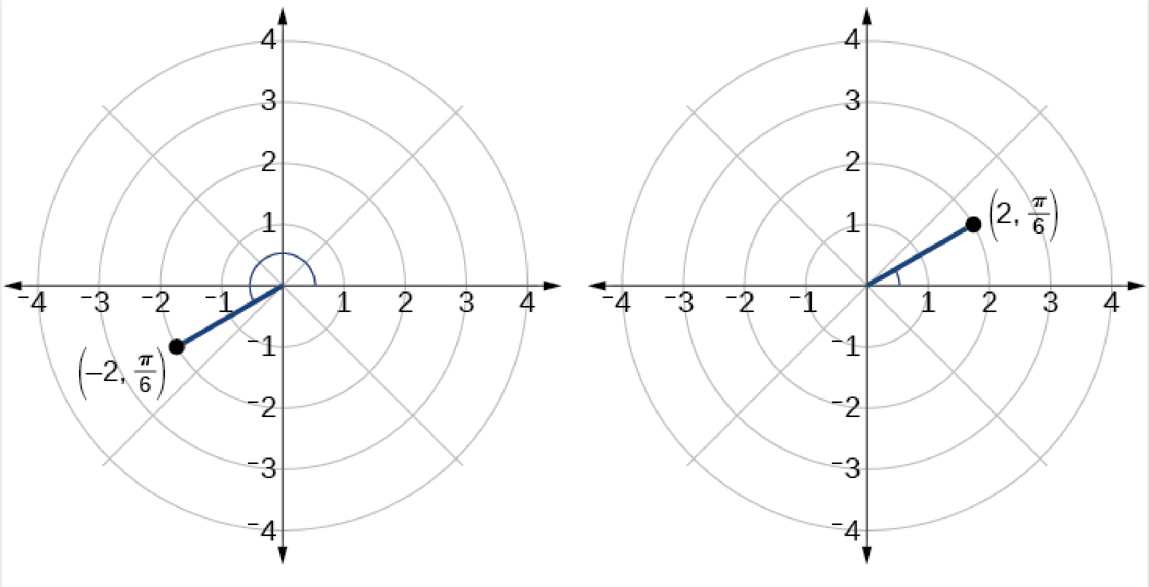
Plot the point $\left(-2,\ \frac{\pi}{6}\right)$ on the polar grid.

Solution:

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See [link](a). Compare this to the graph of the polar coordinate $\left(2, \frac{\pi}{6}\right)$ shown in [link](b).

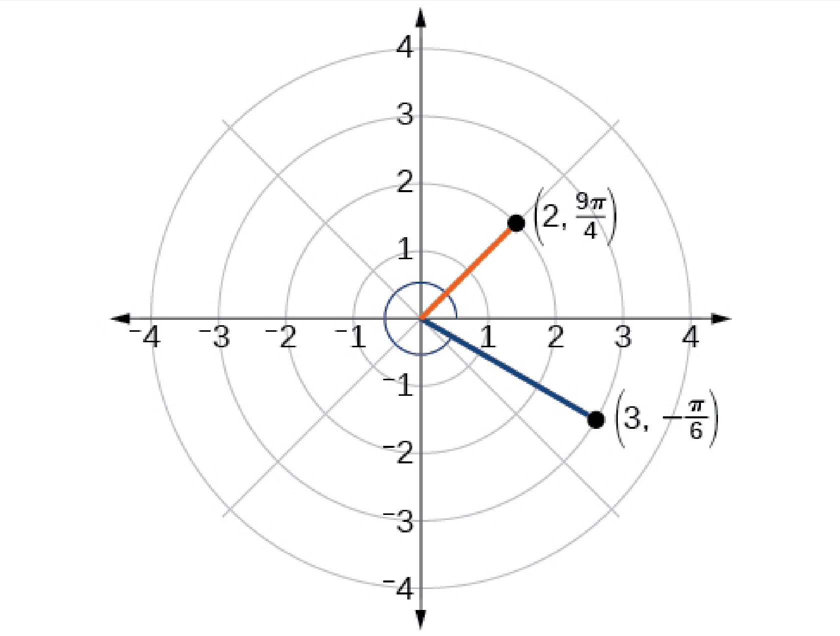

$\left(-2, \frac{\pi}{6}\right)$
$\left(2, \frac{\pi}{6}\right)$
4
3
2
1
−1
−2
−3
−4

(a) (b)

Note:
Exercise:

Problem: Plot the points $\left(3, -\frac{\pi}{6}\right)$ and $\left(2, \frac{9\pi}{4}\right)$ on the same polar grid.

Solution:


$\left(2, \frac{9\pi}{4}\right)$
$\left(3, -\frac{\pi}{6}\right)$
4
3
2
1
−1
−2
−3
−4

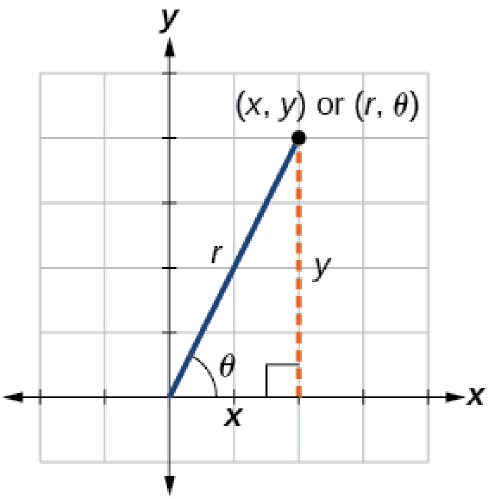
Converting from Polar Coordinates to Rectangular Coordinates

When given a set of polar coordinates, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables $x, \ y, \ r$, and θ.

Equation:

$$\begin{aligned} \cos \ \theta = \tfrac{x}{r} \rightarrow x = r \cos \ \theta \\ \sin \ \theta = \tfrac{y}{r} \rightarrow y = r \sin \ \theta \end{aligned}$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in [link]. An easy way to remember the equations above is to think of $\cos \ \theta$ as the adjacent side over the hypotenuse and $\sin \ \theta$ as the opposite side over the hypotenuse.



Note:
Converting from Polar Coordinates to Rectangular Coordinates
To convert polar coordinates $(r, \ \theta)$ to rectangular coordinates $(x, \ y)$, let
Equation:

$$\cos\ \theta = \frac{x}{r} \rightarrow x = r\cos\ \theta$$

Equation:

$$\sin\ \theta = \frac{y}{r} \rightarrow y = r\sin\ \theta$$

Note:

Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos\ \theta$ and $y = r\sin\ \theta$.
2. Evaluate $\cos\ \theta$ and $\sin\ \theta$.
3. Multiply $\cos\ \theta$ by r to find the *x*-coordinate of the rectangular form.
4. Multiply $\sin\ \theta$ by r to find the *y*-coordinate of the rectangular form.

Example:

Exercise:

Problem:

Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3, \frac{\pi}{2}\right)$ as rectangular coordinates.

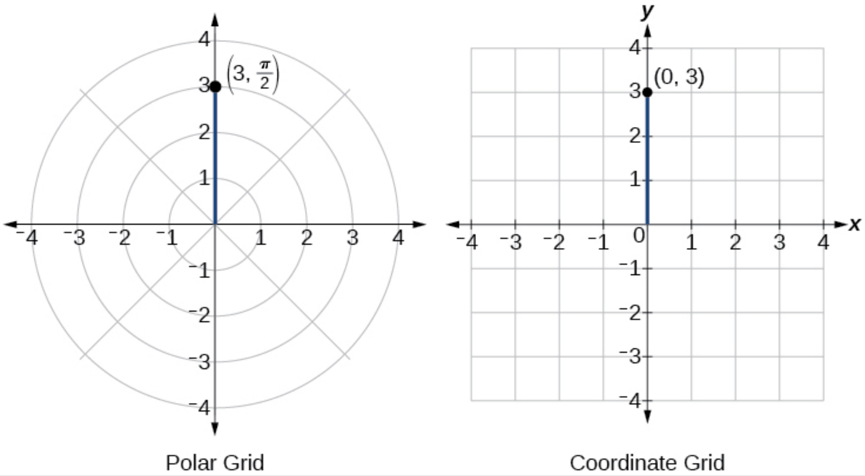
Solution:

Use the equivalent relationships.

Equation:

$$\begin{aligned} x &= r\cos\ \theta \\ x &= 3\cos\ \tfrac{\pi}{2} = 0 \\ y &= r\sin\ \theta \\ y &= 3\sin\ \tfrac{\pi}{2} = 3 \end{aligned}$$

The rectangular coordinates are $(0, 3)$. See [link].


$(3, \frac{\pi}{2})$
(0, 3)
y
x
Polar Grid
Coordinate Grid

Example:
Exercise:

Problem:
Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

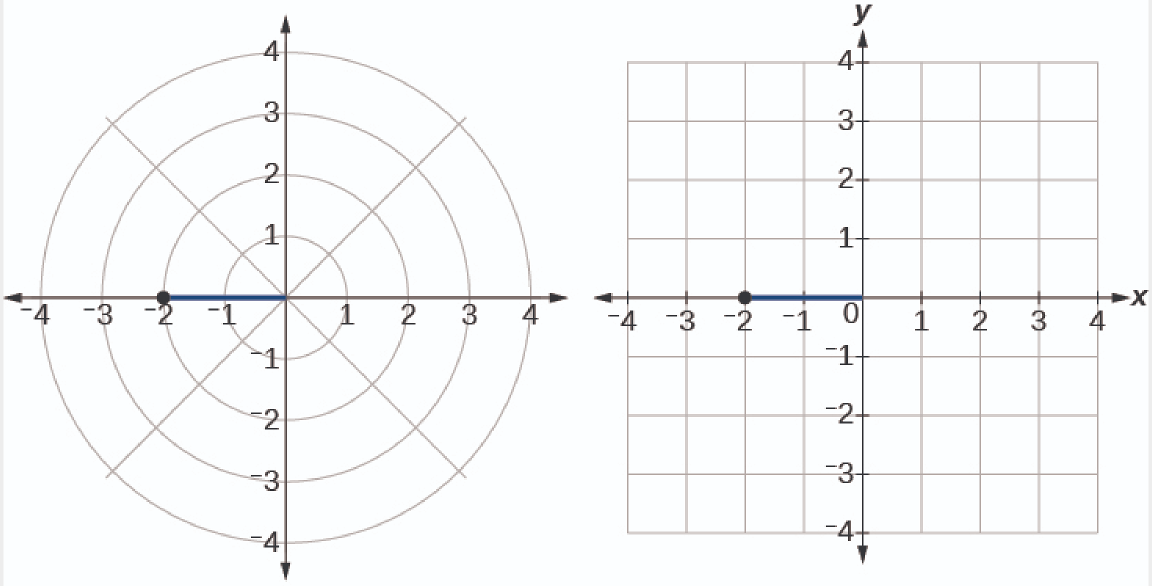
Solution:

See [link]. Writing the polar coordinates as rectangular, we have
Equation:

$$\begin{aligned} x &= r\cos\ \theta \\ x &= -2\cos(0) = -2 \end{aligned}$$

$$\begin{aligned} y &= r\sin\ \theta \\ y &= -2\sin(0) = 0 \end{aligned}$$

The rectangular coordinates are also $(-2, 0)$.



Note:
Exercise:

Problem: Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Solution:

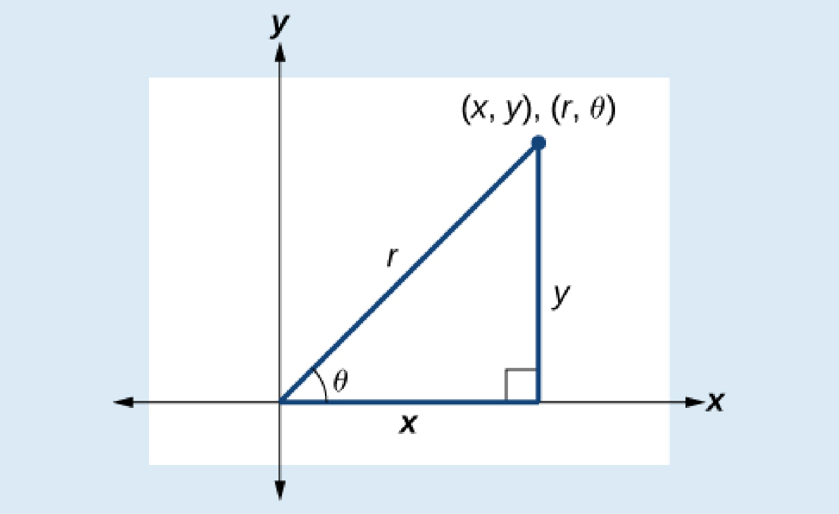
$$(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

Note:
Converting from Rectangular Coordinates to Polar Coordinates
Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in [link].
Equation:

$$
\begin{aligned}
&\cos\ \theta = \frac{x}{r} x = r\cos\ \theta \\
&\sin\ \theta = \frac{y}{r} \quad \text{or} \quad y = r\sin\ \theta \\
&r^2 = x^2 + y^2 \\
&\tan\ \theta = \frac{y}{x}
\end{aligned}
$$



Example:
Exercise:

Problem:
Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Solution:

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\ \theta = \frac{y}{x}$. This gives

Equation:

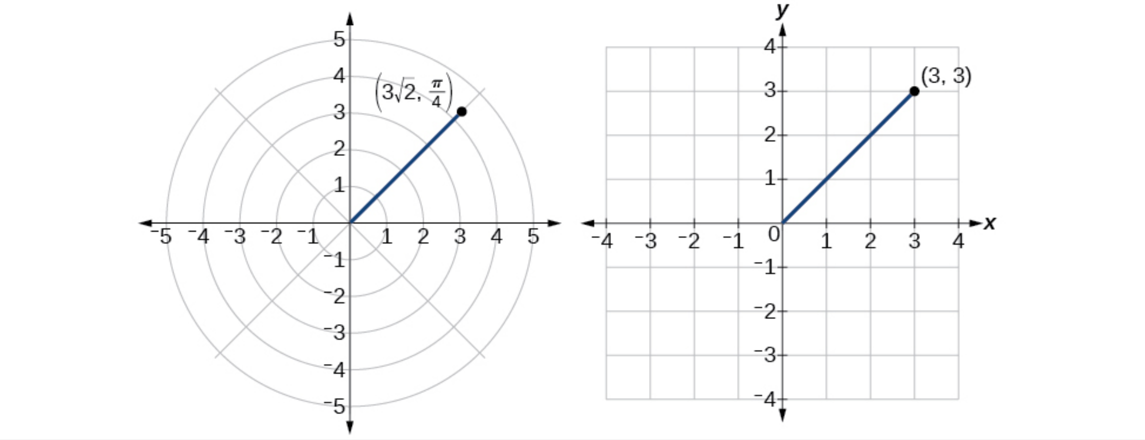
$$
\begin{aligned}
&\tan\ \theta = \frac{3}{3} \\
&\tan\ \theta = 1 \\
&\theta = \tan^{-1}(1) \\
&\theta = \frac{\pi}{4}
\end{aligned}
$$

To find r, we substitute the values for x and y into the formula $r=\sqrt{x^2+y^2}$. We know that r must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

Equation:

$$\begin{aligned} r &= \sqrt{3^2+3^2} \\ r &= \sqrt{9+9} \\ r &= \sqrt{18}=3\sqrt{2} \end{aligned}$$

So, $r=3\sqrt{2}$ and $\theta=\frac{\pi}{4}$, giving us the polar point $\left(3\sqrt{2},\frac{\pi}{4}\right)$. See [link].



Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2},\ \frac{5\pi}{4}\right)$ and $\left(3\sqrt{2},-\frac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2},\ \frac{\pi}{4}\right)$. The point $\left(-3\sqrt{2},\ \frac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\frac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\frac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2},\ \frac{\pi}{4}\right)$. The point $\left(3\sqrt{2},\ -\frac{7\pi}{4}\right)$ is a move further clockwise by $-\frac{7\pi}{4}$, from $\frac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

Note:
Given an equation in polar form, graph it using a graphing calculator.

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: $r_1,\ r_2,\ .\ .\ .\ ,\ r_6.$
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example:
Exercise:

Problem:
Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2 + y^2 = 9$ in polar form.

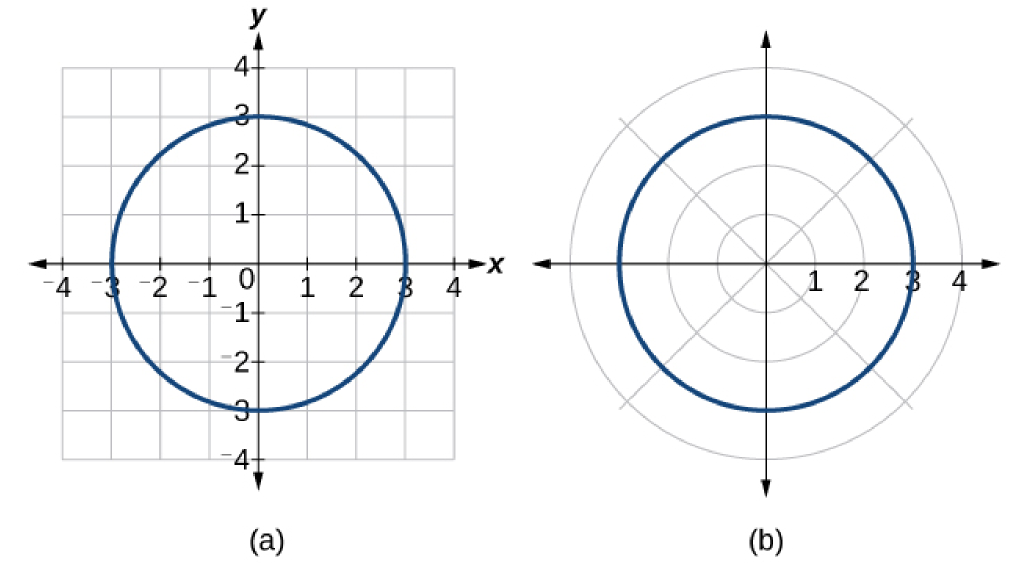
Solution:

The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x, y) and (r, θ). Since $x = r\cos\ \theta$ and $y = r\sin\ \theta$, we can substitute and solve for r.

Equation:

$$\begin{aligned}
(r\cos\ \theta)^2 + (r\sin\ \theta)^2 &= 9 \\
r^2\cos^2\theta + r^2\sin^2\theta &= 9 \\
r^2(\cos^2\theta + \sin^2\theta) &= 9 \\
r^2(1) &= 9 && \text{Substitute } \cos^2\theta + \sin^2\theta = 1. \\
r &= \pm 3 && \text{Use the square root property.}
\end{aligned}$$

Thus, $x^2+y^2=9, r=3$, and $r=-3$ should generate the same graph. See [link].



(a) Cartesian form $x^2+y^2=9$ (b) Polar form $r=3$

To graph a circle in rectangular form, we must first solve for y.

Equation:

$$\begin{aligned} x^2+y^2&=9 \\ y^2&=9-x^2 \\ y&=\pm\sqrt{9-x^2} \end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1=\sqrt{9-x^2}$ and $Y_2=-\sqrt{9-x^2}$. Press **GRAPH.**

Example:

Exercise:

Problem:

Rewriting a Cartesian Equation as a Polar Equation

Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

Solution:

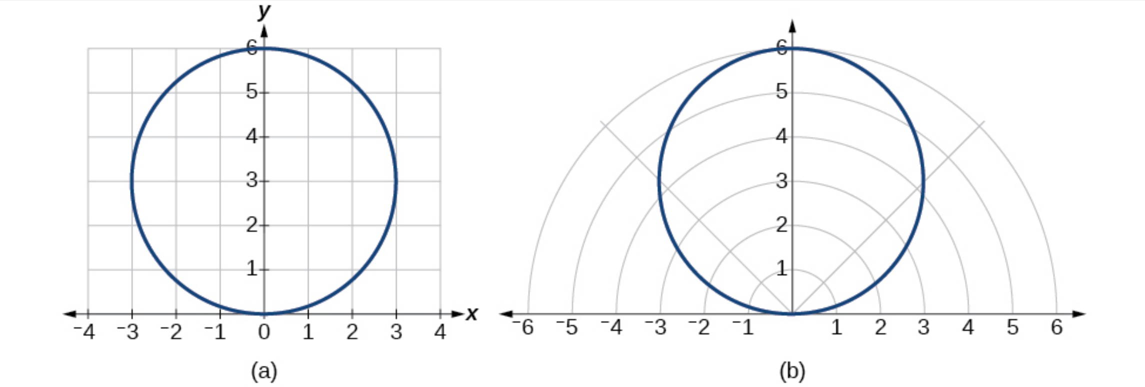
This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

Equation:

$$\begin{array}{ll} r^2 = 6y & \text{Use } x^2 + y^2 = r^2. \\ r^2 = 6r\sin\ \theta & \text{Substitute } y = r\sin\ \theta. \\ r^2 - 6r\sin\ \theta = 0 & \text{Set equal to 0.} \\ r(r - 6\sin\ \theta) = 0 & \text{Factor and solve.} \\ r = 0 & \text{We reject } r = 0, \ \ \text{as it only represents one point, } (0,0). \\ r = 6\sin\ \theta & \end{array}$$

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\ \theta$ should give us the same graph. See [link].



(a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\ \theta$

The Cartesian or rectangular equation is plotted on the rectangular grid, and the polar equation is plotted on the polar grid. Clearly, the graphs are identical.

Example:
Exercise:

Problem:
Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y=3x+2$ as a polar equation.

Solution:

We will use the relationships $x=r\cos\ \theta$ and $y=r\sin\ \theta$.
Equation:

$$\begin{array}{ll} y=3x+2 & \\ r\sin\ \theta=3r\cos\ \theta+2 & \\ r\sin\ \theta-3r\cos\ \theta=2 & \\ r(\sin\ \theta-3\cos\ \theta)=2 & \text{Isolate } r. \\ r=\frac{2}{\sin\ \theta-3\cos\ \theta} & \text{Solve for } r. \end{array}$$

Note:
Exercise:

Problem: Rewrite the Cartesian equation $y^2=3-x^2$ in polar form.

Solution:

$r=\sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example:
Exercise:

Problem:
Graphing a Polar Equation by Converting to a Rectangular Equation

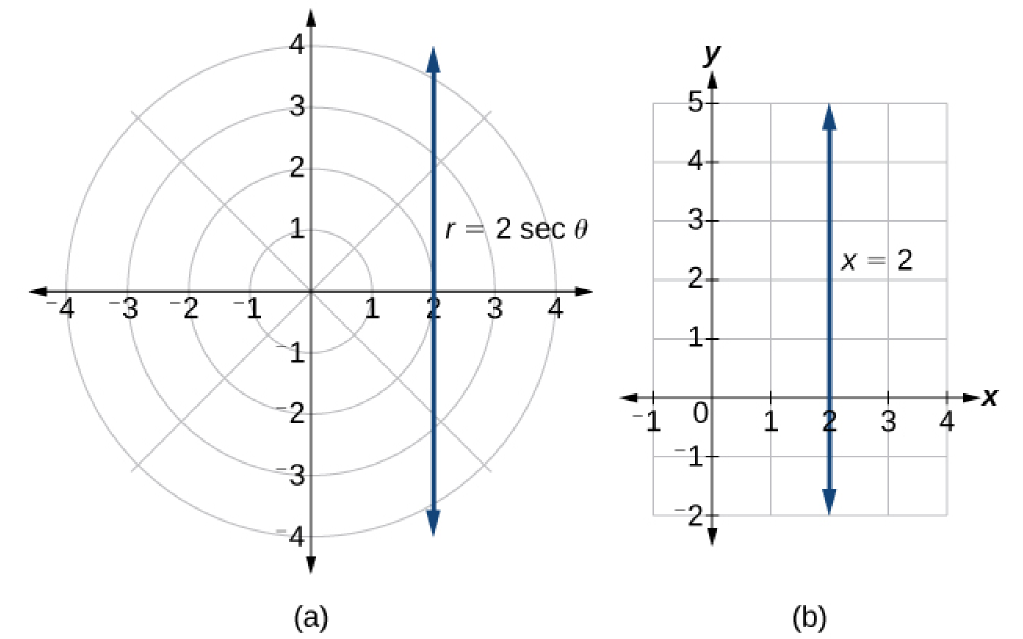
Covert the polar equation $r = 2\sec\ \theta$ to a rectangular equation, and draw its corresponding graph.

Solution:

The conversion is
Equation:

$$\begin{aligned} r &= 2\sec\ \theta \\ r &= \frac{2}{\cos\ \theta} \\ r\cos\ \theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\ \theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see [link]). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\ \theta$ is the standard form for a vertical line in polar form.



(a) Polar grid (b) Rectangular coordinate system

A similar discussion would demonstrate that the graph of the function $r = 2\csc\ \theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\ \theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example:
Exercise:

Problem:
Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \frac{3}{1-2\cos\ \theta}$ as a Cartesian equation.

Solution:

The goal is to eliminate θ and r, and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

Equation:

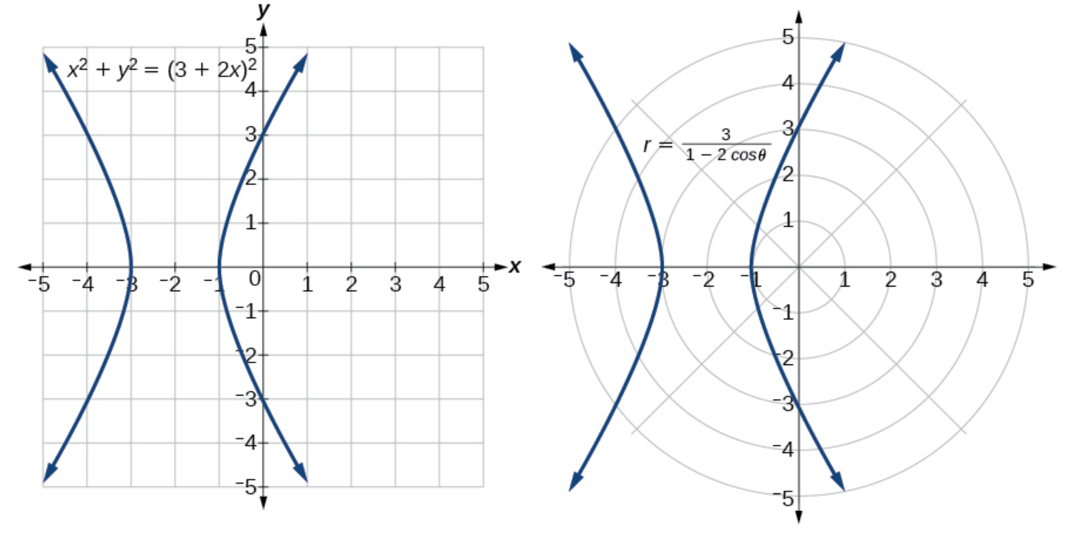
$$\begin{aligned}
r &= \frac{3}{1-2\cos\ \theta} \\
r(1 - 2\cos\ \theta) &= 3 \\
r\left(1 - 2\left(\frac{x}{r}\right)\right) &= 3 && \text{Use } \cos\ \theta = \frac{x}{r} \text{ to eliminate } \theta. \\
r - 2x &= 3 \\
r &= 3 + 2x && \text{Isolate } r. \\
r^2 &= (3 + 2x)^2 && \text{Square both sides.} \\
x^2 + y^2 &= (3 + 2x)^2 && \text{Use } x^2 + y^2 = r^2.
\end{aligned}$$

The Cartesian equation is $x^2 + y^2 = (3 + 2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

Equation:

$$\begin{aligned}
x^2 + y^2 &= (3 + 2x)^2 \\
y^2 &= (3 + 2x)^2 - x^2 \\
y &= \pm\sqrt{(3 + 2x)^2 - x^2}
\end{aligned}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See [link].



Coordinate Grid Polar Grid

The “hour-glass” shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola’s standard form. To do this, we can start with the initial equation.

Equation:

$$\begin{array}{ll}
x^2 + y^2 = (3 + 2x)^2 & \\
x^2 + y^2 - (3 + 2x)^2 = 0 & \\
x^2 + y^2 - (9 + 12x + 4x^2) = 0 & \\
x^2 + y^2 - 9 - 12x - 4x^2 = 0 & \\
-3x^2 - 12x + y^2 = 9 & \text{Multiply through by } -1. \\
3x^2 + 12x - y^2 = -9 & \\
3(x^2 + 4x + \quad) - y^2 = -9 & \text{Organize terms to complete the square for } x. \\
3(x^2 + 4x + 4) - y^2 = -9 + 12 & \\
\qquad 3(x + 2)^2 - y^2 = 3 & \\
(x + 2)^2 - \frac{y^2}{3} = 1 &
\end{array}$$

Note:
Exercise:

Problem: Rewrite the polar equation $r = 2\sin\ \theta$ in Cartesian form.

Solution:

$x^2 + y^2 = 2y$ or, in the standard form for a circle, $x^2 + (y-1)^2 = 1$

Example:
Exercise:

Problem:
Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \sin\left(2\theta\right)$ in Cartesian form.

Solution:
Equation:

$$\begin{array}{rcll}
r & = & \sin(2\theta) & \text{Use the double angle identity for sine.} \\
r & = & 2\sin\ \theta\cos\ \theta & \text{Use } \cos\ \theta = \frac{x}{r} \text{ and } \sin\ \theta = \frac{y}{r}. \\
r & = & 2\left(\frac{x}{r}\right)\left(\frac{y}{r}\right) & \text{Simplify.} \\
r & = & \frac{2xy}{r^2} & \text{Multiply both sides by } r^2. \\
r^3 & = & 2xy & \\
\left(\sqrt{x^2+y^2}\right)^3 & = & 2xy & \text{As } x^2+y^2 = r^2, r = \sqrt{x^2+y^2}.
\end{array}$$

This equation can also be written as
Equation:

$$\left(x^2+y^2\right)^{\frac{3}{2}} = 2xy \text{ or } x^2+y^2 = \left(2xy\right)^{\frac{2}{3}}$$

Note:

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates
- Comparing Polar and Rectangular Coordinates

Key Equations

Conversion formulas	$\cos\ \theta = \frac{x}{r} \rightarrow x = r\cos\ \theta$ $\sin\ \theta = \frac{y}{r} \rightarrow y = r\sin\ \theta$ $r^2 = x^2 + y^2$ $\tan\ \theta = \frac{y}{x}$

Key Concepts

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form $(r,\theta),\ \ \theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See [link].
- If r is negative, extend the directed line segment in the opposite direction of θ. See [link].
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\ \theta$ and $y = r\sin\ \theta$. See [link] and [link].
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\ \theta = \frac{x}{r}$, $\sin\ \theta = \frac{y}{r}$, $\tan\ \theta = \frac{y}{x}$, and $r = \sqrt{x^2 + y^2}$. See [link].
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See [link], [link], and [link].
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See [link], [link],

and [link].

Section Exercises

Verbal

Exercise:

Problem: How are polar coordinates different from rectangular coordinates?

Solution:

For polar coordinates, the point in the plane depends on the angle from the positive *x*-axis and distance from the origin, while in Cartesian coordinates, the point represents the horizontal and vertical distances from the origin. For each point in the coordinate plane, there is one representation, but for each point in the polar plane, there are infinite representations.

Exercise:

Problem:

How are the polar axes different from the *x*- and *y*-axes of the Cartesian plane?

Exercise:

Problem: Explain how polar coordinates are graphed.

Solution:

Determine θ for the point, then move r units from the pole to plot the point. If r is negative, move r units from the pole in the opposite direction but along the same angle. The point is a distance of r away from the origin at an angle of θ from the polar axis.

Exercise:

Problem: How are the points $\left(3,\ \frac{\pi}{2}\right)$ and $\left(-3,\ \frac{\pi}{2}\right)$ related?

Exercise:

Problem: Explain why the points $\left(-3,\ \frac{\pi}{2}\right)$ and $\left(3,\ -\frac{\pi}{2}\right)$ are the same.

Solution:

The point $\left(-3, \frac{\pi}{2}\right)$ has a positive angle but a negative radius and is plotted by moving to an angle of $\frac{\pi}{2}$ and then moving 3 units in the negative direction. This places the point 3 units down the negative *y*-axis. The point $\left(3, -\frac{\pi}{2}\right)$ has a negative angle and a positive radius and is plotted by first moving to an angle of $-\frac{\pi}{2}$ and then moving 3 units down, which is the positive direction for a negative angle. The point is also 3 units down the negative *y*-axis.

Algebraic

For the following exercises, convert the given polar coordinates to Cartesian coordinates with $r > 0$ and $0 \leq \theta \leq 2\pi$. Remember to consider the quadrant in which the given point is located when determining θ for the point.

Exercise:

Problem: $\left(7, \frac{7\pi}{6}\right)$

Exercise:

Problem: $(5, \pi)$

Solution:

$(-5, 0)$

Exercise:

Problem: $\left(6, -\frac{\pi}{4}\right)$

Exercise:

Problem: $\left(-3, \frac{\pi}{6}\right)$

Solution:

$\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

Exercise:

Problem: $\left(4, \frac{7\pi}{4}\right)$

For the following exercises, convert the given Cartesian coordinates to polar coordinates with $r > 0, \ \ 0 \leq \theta < 2\pi$. Remember to consider the quadrant in which the given point is located.

Exercise:

Problem: $(4, 2)$

Solution:

$$\left(2\sqrt{5}, \quad 0.464\right)$$

Exercise:

Problem: $(-4, 6)$

Exercise:

Problem: $(3, -5)$

Solution:

$$\left(\sqrt{34}, 5.253\right)$$

Exercise:

Problem: $(-10, -13)$

Exercise:

Problem: $(8, 8)$

Solution:

$$\left(8\sqrt{2}, \frac{\pi}{4}\right)$$

For the following exercises, convert the given Cartesian equation to a polar equation.

Exercise:

Problem: $x = 3$

Exercise:

Problem: $y = 4$

Solution:

$r = 4\csc\theta$

Exercise:

Problem: $y = 4x^2$

Exercise:

Problem: $y = 2x^4$

Solution:

$r = \sqrt[3]{\frac{sin\theta}{2cos^4\theta}}$

Exercise:

Problem: $x^2 + y^2 = 4y$

Exercise:

Problem: $x^2 + y^2 = 3x$

Solution:

$r = 3\cos\theta$

Exercise:

Problem: $x^2 - y^2 = x$

Exercise:

Problem: $x^2 - y^2 = 3y$

Solution:

$r = \frac{3\sin\theta}{\cos(2\theta)}$

Exercise:

Exercise:

Problem: $x^2 + y^2 = 9$

Exercise:

Problem: $x^2 = 9y$

Solution:

$r = \frac{9 \sin \theta}{\cos^2 \theta}$

Exercise:

Problem: $y^2 = 9x$

Exercise:

Problem: $9xy = 1$

Solution:

$r = \sqrt{\frac{1}{9 \cos \theta \sin \theta}}$

For the following exercises, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

Exercise:

Problem: $r = 3 \sin \ \theta$

Exercise:

Problem: $r = 4 \cos \ \theta$

Solution:

$x^2 + y^2 = 4x$ or $\frac{(x-2)^2}{4} + \frac{y^2}{4} = 1$; circle

Exercise:

Problem: $r = \frac{4}{\sin \ \theta + 7 \cos \ \theta}$

Exercise:

Problem: $r = \frac{6}{\cos\ \theta + 3\sin\ \theta}$

Solution:

$3y + x = 6$; line

Exercise:

Problem: $r = 2\sec\ \theta$

Exercise:

Problem: $r = 3\csc\ \theta$

Solution:

$y = 3$; line

Exercise:

Problem: $r = \sqrt{r\cos\ \theta + 2}$

Exercise:

Problem: $r^2 = 4\sec\ \theta\ \csc\ \theta$

Solution:

$xy = 4$; hyperbola

Exercise:

Problem: $r = 4$

Exercise:

Problem: $r^2 = 4$

Solution:

$x^2 + y^2 = 4$; circle

Exercise:

Problem: $r = \frac{1}{4\cos\ \theta - 3\sin\ \theta}$

Exercise:

Problem: $r = \frac{3}{\cos\ \theta - 5\sin\ \theta}$

Solution:

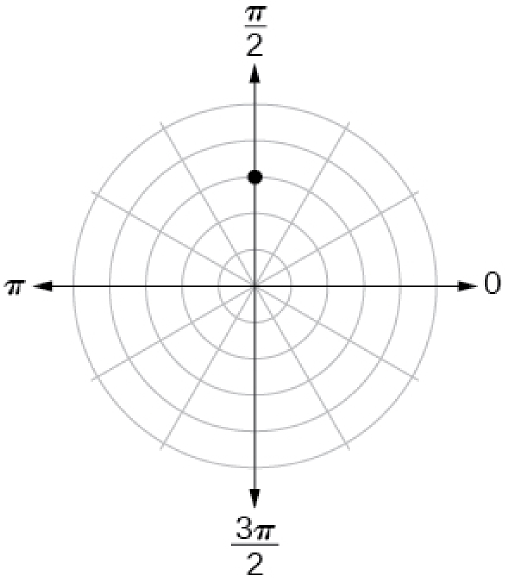
$x - 5y = 3$; line

Graphical

For the following exercises, find the polar coordinates of the point.

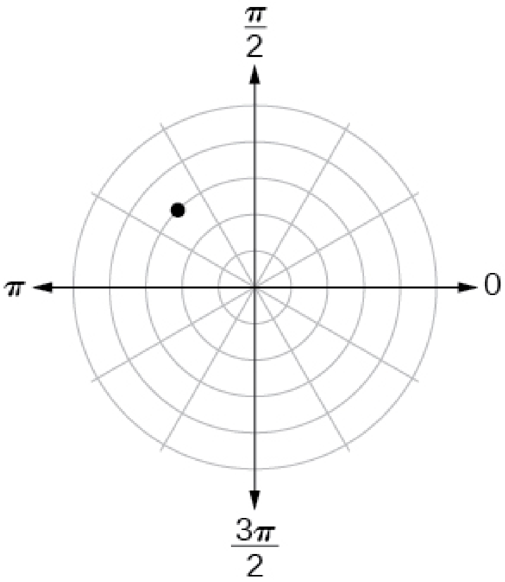
Exercise:

Problem:



Exercise:

Problem:



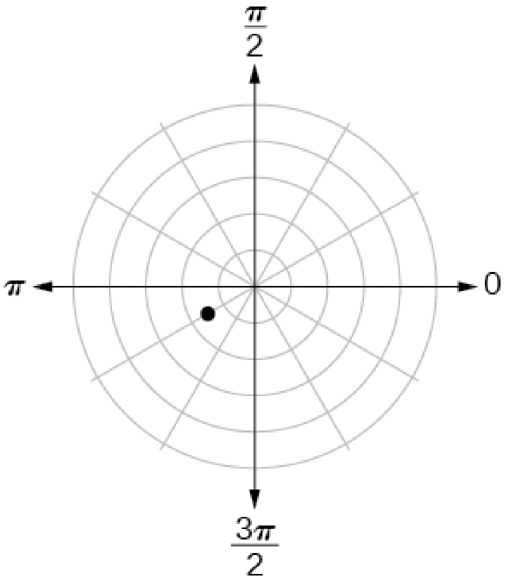
$\frac{\pi}{2}$
π
0
$\frac{3\pi}{2}$

Solution:

$\left(3, \frac{3\pi}{4}\right)$

Exercise:

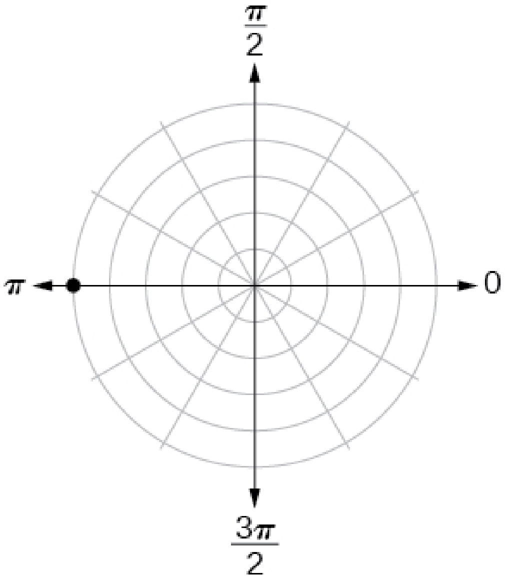
Problem:



$\frac{\pi}{2}$
π
0
$\frac{3\pi}{2}$

Exercise:

Problem:

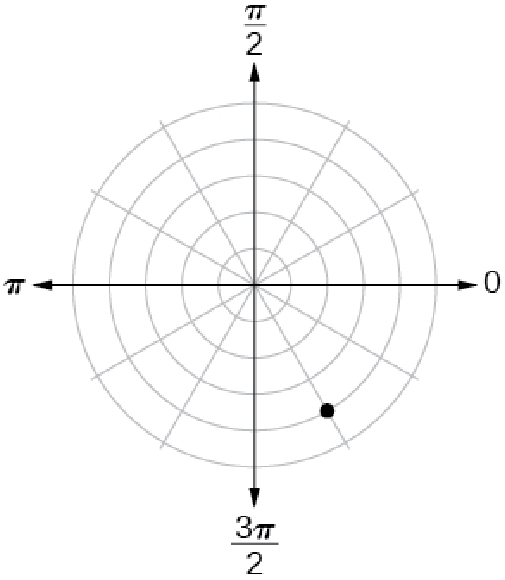

$\frac{\pi}{2}$
π
0
$\frac{3\pi}{2}$

Solution:

$(5, \pi)$

Exercise:

Problem:

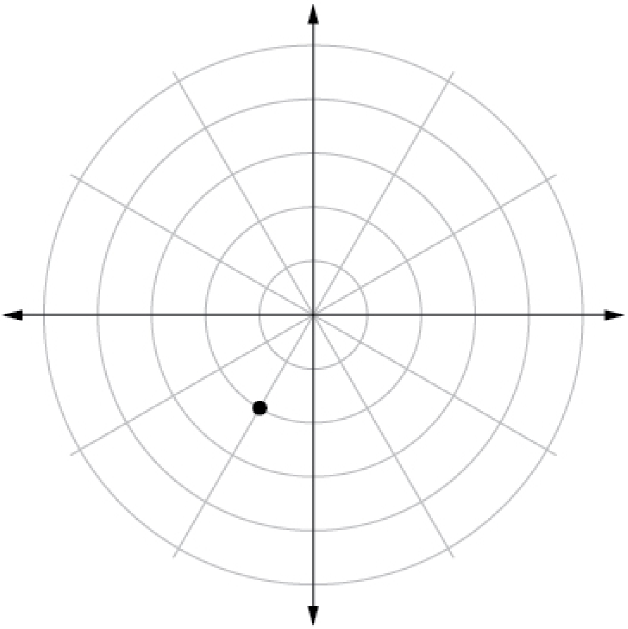

$\frac{\pi}{2}$
π
0
$\frac{3\pi}{2}$

For the following exercises, plot the points.

Exercise:

Problem: $\left(-2, \frac{\pi}{3}\right)$

Solution:



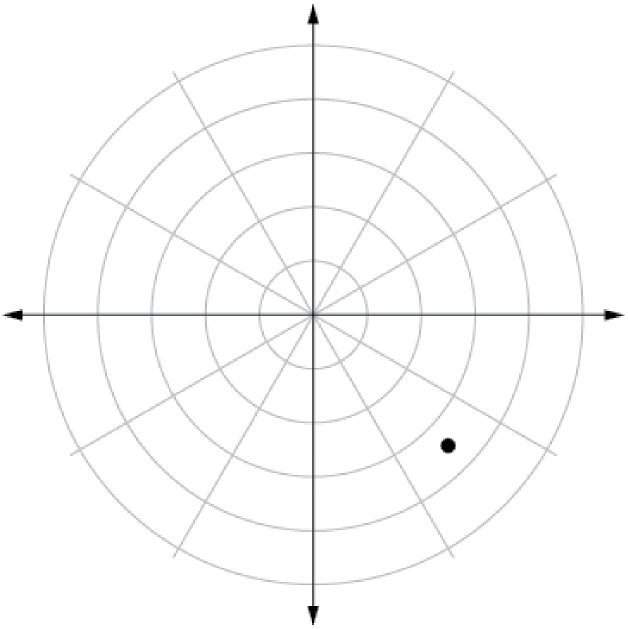
Exercise:

Problem: $\left(-1, -\frac{\pi}{2}\right)$

Exercise:

Problem: $\left(3.5, \frac{7\pi}{4}\right)$

Solution:



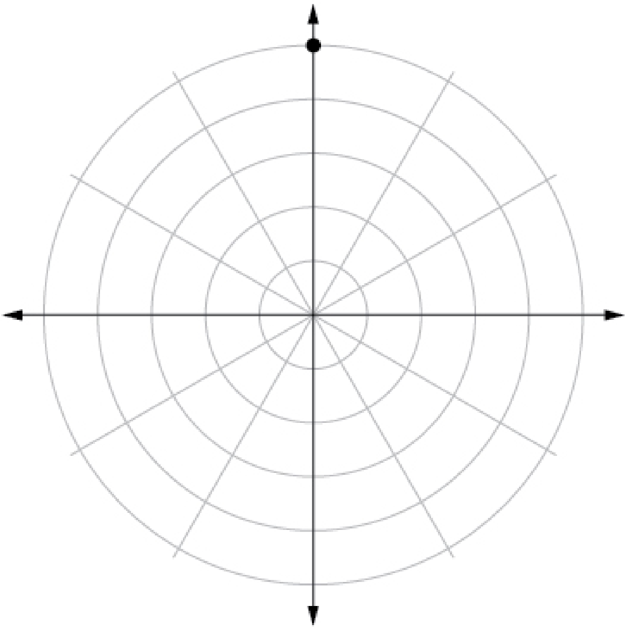
Exercise:

Problem: $\left(-4, \frac{\pi}{3}\right)$

Exercise:

Problem: $\left(5, \frac{\pi}{2}\right)$

Solution:



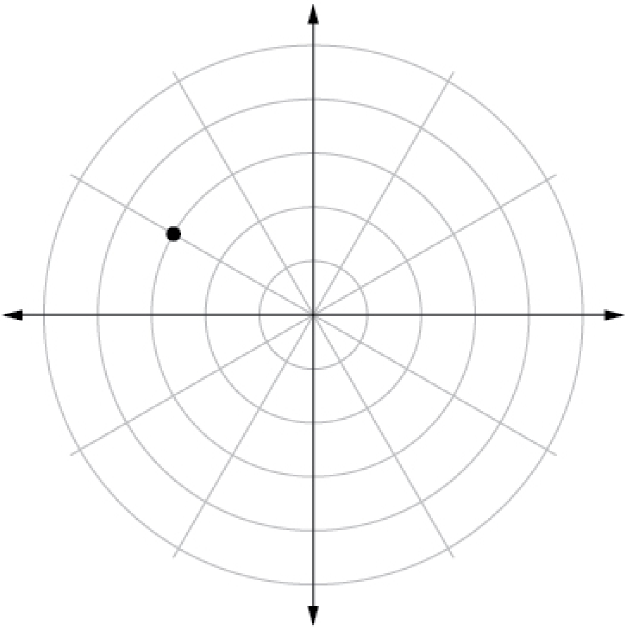
Exercise:

Problem: $\left(4, \frac{-5\pi}{4}\right)$

Exercise:

Problem: $\left(3, \frac{5\pi}{6}\right)$

Solution:



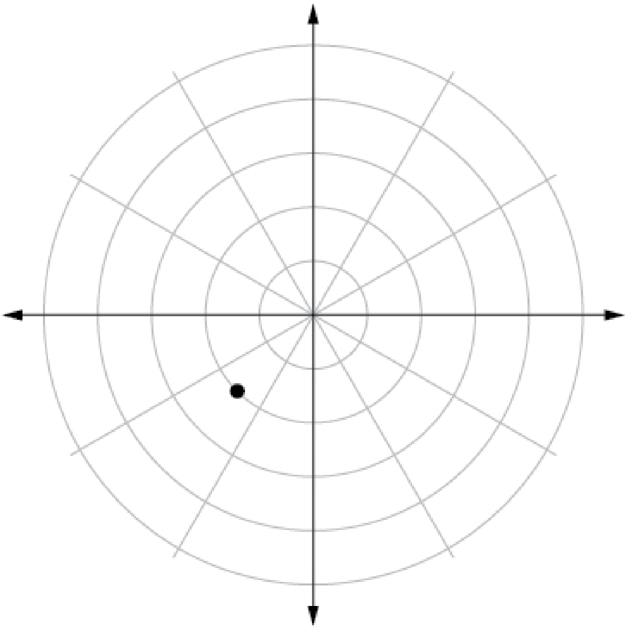
Exercise:

Problem: $\left(-1.5, \frac{7\pi}{6}\right)$

Exercise:

Problem: $\left(-2, \frac{\pi}{4}\right)$

Solution:



Exercise:

Problem: $\left(1, \frac{3\pi}{2}\right)$

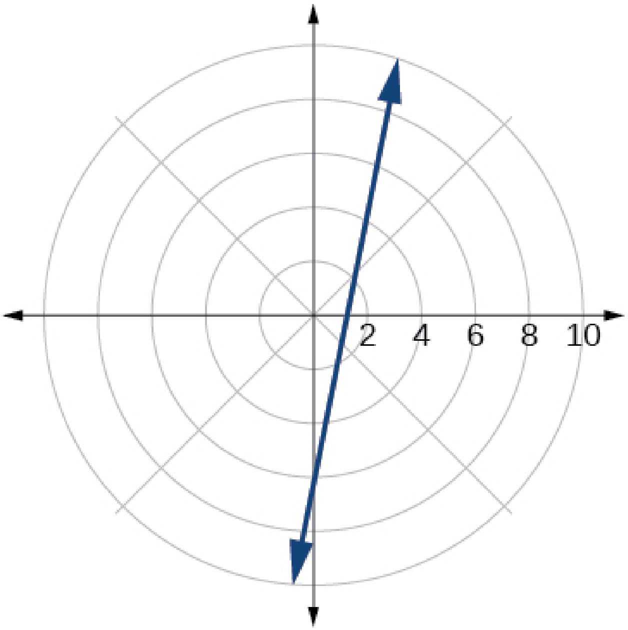
For the following exercises, convert the equation from rectangular to polar form and graph on the polar axis.

Exercise:

Problem: $5x - y = 6$

Solution:

$r = \frac{6}{5\cos\theta - \sin\theta}$



Exercise:

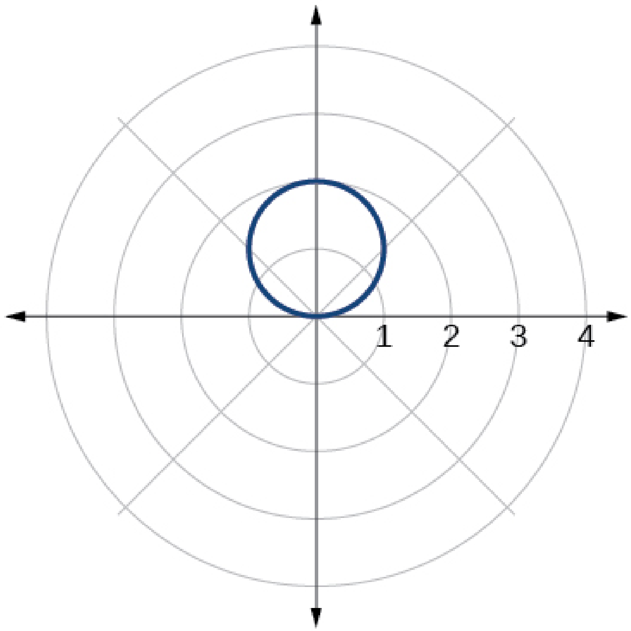
Problem: $2x+7y=-3$

Exercise:

Problem: $x^2+(y-1)^2=1$

Solution:

$r=2\sin\theta$



Exercise:

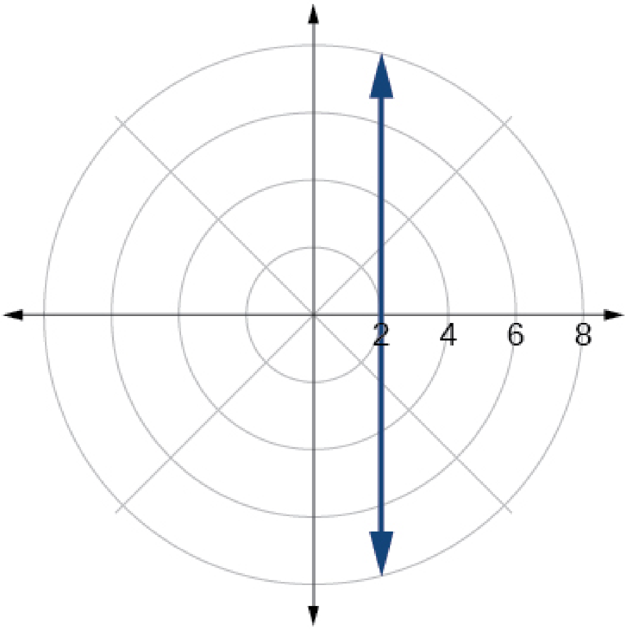
Problem: $(x+2)^2+(y+3)^2=13$

Exercise:

Problem: $x=2$

Solution:

$r=\frac{2}{\cos\theta}$



Exercise:

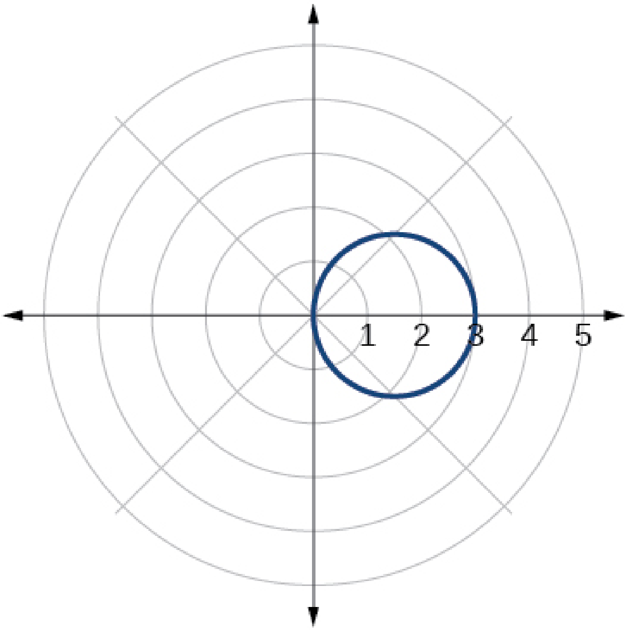
Problem: $x^2 + y^2 = 5y$

Exercise:

Problem: $x^2 + y^2 = 3x$

Solution:

$r = 3\cos\theta$



For the following exercises, convert the equation from polar to rectangular form and graph on the rectangular plane.

Exercise:

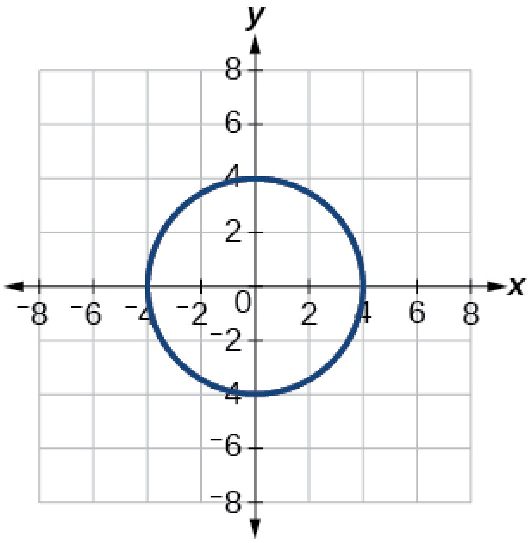
Problem: $r = 6$

Exercise:

Problem: $r = -4$

Solution:

$x^2 + y^2 = 16$



Exercise:

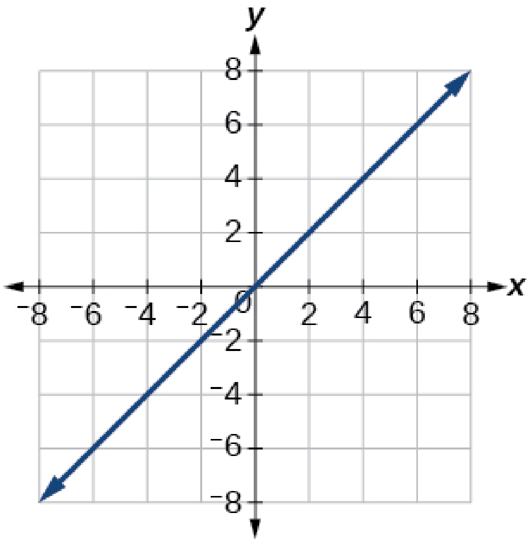
Problem: $\theta = -\frac{2\pi}{3}$

Exercise:

Problem: $\theta = \frac{\pi}{4}$

Solution:

$y = x$



Exercise:

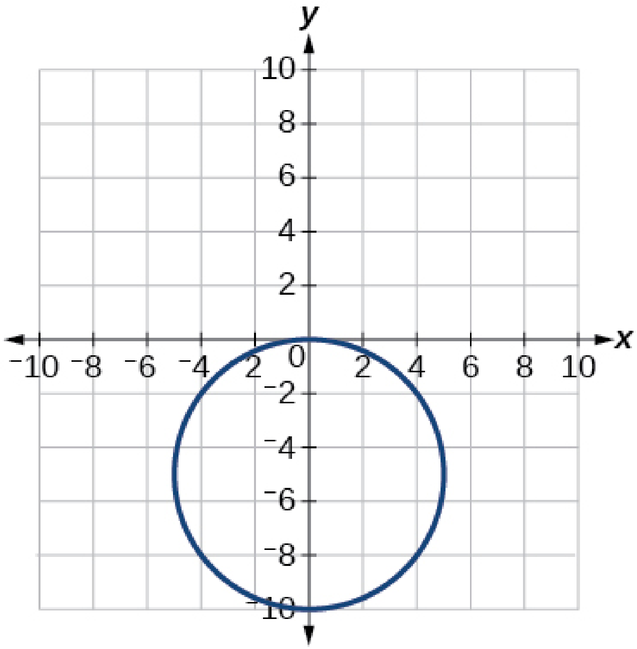
Problem: $r = \sec\ \theta$

Exercise:

Problem: $r = -10\sin\ \theta$

Solution:

$x^2 + (y+5)^2 = 25$



Exercise:

Problem: $r = 3\cos\ \theta$

Technology

Exercise:

Problem:

Use a graphing calculator to find the rectangular coordinates of $\left(2, -\frac{\pi}{5}\right)$. Round to the nearest thousandth.

Solution:

$(1.618, -1.176)$

Exercise:

Problem:

Use a graphing calculator to find the rectangular coordinates of $\left(-3, \frac{3\pi}{7}\right)$. Round to the nearest thousandth.

Exercise:

Problem:

Use a graphing calculator to find the polar coordinates of $(-7, 8)$ in degrees. Round to the nearest thousandth.

Solution:

$(10.630, 131.186^\circ)$

Exercise:

Problem:

Use a graphing calculator to find the polar coordinates of $(3, -4)$ in degrees. Round to the nearest hundredth.

Exercise:

Problem:

Use a graphing calculator to find the polar coordinates of $(-2, 0)$ in radians. Round to the nearest hundredth.

Solution:

$(2, 3.14) or\ (2, \pi)$

Extensions

Exercise:

Problem: Describe the graph of $r = a\sec\ \theta; a > 0$.

Exercise:

Problem: Describe the graph of $r = a\sec\ \theta; a < 0$.

Solution:

A vertical line with a units left of the *y*-axis.

Exercise:

Problem: Describe the graph of $r = a\csc\ \theta; a > 0$.

Exercise:

Problem: Describe the graph of $r = a\csc\ \theta; a < 0$.

Solution:

A horizontal line with a units below the *x*-axis.

Exercise:

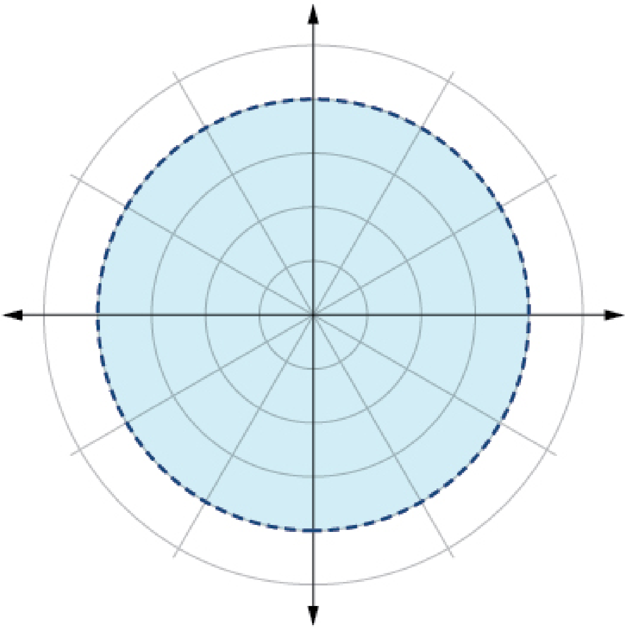
Problem: What polar equations will give an oblique line?

For the following exercise, graph the polar inequality.

Exercise:

Problem: $r < 4$

Solution:



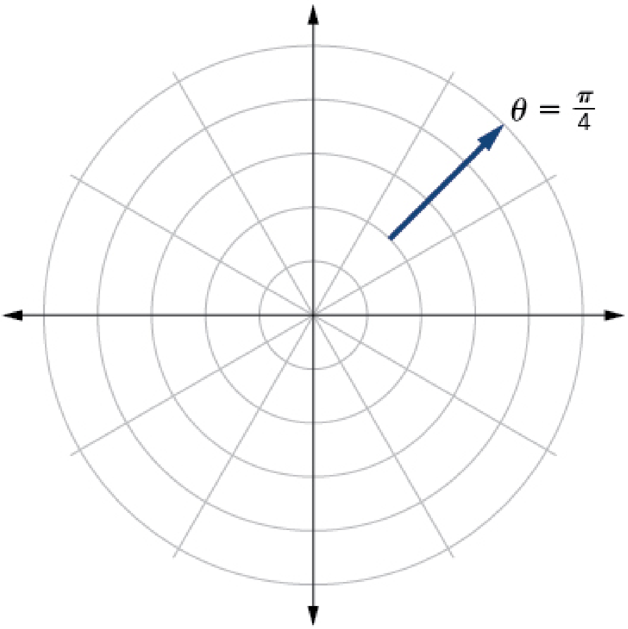
Exercise:

Problem: $0 \leq \theta \leq \frac{\pi}{4}$

Exercise:

Problem: $\theta = \frac{\pi}{4},\ \ r\ \geq\ 2$

Solution:


$\theta = \frac{\pi}{4}$

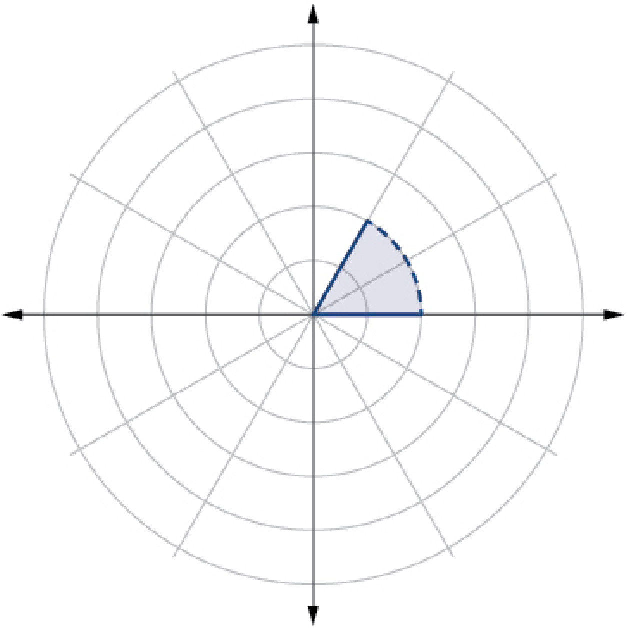
Exercise:

Problem: $\theta = \frac{\pi}{4}, \;\; r \;\geq -3$

Exercise:

Problem: $0 \leq \theta \leq \frac{\pi}{3}, \;\; r \;<\; 2$

Solution:



Exercise:

Problem: $\frac{-\pi}{6} < \theta \leq \frac{\pi}{3}, -3 < r \ < \ 2$

Glossary

polar axis
: on the polar grid, the equivalent of the positive *x*-axis on the rectangular grid

polar coordinates
: on the polar grid, the coordinates of a point labeled (r, θ), where θ indicates the angle of rotation from the polar axis and r represents the radius, or the distance of the point from the pole in the direction of θ

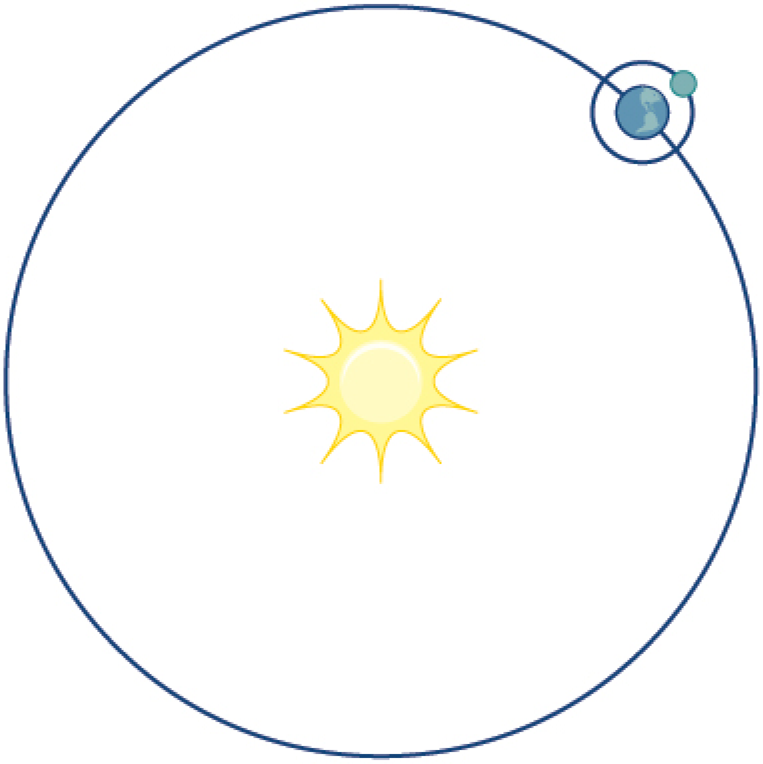
pole
: the origin of the polar grid

Parametric Equations

In this section, you will:

- Parameterize a curve.
- Eliminate the parameter.
- Find a rectangular equation for a curve defined parametrically.
- Find parametric equations for curves defined by rectangular equations.

Consider the path a moon follows as it orbits a planet, which simultaneously rotates around the sun, as seen in [link]. At any moment, the moon is located at a particular spot relative to the planet. But how do we write and solve the equation for the position of the moon when the distance from the planet, the speed of the moon's orbit around the planet, and the speed of rotation around the sun are all unknowns? We can solve only for one variable at a time.



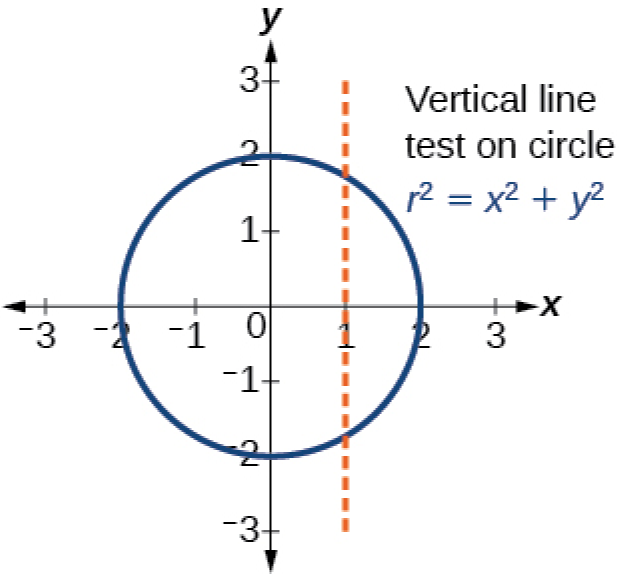
In this section, we will consider sets of equations given by $x\left(t\right)$ and $y\left(t\right)$ where t is the independent variable of time. We can use these parametric equations in a number of applications when we are looking for not only a particular position but also the direction of the movement. As we trace out successive values of t, the orientation of the curve becomes clear. This is one of the primary advantages of using parametric equations: we are able to trace the movement of an object along a path according to time. We begin this section with a look at the basic components of parametric equations and what it means to parameterize a curve. Then we will learn how to eliminate the parameter, translate the equations of a curve defined parametrically into rectangular equations, and find the parametric equations for curves defined by rectangular equations.

Parameterizing a Curve

When an object moves along a curve—or curvilinear path—in a given direction and in a given amount of time, the position of the object in the plane is given by the *x*-coordinate and the *y*-coordinate. However, both x and y vary over time and so are functions of time. For this reason, we add another variable, the **parameter**, upon which both x and y are dependent functions. In the example in the section opener, the parameter is time, t. The x position of the moon at time, t, is represented as the function $x(t)$, and the y position of the moon at time, t, is represented as the function $y(t)$. Together, $x(t)$ and $y(t)$ are called parametric equations, and generate an ordered pair $(x(t),\ y(t))$. Parametric equations primarily describe motion and direction.

When we parameterize a curve, we are translating a single equation in two variables, such as x and y , into an equivalent pair of equations in three variables, x, y, and t. One of the reasons we parameterize a curve is because the parametric equations yield more information: specifically, the direction of the object’s motion over time.

When we graph parametric equations, we can observe the individual behaviors of x and of y. There are a number of shapes that cannot be represented in the form $y = f(x)$, meaning that they are not functions. For example, consider the graph of a circle, given as $r^2 = x^2 + y^2$. Solving for y gives $y = \pm\sqrt{r^2 - x^2}$, or two equations: $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$. If we graph y_1 and y_2 together, the graph will not pass the vertical line test, as shown in [link]. Thus, the equation for the graph of a circle is not a function.



However, if we were to graph each equation on its own, each one would pass the vertical line test and therefore would represent a function. In some instances, the concept of breaking up the equation for a circle into two functions is similar to the concept of creating parametric equations, as we use two functions to produce a non-function. This will become clearer as we move forward.

Note:

Parametric Equations

Suppose t is a number on an interval, I. The set of ordered pairs, $(x(t), y(t))$, where $x = f(t)$ and $y = g(t)$, forms a plane curve based on the parameter t. The equations $x = f(t)$ and $y = g(t)$ are the parametric equations.

Example:

Exercise:

Problem:
Parameterizing a Curve

Parameterize the curve $y = x^2 - 1$ letting $x(t) = t$. Graph both equations.

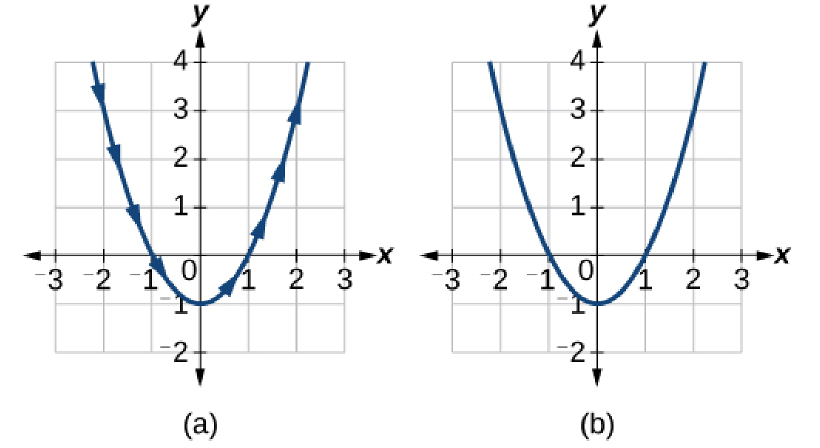
Solution:

If $x\left(t\right) = t$, then to find $y\left(t\right)$ we replace the variable x with the expression given in $x\left(t\right)$. In other words, $y\left(t\right) = t^2 - 1$. Make a table of values similar to [link], and sketch the graph.

t	$x(t)$	$y(t)$
-4	-4	$y\left(-4\right) = {\left(-4\right)}^2 - 1 = 15$
-3	-3	$y\left(-3\right) = {\left(-3\right)}^2 - 1 = 8$
-2	-2	$y\left(-2\right) = {\left(-2\right)}^2 - 1 = 3$
-1	-1	$y\left(-1\right) = {\left(-1\right)}^2 - 1 = 0$
0	0	$y\left(0\right) = {\left(0\right)}^2 - 1 = -1$
1	1	$y\left(1\right) = {\left(1\right)}^2 - 1 = 0$
2	2	$y\left(2\right) = {\left(2\right)}^2 - 1 = 3$
3	3	$y\left(3\right) = {\left(3\right)}^2 - 1 = 8$

t	$x(t)$	$y(t)$
4	4	$y(4) = (4)^2 - 1 = 15$

See the graphs in [link]. It may be helpful to use the **TRACE** feature of a graphing calculator to see how the points are generated as t increases.



(a) Parametric $y(t) = t^2 - 1$ (b) Rectangular $y = x^2 - 1$

Analysis

The arrows indicate the direction in which the curve is generated. Notice the curve is identical to the curve of $y = x^2 - 1$.

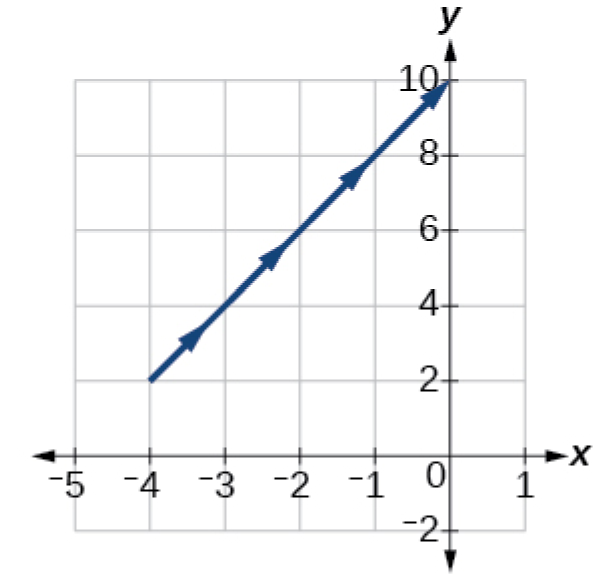
Note:
Exercise:

Problem:

Construct a table of values and plot the parametric equations: $x(t) = t - 3,\, y(t) = 2t + 4; \quad -1 \le t \le 2.$

Solution:

t	$x\left(t\right)$	$y\left(t\right)$
-1	-4	2
0	-3	4
1	-2	6
2	-1	8



Example:
Exercise:

Problem:
Finding a Pair of Parametric Equations

Find a pair of parametric equations that models the graph of $y = 1 - x^2$, using the parameter $x\left(t\right) = t$. Plot some points and sketch the graph.

Solution:

If $x(t) = t$ and we substitute t for x into the y equation, then $y\left(t\right) = 1 - t^2$. Our pair of parametric equations is
Equation:

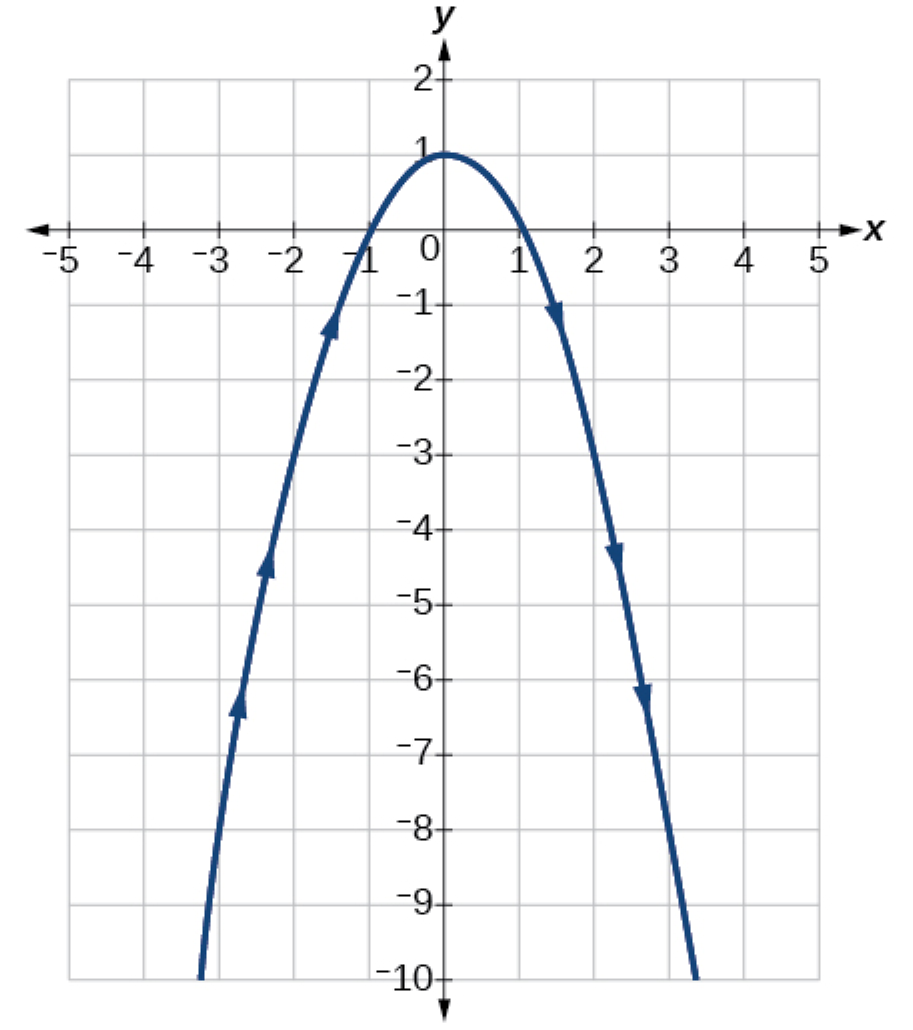
$$\begin{aligned} x(t) &= t \\ y(t) &= 1 - t^2 \end{aligned}$$

To graph the equations, first we construct a table of values like that in [link]. We can choose values around $t = 0$, from $t = -3$ to $t = 3$. The values in the $x(t)$ column will be the same as those in the t column because $x(t) = t$. Calculate values for the column $y(t)$.

t	$x(t) = t$	$y(t) = 1 - t^2$
-3	-3	$y\left(-3\right) = 1 - \left(-3\right)^2 = -8$
-2	-2	$y\left(-2\right) = 1 - \left(-2\right)^2 = -3$
-1	-1	$y\left(-1\right) = 1 - \left(-1\right)^2 = 0$
0	0	$y(0) = 1 - 0 = 1$

t	$x(t)=t$	$y(t)=1-t^2$
1	1	$y(1)=1-(1)^2=0$
2	2	$y(2)=1-(2)^2=-3$
3	3	$y(3)=1-(3)^2=-8$

The graph of $y=1-t^2$ is a parabola facing downward, as shown in [link]. We have mapped the curve over the interval $[-3,\ 3]$, shown as a solid line with arrows indicating the orientation of the curve according to t. Orientation refers to the path traced along the curve in terms of increasing values of t. As this parabola is symmetric with respect to the line $x=0$, the values of x are reflected across the y-axis.



Note:
Exercise:

Problem: Parameterize the curve given by $x = y^3 - 2y$.

Solution:

$x(t) = t^3 - 2t$
$y(t) = t$

Example:
Exercise:

Problem:
Finding Parametric Equations That Model Given Criteria

An object travels at a steady rate along a straight path $(-5,\ 3)$ to $(3,\ -1)$ in the same plane in four seconds. The coordinates are measured in meters. Find parametric equations for the position of the object.

Solution:

The parametric equations are simple linear expressions, but we need to view this problem in a step-by-step fashion. The *x*-value of the object starts at -5 meters and goes to 3 meters. This means the distance *x* has changed by 8 meters in 4 seconds, which is a rate of $\frac{8\text{ m}}{4\text{ s}}$, or $2\ \text{m}/\text{s}$. We can write the *x*-coordinate as a linear function with respect to time as $x(t)=2t-5$. In the linear function template $y=mx+b, 2t=mx$ and $-5=b$.

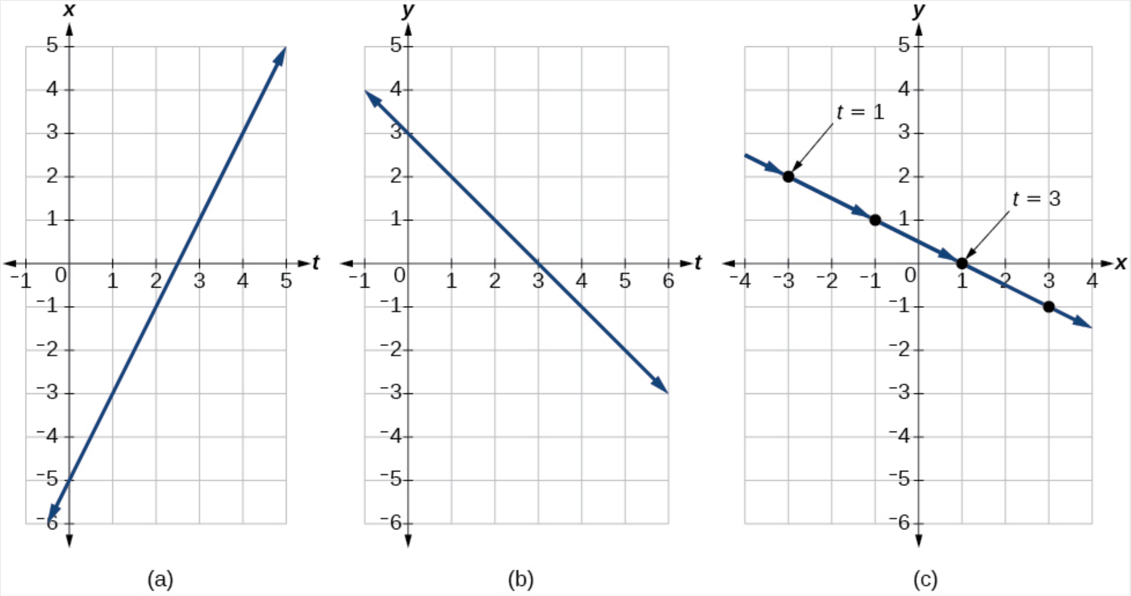
Similarly, the *y*-value of the object starts at 3 and goes to -1, which is a change in the distance *y* of −4 meters in 4 seconds, which is a rate of $\frac{-4\text{ m}}{4\text{ s}}$, or $-1\text{m}/\text{s}$. We can also write the *y*-coordinate as the linear function $y(t)=-t+3$. Together, these are the parametric equations for the position of the object, where x and y are expressed in meters and t represents time:
Equation:

$$\begin{aligned}x(t)&=2t-5\\y(t)&=-t+3\end{aligned}$$

Using these equations, we can build a table of values for t, x, and y (see [link]). In this example, we limited values of t to non-negative numbers. In general, any value of t can be used.

t	$x(t)=2t-5$	$y(t)=-t+3$
0	$x=2(0)-5=-5$	$y=-(0)+3=3$
1	$x=2(1)-5=-3$	$y=-(1)+3=2$
2	$x=2(2)-5=-1$	$y=-(2)+3=1$
3	$x=2(3)-5=1$	$y=-(3)+3=0$
4	$x=2(4)-5=3$	$y=-(4)+3=-1$

From this table, we can create three graphs, as shown in [link].



(a) A graph of x vs. t, representing the horizontal position over time. (b) A graph of y vs. t, representing the vertical position over time. (c) A graph of y vs. x, representing the position of the object in the plane at time t.

Analysis

Again, we see that, in [link](c), when the parameter represents time, we can indicate the movement of the object along the path with arrows.

Eliminating the Parameter

In many cases, we may have a pair of parametric equations but find that it is simpler to draw a curve if the equation involves only two variables, such as x and y. Eliminating the parameter is a method that may make graphing some curves easier. However, if we are concerned with the mapping of the equation according to time, then it will be necessary to indicate the orientation of the curve as well. There are various methods for eliminating the parameter t from a set of parametric equations; not every method works for every type of equation. Here we will review the methods for the most common types of equations.

Eliminating the Parameter from Polynomial, Exponential, and Logarithmic Equations

For polynomial, exponential, or logarithmic equations expressed as two parametric equations, we choose the equation that is most easily manipulated and solve for t. We substitute the resulting expression for t into the second equation. This gives one equation in x and y.

Example:
Exercise:

Problem:
Eliminating the Parameter in Polynomials

Given $x(t) = t^2 + 1$ and $y(t) = 2 + t$, eliminate the parameter, and write the parametric equations as a Cartesian equation.

Solution:

We will begin with the equation for y because the linear equation is easier to solve for t.

Equation:

$$\begin{aligned} y &= 2 + t \\ y - 2 &= t \end{aligned}$$

Next, substitute $y - 2$ for t in $x(t)$.

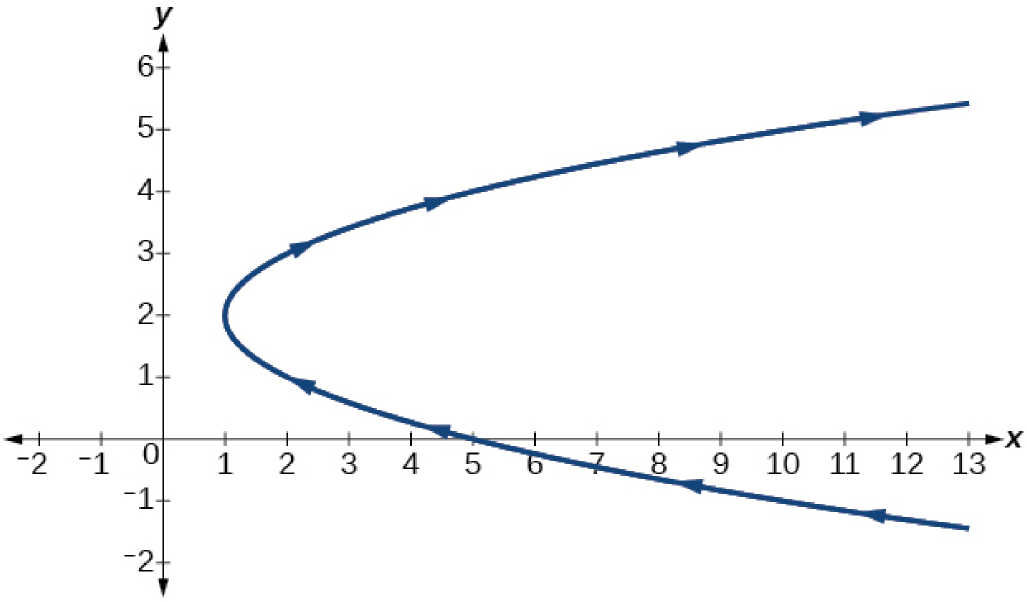
Equation:

$$\begin{array}{ll} x = t^2 + 1 & \\ x = {(y - 2)}^2 + 1 & \text{Substitute the expression for } t \text{ into } x. \\ x = y^2 - 4y + 4 + 1 & \\ x = y^2 - 4y + 5 & \\ x = y^2 - 4y + 5 & \end{array}$$

The Cartesian form is $x = y^2 - 4y + 5$.

Analysis

This is an equation for a parabola in which, in rectangular terms, x is dependent on y. From the curve's vertex at $(1, 2)$, the graph sweeps out to the right. See [link]. In this section, we consider sets of equations given by the functions $x\left(t\right)$ and $y\left(t\right)$, where t is the independent variable of time. Notice, both x and y are functions of time; so in general y is not a function of x.



Note:

Exercise:

Problem:

Given the equations below, eliminate the parameter and write as a rectangular equation for y as a function of x.

Equation:

$$x(t) = 2t^2 + 6$$
$$y(t) = 5 - t$$

Solution:

$y = 5 - \sqrt{\frac{1}{2}x - 3}$

Example:
Exercise:

Problem:
Eliminating the Parameter in Exponential Equations

Eliminate the parameter and write as a Cartesian equation: $x(t)=e^{-t}$ and $y(t)=3e^{t}$, $t>0$.

Solution:

Isolate e^{t}.
Equation:

$$\begin{aligned} x &= e^{-t} \\ e^{t} &= \frac{1}{x} \end{aligned}$$

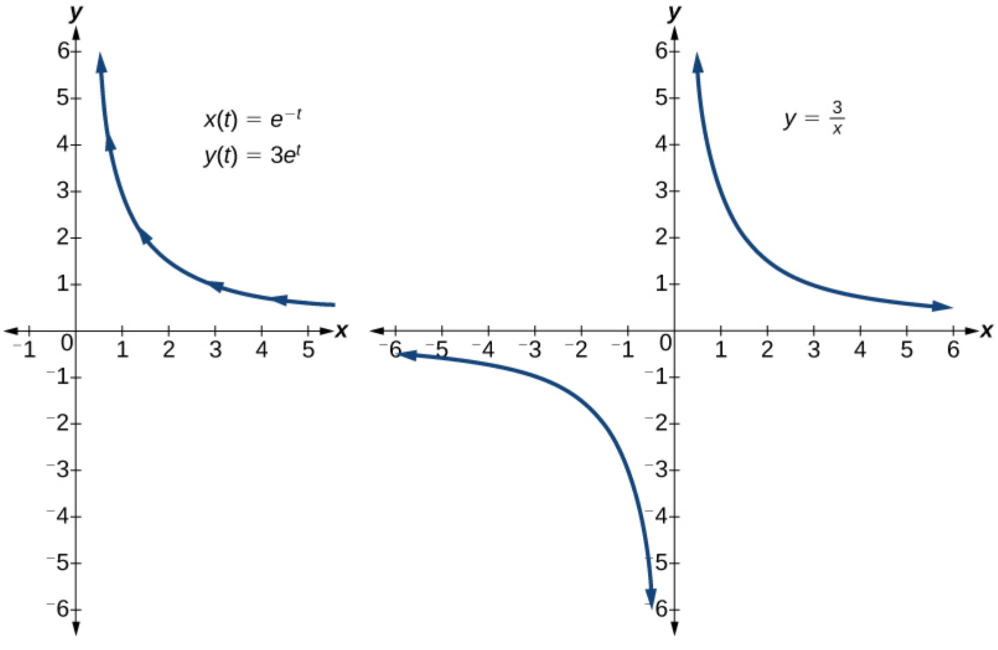
Substitute the expression into $y(t)$.
Equation:

$$\begin{aligned} y &= 3e^{t} \\ y &= 3\left(\frac{1}{x}\right) \\ y &= \frac{3}{x} \end{aligned}$$

The Cartesian form is $y=\frac{3}{x}$.

Analysis

The graph of the parametric equation is shown in [link]**(a)**. The domain is restricted to $t>0$. The Cartesian equation, $y=\frac{3}{x}$ is shown in [link]**(b)** and has only one restriction on the domain, $x\neq 0$.



(a) (b)

Example:
Exercise:

Problem:
Eliminating the Parameter in Logarithmic Equations

Eliminate the parameter and write as a Cartesian equation: $x(t)=\sqrt{t}+2$ and $y(t)=\log(t)$.

Solution:

Solve the first equation for t.
Equation:

$$\begin{aligned} x &= \sqrt{t}+2 \\ x-2 &= \sqrt{t} \\ (x-2)^2 &= t \qquad \text{Square both sides.} \end{aligned}$$

Then, substitute the expression for t into the y equation.
Equation:

$$\begin{aligned} y &= \log\left(t\right) \\ y &= \log\left(x-2\right)^2 \end{aligned}$$

The Cartesian form is $y=\log\left(x-2\right)^2$.

Analysis

To be sure that the parametric equations are equivalent to the Cartesian equation, check the domains. The parametric equations restrict the domain on $x=\sqrt{t}+2$ to $t>0$; we restrict the domain on x to $x>2$. The domain for the parametric equation $y=\log(t)$ is restricted to $t>0$; we limit the domain on $y=\log\left(x-2\right)^2$ to $x>2$.

Note:
Exercise:

Problem: Eliminate the parameter and write as a rectangular equation.
Equation:

$$\begin{aligned} x(t) &= t^2 \\ y(t) &= \ln\ t \quad t>0 \end{aligned}$$

Solution:

$y=\ln\sqrt{x}$

Eliminating the Parameter from Trigonometric Equations

Eliminating the parameter from trigonometric equations is a straightforward substitution. We can use a few of the familiar trigonometric identities and the Pythagorean Theorem.

First, we use the identities:

Equation:

$$x\left(t\right)=a\cos\ t$$
$$y\left(t\right)=b\sin\ t$$

Solving for $\cos\ t$ and $\sin\ t$, we have

Equation:

$$\frac{x}{a}=\cos\ t$$
$$\frac{y}{b}=\sin\ t$$

Then, use the Pythagorean Theorem:

Equation:

$$\cos^2 t+\sin^2 t=1$$

Substituting gives

Equation:

$$\cos^2 t+\sin^2 t=\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

Example:

Exercise:

Problem:

Eliminating the Parameter from a Pair of Trigonometric Parametric Equations

Eliminate the parameter from the given pair of trigonometric equations where $0 \leq t \leq 2\pi$ and sketch the graph.
Equation:

$$\begin{array}{c} x(t) = 4\cos\ t \\ y(t) = 3\sin\ t \end{array}$$

Solution:

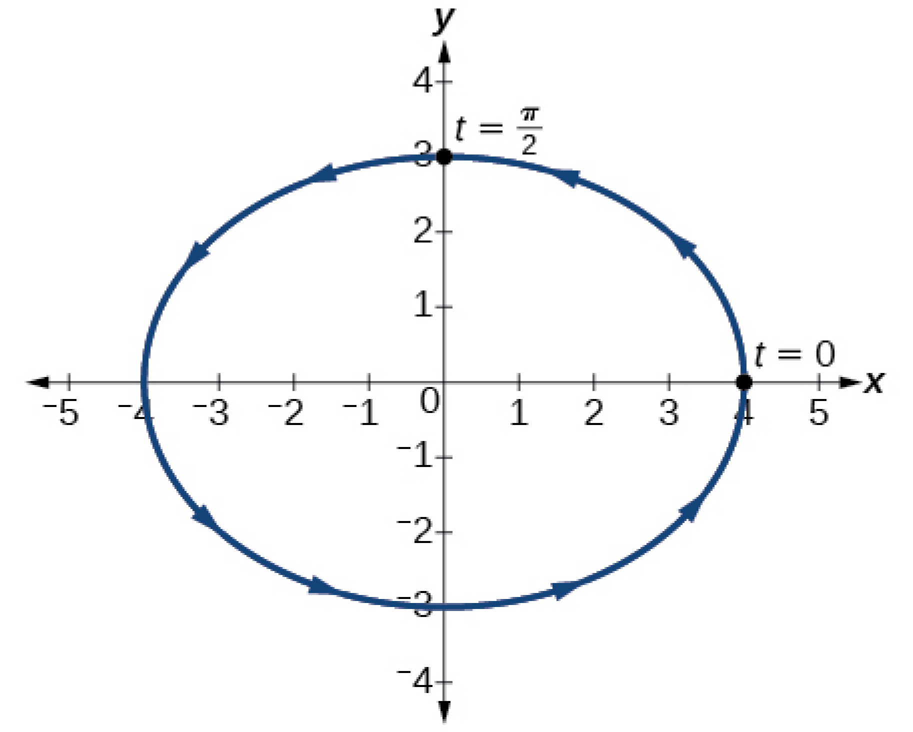
Solving for $\cos\ t$ and $\sin\ t$, we have
Equation:

$$\begin{array}{c} x = 4\cos\ t \\ \frac{x}{4} = \cos\ t \\ y = 3\sin\ t \\ \frac{y}{3} = \sin\ t \end{array}$$

Next, use the Pythagorean identity and make the substitutions.
Equation:

$$\begin{array}{r} \cos^2 t + \sin^2 t = 1 \\ \left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1 \\ \frac{x^2}{16} + \frac{y^2}{9} = 1 \end{array}$$

The graph for the equation is shown in [link].



Analysis

Applying the general equations for conic sections (introduced in Analytic Geometry, we can identify $\frac{x^2}{16}+\frac{y^2}{9}=1$ as an ellipse centered at $(0,0)$. Notice that when $t=0$ the coordinates are $(4,0)$, and when $t=\frac{\pi}{2}$ the coordinates are $(0,3)$. This shows the orientation of the curve with increasing values of t.

Note:
Exercise:

Problem:

Eliminate the parameter from the given pair of parametric equations and write as a Cartesian equation: $x(t)=2\cos\ t$ and $y(t)=3\sin\ t$.

Solution:

$\frac{x^2}{4}+\frac{y^2}{9}=1$

Finding Cartesian Equations from Curves Defined Parametrically

When we are given a set of parametric equations and need to find an equivalent Cartesian equation, we are essentially “eliminating the parameter.” However, there are various methods we can use to rewrite a set of parametric equations as a Cartesian equation. The simplest method is to set one equation equal to the parameter, such as $x\left(t\right)=t$. In this case, $y\left(t\right)$ can be any expression. For example, consider the following pair of equations.

Equation:

$$\begin{gathered} x\left(t\right)=t \\ y\left(t\right)=t^{2}-3 \end{gathered}$$

Rewriting this set of parametric equations is a matter of substituting x for t. Thus, the Cartesian equation is $y=x^{2}-3$.

Example:
Exercise:

Problem:
Finding a Cartesian Equation Using Alternate Methods

Use two different methods to find the Cartesian equation equivalent to the given set of parametric equations.

Equation:

$$\begin{gathered} x(t)=3t-2 \\ y(t)=t+1 \end{gathered}$$

Solution:

Method 1. First, let's solve the x equation for t. Then we can substitute the result into the y equation.

Equation:

$$\begin{aligned} x &= 3t - 2 \\ x + 2 &= 3t \\ \tfrac{x+2}{3} &= t \end{aligned}$$

Now substitute the expression for t into the y equation.

Equation:

$$\begin{aligned} y &= t + 1 \\ y &= \left(\tfrac{x+2}{3}\right) + 1 \\ y &= \tfrac{x}{3} + \tfrac{2}{3} + 1 \\ y &= \tfrac{1}{3}x + \tfrac{5}{3} \end{aligned}$$

Method 2. Solve the y equation for t and substitute this expression in the x equation.

Equation:

$$\begin{aligned} y &= t + 1 \\ y - 1 &= t \end{aligned}$$

Make the substitution and then solve for y.

Equation:

$$\begin{aligned}x &= 3(y-1)-2\\ x &= 3y-3-2\\ x &= 3y-5\\ x+5 &= 3y\\ \tfrac{x+5}{3} &= y\\ y &= \tfrac{1}{3}x+\tfrac{5}{3}\end{aligned}$$

Note:
Exercise:

Problem:

Write the given parametric equations as a Cartesian equation: $x(t)=t^3$ and $y(t)=t^6$.

Solution:

$y=x^2$

Finding Parametric Equations for Curves Defined by Rectangular Equations

Although we have just shown that there is only one way to interpret a set of parametric equations as a rectangular equation, there are multiple ways to interpret a rectangular equation as a set of parametric equations. Any strategy we may use to find the parametric equations is valid if it produces equivalency. In other words, if we choose an expression to represent x, and then substitute it into the y equation, and it produces the same graph over the same domain as the rectangular equation, then the set of parametric equations is valid. If the domain becomes restricted in the set of parametric equations, and the function does not allow the same values for x as the domain of the rectangular equation, then the graphs will be different.

Example:
Exercise:

Problem:
Finding a Set of Parametric Equations for Curves Defined by Rectangular Equations

Find a set of equivalent parametric equations for $y=(x+3)^2+1$.

Solution:

An obvious choice would be to let $x(t)=t$. Then $y(t)=(t+3)^2+1$. But let’s try something more interesting. What if we let $x=t+3$? Then we have

Equation:

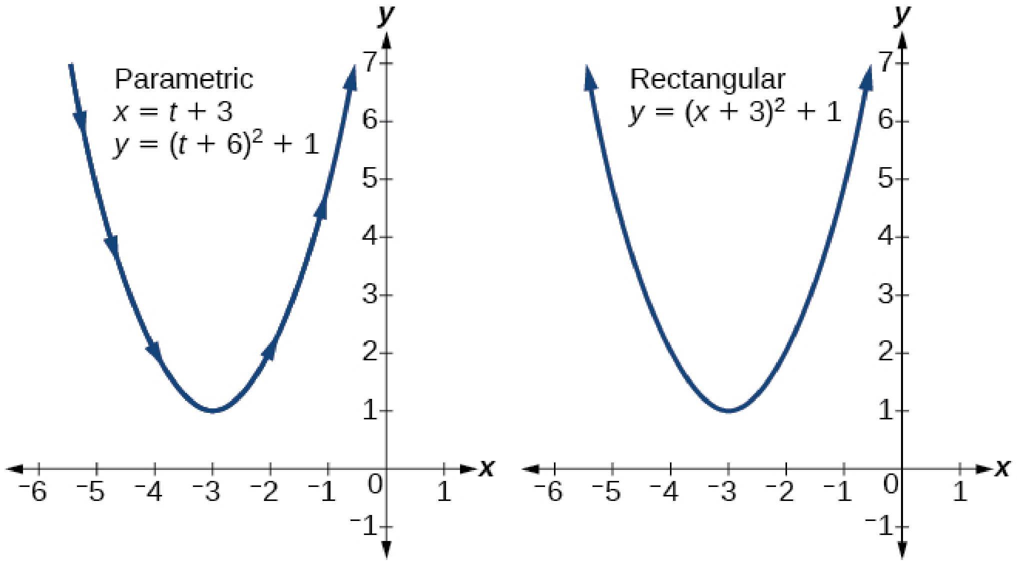
$$\begin{aligned} y &= (x+3)^2+1 \\ y &= ((t+3)+3)^2+1 \\ y &= (t+6)^2+1 \end{aligned}$$

The set of parametric equations is

Equation:

$$\begin{aligned} x(t) &= t+3 \\ y(t) &= (t+6)^2+1 \end{aligned}$$

See [link].



(a) (b)

Note:
Access these online resources for additional instruction and practice with parametric equations.

- Introduction to Parametric Equations
- Converting Parametric Equations to Rectangular Form

Key Concepts

- Parameterizing a curve involves translating a rectangular equation in two variables, x and y, into two equations in three variables, x, y, and t. Often, more information is obtained from a set of parametric equations. See [link], [link], and [link].
- Sometimes equations are simpler to graph when written in rectangular form. By eliminating t, an equation in x and y is the result.
- To eliminate t, solve one of the equations for t, and substitute the expression into the second equation. See [link], [link], [link], and [link].

- Finding the rectangular equation for a curve defined parametrically is basically the same as eliminating the parameter. Solve for t in one of the equations, and substitute the expression into the second equation. See [link].
- There are an infinite number of ways to choose a set of parametric equations for a curve defined as a rectangular equation.
- Find an expression for x such that the domain of the set of parametric equations remains the same as the original rectangular equation. See [link].

Section Exercises

Verbal

Exercise:

Problem: What is a system of parametric equations?

Solution:

A pair of functions that is dependent on an external factor. The two functions are written in terms of the same parameter. For example, $x = f(t)$ and $y = f(t)$.

Exercise:

Problem:

Some examples of a third parameter are time, length, speed, and scale. Explain when time is used as a parameter.

Exercise:

Problem:

Explain how to eliminate a parameter given a set of parametric equations.

Solution:

Choose one equation to solve for t, substitute into the other equation and simplify.

Exercise:

Problem:

What is a benefit of writing a system of parametric equations as a Cartesian equation?

Exercise:

Problem: What is a benefit of using parametric equations?

Solution:

Some equations cannot be written as functions, like a circle. However, when written as two parametric equations, separately the equations are functions.

Exercise:

Problem:

Why are there many sets of parametric equations to represent on Cartesian function?

Algebraic

For the following exercises, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

Exercise:

Problem: $\begin{cases} x(t) = 5 - t \\ y(t) = 8 - 2t \end{cases}$

Solution:

$y = -2 + 2x$

Exercise:

Problem: $\begin{cases} x\left(t\right) = 6 - 3t \\ y\left(t\right) = 10 - t \end{cases}$

Exercise:

Problem: $\begin{cases} x\left(t\right) = 2t + 1 \\ y\left(t\right) = 3\sqrt{t} \end{cases}$

Solution:

$y = 3\sqrt{\frac{x-1}{2}}$

Exercise:

Problem: $\begin{cases} x\left(t\right) = 3t - 1 \\ y\left(t\right) = 2t^2 \end{cases}$

Exercise:

Problem: $\begin{cases} x\left(t\right) = 2e^t \\ y\left(t\right) = 1 - 5t \end{cases}$

Solution:

$x = 2e^{\frac{1-y}{5}}$ or $y = 1 - 5ln\left(\frac{x}{2}\right)$

Exercise:

Problem: $\begin{cases} x\left(t\right) = e^{-2t} \\ y\left(t\right) = 2e^{-t} \end{cases}$

Exercise:

Problem: $\begin{cases} x(t) = 4\log(t) \\ y(t) = 3 + 2t \end{cases}$

Solution:

$$x = 4\log\left(\frac{y-3}{2}\right)$$

Exercise:

Problem: $\begin{cases} x(t) = \log(2t) \\ y(t) = \sqrt{t-1} \end{cases}$

Exercise:

Problem: $\begin{cases} x\left(t\right) = t^3 - t \\ y\left(t\right) = 2t \end{cases}$

Solution:

$$x = \left(\frac{y}{2}\right)^3 - \frac{y}{2}$$

Exercise:

Problem: $\begin{cases} x\left(t\right) = t - t^4 \\ y\left(t\right) = t + 2 \end{cases}$

Exercise:

Problem: $\begin{cases} x\left(t\right) = e^{2t} \\ y\left(t\right) = e^{6t} \end{cases}$

Solution:

$$y = x^3$$

Exercise:

Problem: $\begin{cases} x\left(t\right)=t^5 \\ y\left(t\right)=t^{10} \end{cases}$

Exercise:

Problem: $\begin{cases} x(t)=4\text{cos }t \\ y(t)=5\sin\ t \end{cases}$

Solution:

$\left(\frac{x}{4}\right)^2+\left(\frac{y}{5}\right)^2=1$

Exercise:

Problem: $\begin{cases} x\left(t\right)=3\sin\ t \\ y\left(t\right)=6\cos\ t \end{cases}$

Exercise:

Problem: $\begin{cases} x(t)=2\text{cos}^2 t \\ y(t)=-\sin\ t \end{cases}$

Solution:

$y^2=1-\frac{1}{2}x$

Exercise:

Problem: $\begin{cases} x(t)=\cos\ t+4 \\ y(t)=2\sin^2 t \end{cases}$

Exercise:

Problem: $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

Solution:

$y = x^2 + 2x + 1$

Exercise:

Problem: $\begin{cases} x(t) = -t \\ y(t) = t^3 + 1 \end{cases}$

Exercise:

Problem: $\begin{cases} x(t) = 2t - 1 \\ y(t) = t^3 - 2 \end{cases}$

Solution:

$y = \left(\frac{x+1}{2}\right)^3 - 2$

For the following exercises, rewrite the parametric equation as a Cartesian equation by building an x-y table.

Exercise:

Problem: $\begin{cases} x(t) = 2t - 1 \\ y(t) = t + 4 \end{cases}$

Exercise:

Problem: $\begin{cases} x(t) = 4 - t \\ y(t) = 3t + 2 \end{cases}$

Solution:

$y = -3x + 14$

Exercise:

Problem: $\begin{cases} x(t) = 2t - 1 \\ y(t) = 5t \end{cases}$

Exercise:

Problem: $\begin{cases} x(t) = 4t - 1 \\ y(t) = 4t + 2 \end{cases}$

Solution:

$y = x + 3$

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by setting $x\left(t\right) = t$ or by setting $y(t) = t$.

Exercise:

Problem: $y\left(x\right) = 3x^2 + 3$

Exercise:

Problem: $y\left(x\right) = 2\sin\ x + 1$

Solution:

$\begin{cases} x\left(t\right) = t \\ y\left(t\right) = 2\sin t + 1 \end{cases}$

Exercise:

Problem: $x\left(y\right) = 3\log\left(y\right) + y$

Exercise:

Problem: $x\left(y\right)=\sqrt{y}+2y$

Solution:

$$\begin{cases}x\left(t\right)=\sqrt{t}+2t\\ y\left(t\right)=t\end{cases}$$

For the following exercises, parameterize (write parametric equations for) each Cartesian equation by using $x\left(t\right)=a\cos\ t$ and $y(t)=b\sin\ t$. Identify the curve.

Exercise:

Problem: $\frac{x^2}{4}+\frac{y^2}{9}=1$

Exercise:

Problem: $\frac{x^2}{16}+\frac{y^2}{36}=1$

Solution:

$$\begin{cases}x\left(t\right)=4\cos\ t\\ y\left(t\right)=6\sin\ t\end{cases}; \text{Ellipse}$$

Exercise:

Problem: $x^2+y^2=16$

Exercise:

Problem: $x^2+y^2=10$

Solution:

$$\begin{cases} x(t) = \sqrt{10}\cos t \\ y(t) = \sqrt{10}\sin t \end{cases}; \text{ Circle}$$

Exercise:

Problem:

Parameterize the line from $(3,0)$ to $(-2,-5)$ so that the line is at $(3,0)$ at $t=0$, and at $(-2,-5)$ at $t=1$.

Exercise:

Problem:

Parameterize the line from $(-1,0)$ to $(3,-2)$ so that the line is at $(-1,0)$ at $t=0$, and at $(3,-2)$ at $t=1$.

Solution:

$$\begin{cases} x(t) = -1+4t \\ y(t) = -2t \end{cases}$$

Exercise:

Problem:

Parameterize the line from $(-1,5)$ to $(2,3)$ so that the line is at $(-1,5)$ at $t=0$, and at $(2,3)$ at $t=1$.

Exercise:

Problem:

Parameterize the line from $(4,1)$ to $(6,-2)$ so that the line is at $(4,1)$ at $t=0$, and at $(6,-2)$ at $t=1$.

Solution:

$$\begin{cases} x(t) = 4+2t \\ y(t) = 1-3t \end{cases}$$

Technology

For the following exercises, use the table feature in the graphing calculator to determine whether the graphs intersect.

Exercise:

Problem: $\begin{cases} x_1(t) = 3t \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = t + 3 \\ y_2(t) = 4t - 4 \end{cases}$

Exercise:

Problem: $\begin{cases} x_1(t) = t^2 \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = -t + 6 \\ y_2(t) = t + 1 \end{cases}$

Solution:

yes, at $t = 2$

For the following exercises, use a graphing calculator to complete the table of values for each set of parametric equations.

Exercise:

Problem: $\begin{cases} x_1\left(t\right) = 3t^2 - 3t + 7 \\ y_1\left(t\right) = 2t + 3 \end{cases}$

t	x	y
–1		
0		

t	x	y
1		

Exercise:

Problem: $\begin{cases} x_1\left(t\right) = t^2 - 4 \\ y_1\left(t\right) = 2t^2 - 1 \end{cases}$

t	x	y
1		
2		
3		

Solution:

t	x	y
1	-3	1

t	x	y
2	0	7
3	5	17

Exercise:

Problem: $\begin{cases} x_1(t) = t^4 \\ y_1(t) = t^3 + 4 \end{cases}$

t	x	y
-1		
0		
1		
2		

Extensions

Exercise:

Problem:

Find two different sets of parametric equations for $y=(x+1)^2$.

Solution:

answers may vary: $\begin{cases} x(t)=t-1 \\ y(t)=t^2 \end{cases}$ and $\begin{cases} x(t)=t+1 \\ y(t)=(t+2)^2 \end{cases}$

Exercise:

Problem:

Find two different sets of parametric equations for $y=3x-2$.

Exercise:

Problem:

Find two different sets of parametric equations for $y=x^2-4x+4$.

Solution:

answers may vary: , $\begin{cases} x(t)=t \\ y(t)=t^2-4t+4 \end{cases}$ and $\begin{cases} x(t)=t+2 \\ y(t)=t^2 \end{cases}$

Glossary

parameter	a variable, often representing time, upon which x and y are both dependent

Systems of Linear Equations: Two Variables

In this section, you will:

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

LEARNING OBJECTIVES

- Determine whether an ordered pair is a solution of a system of equations (IA 4.1.1)
- Solve a system of linear equations by graphing (IA 4.1.2)

Determine whether an ordered pair is a solution of a system of equations (IA 4.1.1)

A **system** of linear equations is a group of two or more linear equations. For example,
Equation:

$$\begin{cases} y = -2x + 5 \\ y = 2x + 7 \end{cases}$$

is a system of linear equations

A **solution** to a system of linear equations is an ordered pair x,y that is a solution to every equation in the system.

Example:
Determine whether the ordered pairs are solutions to the given system.
Exercise:

Problem: $\begin{cases} 2x - 6y = 0 \\ 3x - y = 5 \end{cases}$ at $(3, 1)$ and $(-3, 4)$

Solution:

We substitute (3, 1) into both equations:

$2x - 6y = 0$	$3x - 4y = 5$
$2(3) - 6(1) = 0$ $6 - 6 = 0$ True $0 = 0$	$3(3) - 4(1) = 5$ $9 - 4 = 5$ True $5 = 5$
$(3, 1)$ is a solution to $2x - 6y = 0$	$(3, 1)$ is a solution to $3x - 4y = 5$
Conclusion: since $(3, 1)$ is a solution to both equations, then it is a solution to the system $\begin{cases} 2x - 6y = 0 \\ 3\text{x} - 4y = 5 \end{cases}$	

Next we substitute (-3, -1) into both equations:

$2x-6y=0$	$3x-4y=5$
$2(-3)-6(-1)=0$ $-6+6=0$ $0=0$ True	$3(-3)-4(-1)=5$ $-9+4=5$ $-5=5$ False
$(-3,-1)$ is a solution to $2x-6y=0$	$(-3,-1)$ is **not** a solution to $3x-4y=5$
Conclusion: Since $(-3,-1)$ is not a solution to one of the equations, then it is not a solution to the system $\begin{cases}2x-6y=0\\3x-4y=5\end{cases}$	

Practice Makes Perfect

Determine whether the ordered pairs are solutions to the given system.

$\begin{cases}3x+y=0\\x+2y=-5\end{cases}$ at $(0,0)$ and $(1,\ -3)$

Exercise:

Problem: At (0, 0):

$3x+y=0$	$x+2y=-5$
______________________	______________________

______________________________	______________________________
______________________________ ______________________________	______________________________ ______________________________
Conclusion: ______________________________	

Exercise:

Problem: At (1, –3)

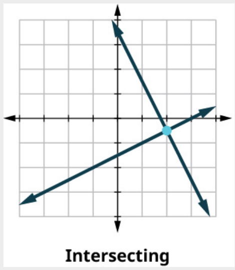
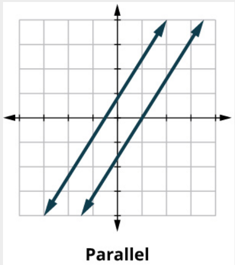
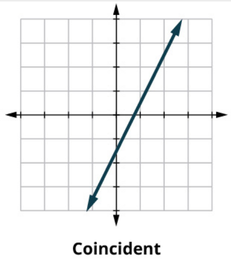
$3x + y = 0$	$x + 2y = -5$
______________________________ ______________________________	______________________________ ______________________________
______________________________ ______________________________	______________________________ ______________________________
Conclusion: ______________________________	

Solve a system of linear equations by graphing (IA 4.1.2)

Note:
Solve a system of linear equations by graphing

Graph the first equation.
Graph the second equation on the same rectangular coordinate

system.
Determine whether the lines intersect, are parallel, or are the same line.
Identify the solution to the system.
Check the solution in both equations.

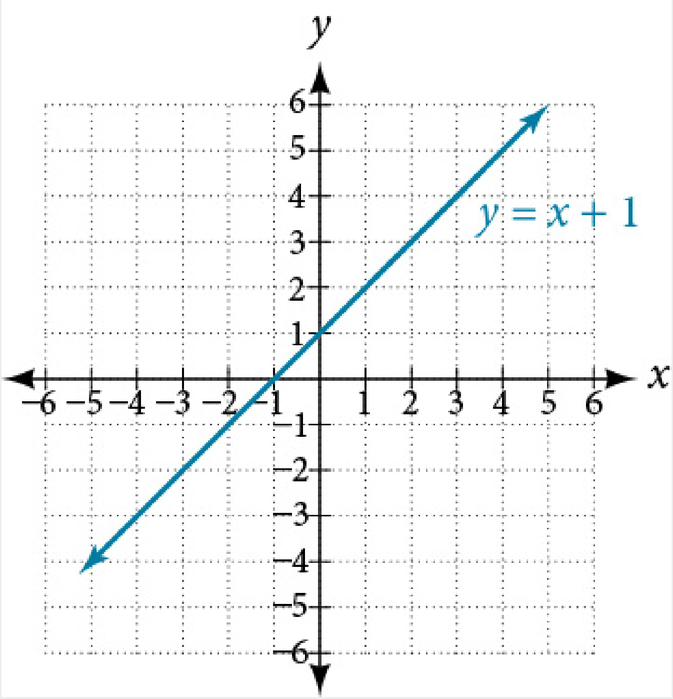
If the lines intersect, identify the point of intersection. This is the solution to the system.	 Intersecting
If the lines are parallel, the system has no solutions.	 Parallel
If the lines are the same, the system has an infinite number of solutions.	 Coincident

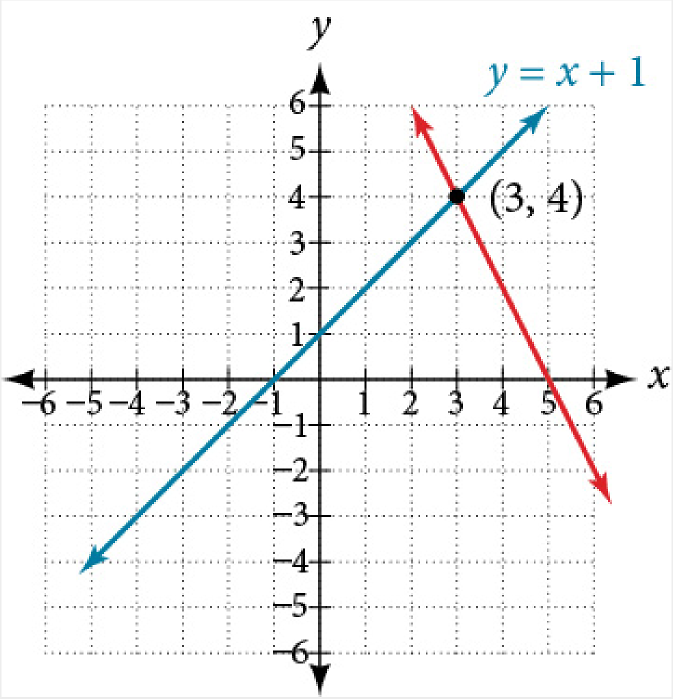
Example:
Solve the system by graphing.
Exercise:

Problem:
Solve the system by graphing.

$$\begin{cases} -x + y = 1 \\ 2x + y = 10 \end{cases}$$

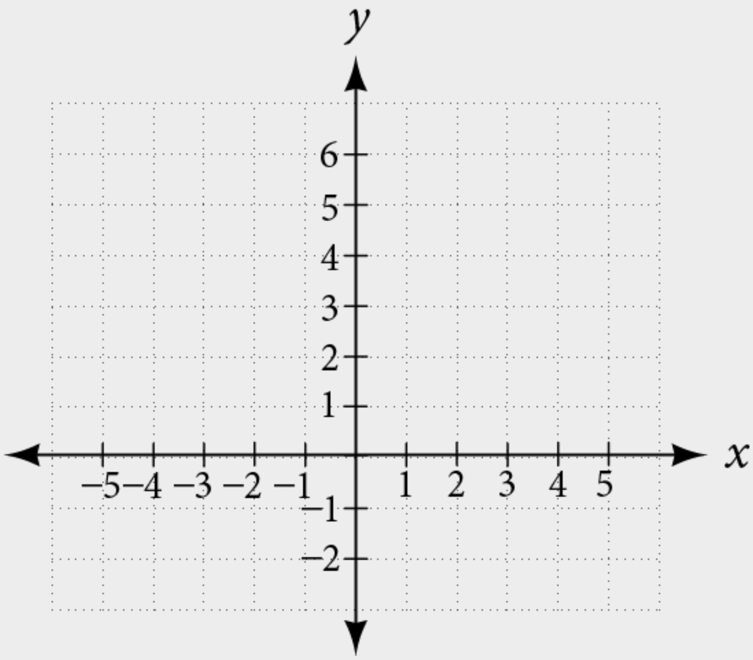
Solution:

Step 1	Graph $-x + y = 1$ We can use the slope intercept – form: $y = x + 1$ Slope = 1 y-intercept: (0, 1) 

Step 2	Graph $2x+y=10$ We can use the slope intercept – form: $y=-2x+10$ Slope = –2 y-intercept: (0, 10) 
Step 3	The lines intersect
Step 4	The solution is the point (3, 4)
Step 5	Let’s check the solution: $-x+y=1 \quad 2x+y=10$ $-3+4=1 \quad 2(3)+4=10$ $1=1 \quad 10=10$ Since (3, 4) is a solution to both equations, then it **is a solution** to the system $-x+y=1$ $2x+y=10$

Note:

Solve a system of linear equations by graphing $\begin{cases} y = -2x + 4 \\ 4x + 2y = 6 \end{cases}$

Step 1	Graph $y = -2x + 4$ Slope = _________ y-intercept = _________
Step 2	Graph $4x + 2y = 6$ Slope = _________ y-intercept = _________  y x 6 5 4 3 2 1 −1 −2 −5 −4 −3 −2 −1 1 2 3 4 5
Step	Do the lines intersect?

3	__________
Step 4	Read from the graph the point of intersection. __________
Step 5	Check the solution in both equations. __________

Practice Makes Perfect

Exercise:

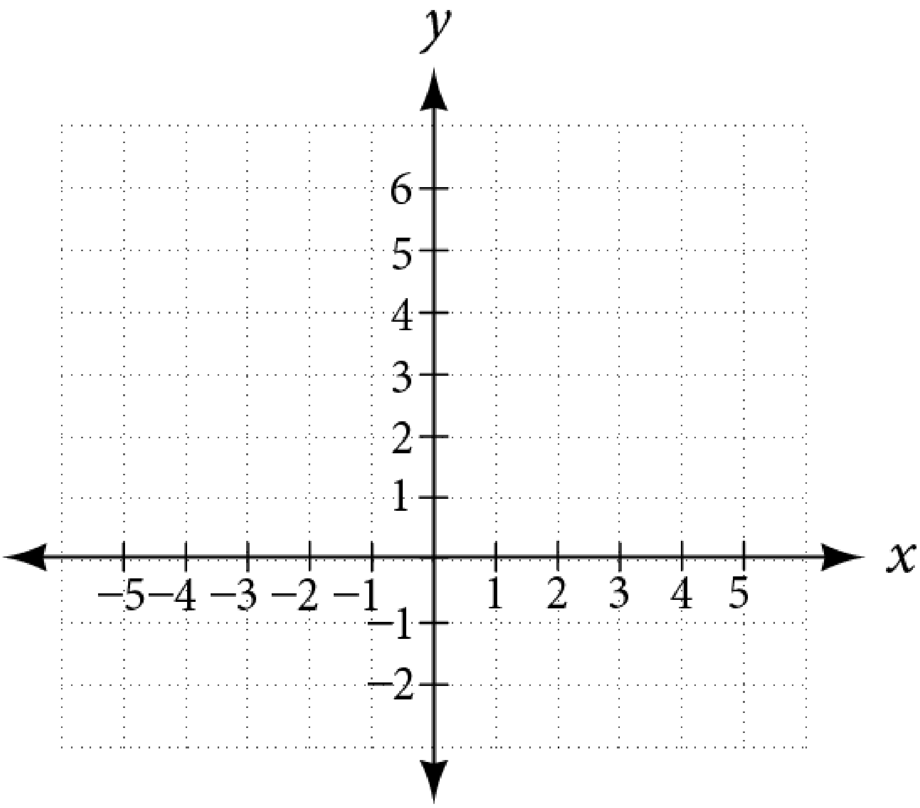
Problem:

Determine whether the ordered pair is a solution to the given system $\begin{cases} x - 3y = -8 \\ -3x - y = 4 \end{cases}$

Exercise:

Problem:

Solve the following system by graphing. $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = 8 \end{cases}$


y
x
6
5
4
3
2
1
−1
−2
−5
−4
−3
−2
−1
1
2
3
4
5

(credit: Thomas Sørenes)

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.
Equation:

$$\begin{aligned} 2x+y= &\quad 15 \\ 3x-y= &\quad 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair (4, 7) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

Equation:

$$\begin{aligned} 2(4) + (7) &= 15 \quad \text{True} \\ 3(4) - (7) &= 5 \quad \text{True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same *y*-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different *y*-intercepts. There are no points common to both lines; hence, there is no solution to the system.

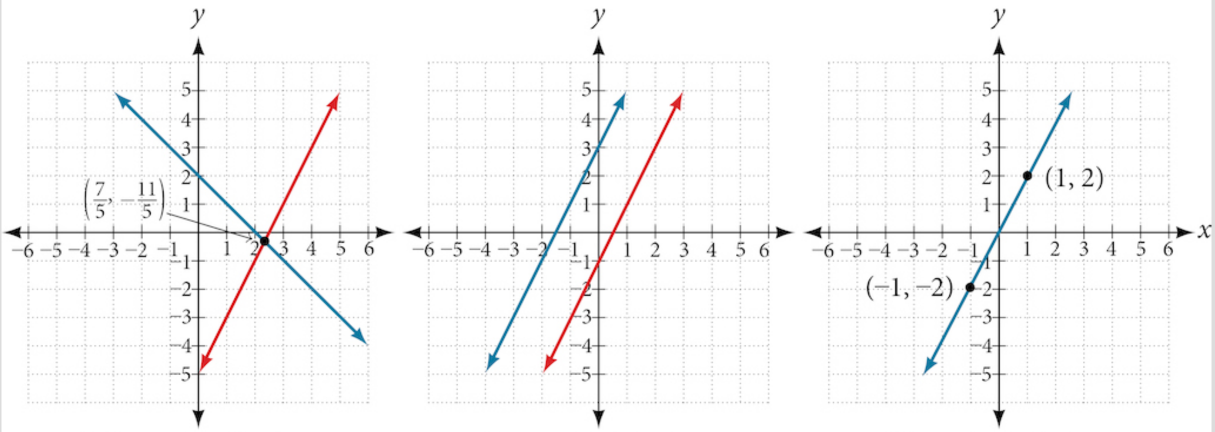
Note:

Types of Linear Systems

There are three types of systems of linear equations in two variables, and three types of solutions.

- An **independent system** has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. Notice that the two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

[link] compares graphical representations of each type of system.



Independent System Inconsistent System Dependent System

Note:

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution.

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example:

Exercise:

Problem:

Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

Equation:

$$\begin{aligned} x + 3y &= 8 \\ 2x - 9 &= y \end{aligned}$$

Solution:

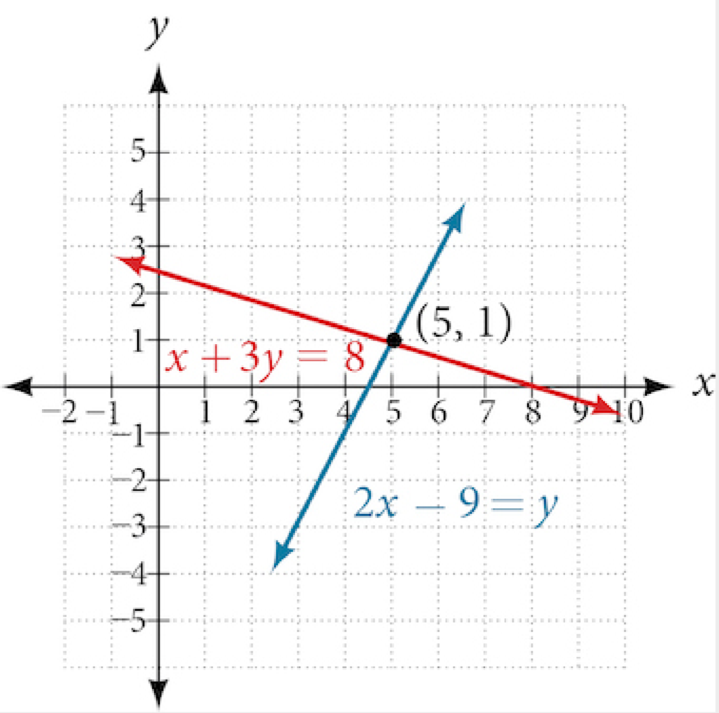
Substitute the ordered pair $(5, 1)$ into both equations.
Equation:

$$\begin{array}{ll} (5) + 3(1) = 8 & \\ 8 = 8 & \text{True} \\ 2(5) - 9 = (1) & \\ 1 = 1 & \text{True} \end{array}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See [link].



Note:
Exercise:

Problem:

Determine whether the ordered pair $(8,5)$ is a solution to the following system.
Equation:

$$\begin{gathered}5x-4y=20\\2x+1=3y\end{gathered}$$

Solution:

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Example:
Exercise:

Problem:
Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.
Equation:

$$\begin{gathered}2x+y=-8\\x-y=-1\end{gathered}$$

Solution:

Solve the first equation for y.

Equation:

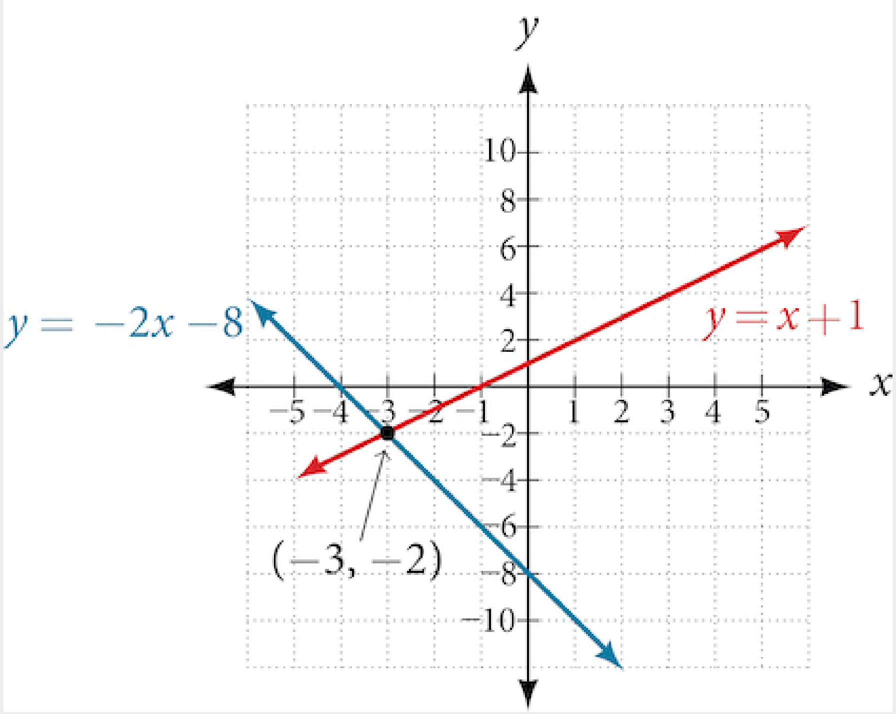
$$2x + y = -8$$
$$y = -2x - 8$$

Solve the second equation for y.

Equation:

$$x - y = -1$$
$$y = x + 1$$

Graph both equations on the same set of axes as in [link].



The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

Equation:

$$\begin{aligned} 2(-3)+(-2) &= -8 & \\ -8 &= -8 & \text{True} \\ (-3)-(-2) &= -1 & \\ -1 &= -1 & \text{True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$, so the system is independent.

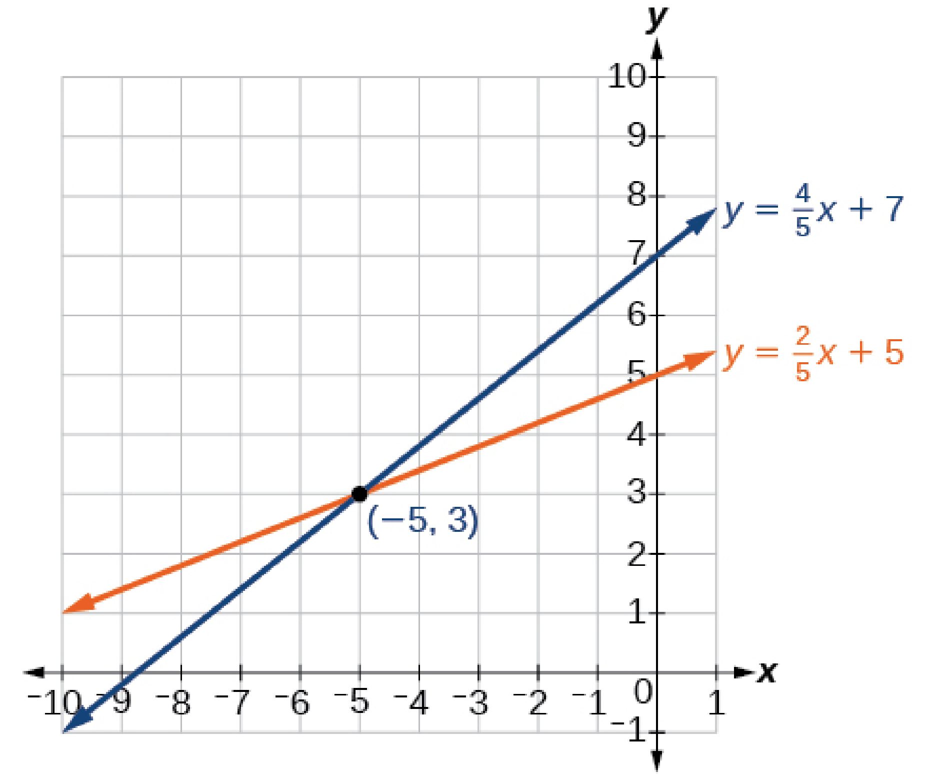
Note:
Exercise:

Problem: Solve the following system of equations by graphing.
Equation:

$$\begin{aligned} 2x-5y &= -25 \\ -4x+5y &= 35 \end{aligned}$$

Solution:

The solution to the system is the ordered pair $(-5,3)$.



Note:

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

Note:
Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example:
Exercise:

Problem:
Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.
Equation:

$$\begin{aligned} -x + y &= -5 \\ 2x - 5y &= 1 \end{aligned}$$

Solution:

First, we will solve the first equation for y.
Equation:

$$-x + y = -5$$
$$y = x - 5$$

Now we can substitute the expression $x - 5$ for y in the second equation.
Equation:

$$\begin{aligned}&2x - 5y = 1\\&2x - 5(x - 5) = 1\\&2x - 5x + 25 = 1\\&-3x = -24\\&x = 8\end{aligned}$$

Now, we substitute $x = 8$ into the first equation and solve for y.
Equation:

$$\begin{aligned}&-(8) + y = -5\\&y = 3\end{aligned}$$

Our solution is $(8, 3)$.

Check the solution by substituting $(8, 3)$ into both equations.
Equation:

$$\begin{aligned}&-x + y = -5\\&-(8) + (3) = -5 && \text{True}\\&2x - 5y = 1\\&2(8) - 5(3) = 1 && \text{True}\end{aligned}$$

Note:
Exercise:

Problem: Solve the following system of equations by substitution.
Equation:

$$x = y + 3$$
$$4 = 3x - 2y$$

Solution:

$(-2, -5)$

Note:
Can the substitution method be used to solve any linear system in two variables?
Yes, but the method works best if one of the equations contains a coefficient of 1 or –1 so that we do not have to deal with fractions.

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the **addition method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

Note:
Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.

2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example:
Exercise:

Problem:
Solving a System by the Addition Method

Solve the given system of equations by addition.
Equation:

$$\begin{array}{l} x+2y=-1 \\ -x+y=3 \end{array}$$

Solution:

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, –1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.
Equation:

$$\begin{array}{c} x+2y=-1 \\ \underline{-x+y=3} \\ 3y=2 \end{array}$$

Now that we have eliminated x, we can solve the resulting equation for y.

Equation:

$$\begin{aligned} 3y &= 2 \\ y &= \frac{2}{3} \end{aligned}$$

Then, we substitute this value for y into one of the original equations and solve for x.

Equation:

$$\begin{aligned} -x + y &= 3 \\ -x + \frac{2}{3} &= 3 \\ -x &= 3 - \frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

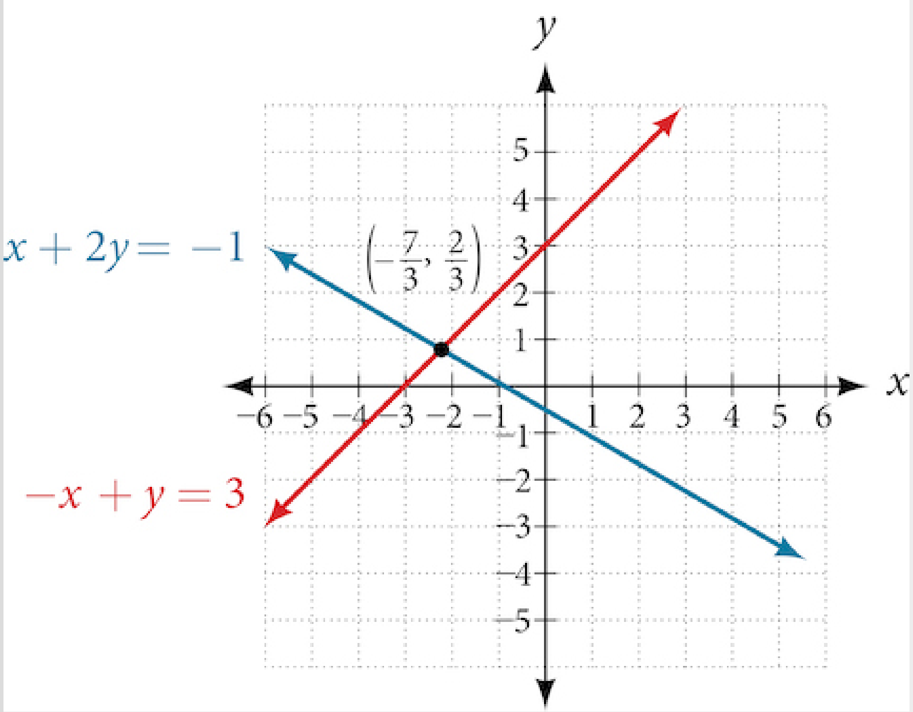
Check the solution in the first equation.

Equation:

$$\begin{aligned} x + 2y &= -1 \\ \left(-\frac{7}{3}\right) + 2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3} + \frac{4}{3} &= \\ -\frac{3}{3} &= \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See [link] to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.



Example:

Exercise:

Problem:

Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

Equation:

$$3x+5y=-11$$

$$x-2y=11$$

Solution:

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by $-3,$ the x-terms will add to zero.

Equation:

$$\begin{array}{ll} x-2y=11 & \\ -3(x-2y)=-3(11) & \text{Multiply both sides by } -3. \\ -3x+6y=-33 & \text{Use the distributive property.} \end{array}$$

Now, let’s add them.

Equation:

$$\begin{array}{r} 3x+5y=-11 \\ -3x+6y=-33 \\ \hline 11y=-44 \\ y=-4 \end{array}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

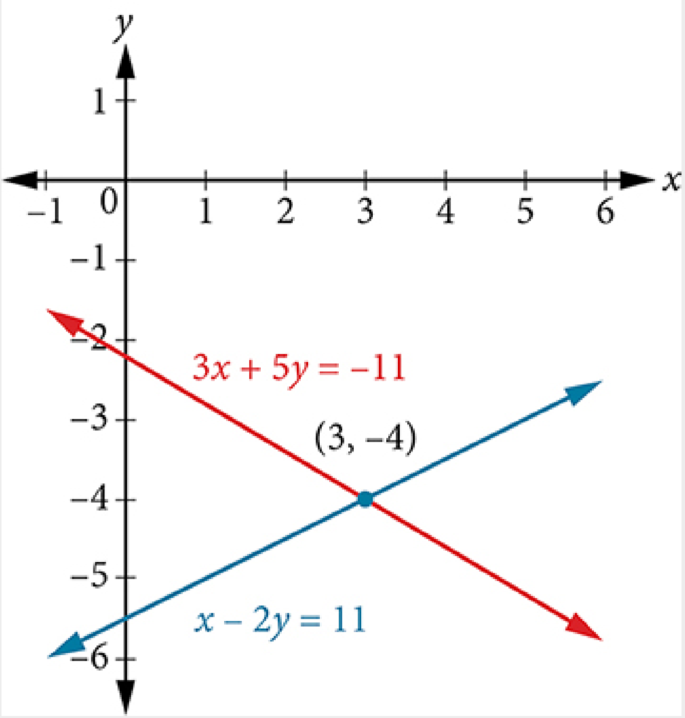
Equation:

$$\begin{array}{c} 3x+5y=-11 \\ 3x+5(-4)=-11 \\ 3x-20=-11 \\ 3x=9 \\ x=3 \end{array}$$

Our solution is the ordered pair $(3,-4)$. See [link]. Check the solution in the original second equation.

Equation:

$$\begin{aligned} x - 2y &= 11 \\ (3) - 2(-4) &= 3 + 8 \\ 11 &= 11 \qquad\qquad \text{True} \end{aligned}$$



Note:
Exercise:

Problem: Solve the system of equations by addition.
Equation:

$$\begin{aligned} 2x - 7y &= 2 \\ 3x + y &= -20 \end{aligned}$$

Solution:

$(-6, -2)$

Example:
Exercise:

Problem:
Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.
Equation:

$$2x+3y=-16$$
$$5x{-}10y=30$$

Solution:

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.
Equation:

$$\begin{aligned}-5(2x+3y)&=-5(-16)\\-10x-15y&=80\\2(5x-10y)&=2(30)\\10x-20y&=60\end{aligned}$$

Then, we add the two equations together.
Equation:

$$\begin{array}{l}-10x{-}15y=80\\ \underline{10x{-}20y=60\qquad\quad}\\ -35y=140\\ y=-4\end{array}$$

Substitute $y = -4$ into the original first equation.

Equation:

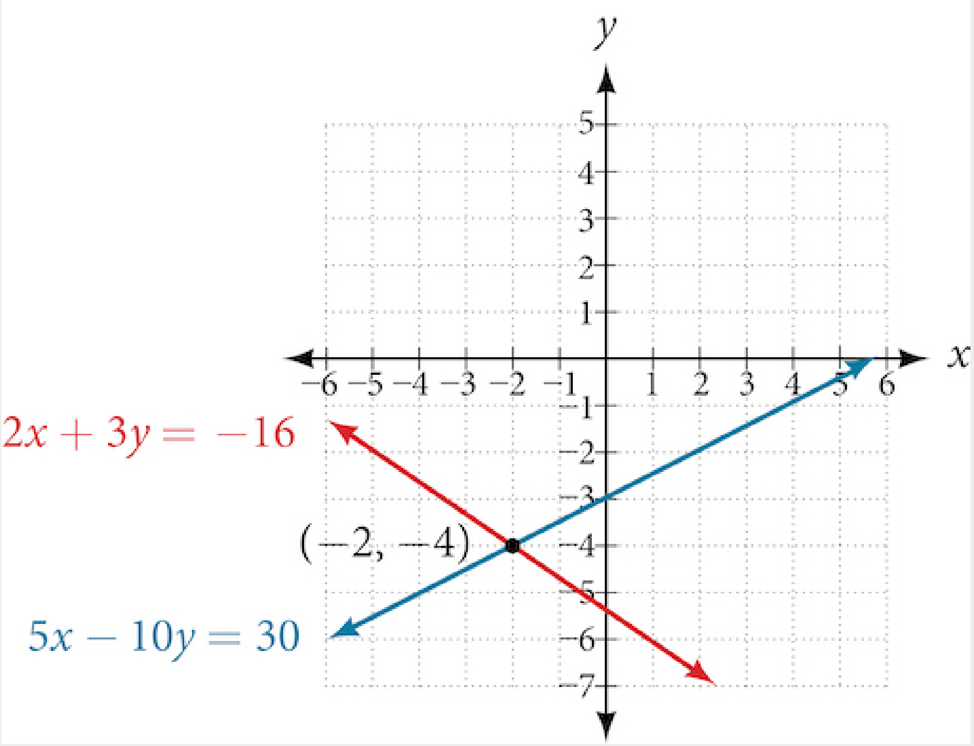
$$\begin{gathered} 2x + 3(-4) = -16 \\ 2x - 12 = -16 \\ 2x = -4 \\ x = -2 \end{gathered}$$

The solution is $(-2, -4)$. Check it in the other equation.

Equation:

$$\begin{aligned} 5x - 10y &= 30 \\ 5(-2) - 10(-4) &= 30 \\ -10 + 40 &= 30 \\ 30 &= 30 \end{aligned}$$

See [link].



Example:
Exercise:

Problem:
Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.
Equation:

$$\begin{aligned}\frac{x}{3}+\frac{y}{6}&=3\\ \frac{x}{2}-\frac{y}{4}&=\quad 1\end{aligned}$$

Solution:

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.
Equation:

$$\begin{aligned}6\left(\frac{x}{3}+\frac{y}{6}\right)&=6(3)\\ 2x+y&=18\\ 4\left(\frac{x}{2}-\frac{y}{4}\right)&=4(1)\\ 2x-y&=4\end{aligned}$$

Now multiply the second equation by -1 so that we can eliminate the *x*-variable.
Equation:

$$\begin{aligned}-1(2x-y)&=-1(4)\\ -2x+y&=-4\end{aligned}$$

Add the two equations to eliminate the *x*-variable and solve the resulting equation.
Equation:

$$\begin{array}{l} 2x + y = 18 \\ \underline{-2x + y = -4} \\ 2y = 14 \\ y = 7 \end{array}$$

Substitute $y = 7$ into the first equation.
Equation:

$$\begin{aligned} 2x + (7) &= 18 \\ 2x &= 11 \\ x &= \frac{11}{2} \\ &= 5.5 \end{aligned}$$

The solution is $\left(\frac{11}{2}, 7\right)$. Check it in the other equation.
Equation:

$$\begin{array}{c} \frac{x}{2} - \frac{y}{4} = 1 \\ \frac{\frac{11}{2}}{2} - \frac{7}{4} = 1 \\ \frac{11}{4} - \frac{7}{4} = 1 \\ \frac{4}{4} = 1 \end{array}$$

Note:
Exercise:

Problem: Solve the system of equations by addition.
Equation:

$$\begin{aligned} 2x + 3y &= 8 \\ 3x + 5y &= 10 \end{aligned}$$

Solution:

$(10, -4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y -intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12 = 0$.

Example:
Exercise:

Problem:
Solving an Inconsistent System of Equations

Solve the following system of equations.
Equation:

$$\begin{aligned} x &= 9-2y \\ x + 2y &= 13 \end{aligned}$$

Solution:

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.
Equation:

$$\begin{aligned} x+2y&=13\\ (9-2y)+2y&=13\\ 9+0y&=13\\ 9&=13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

Equation:

$$\begin{aligned} x&=9-2y\\ 2y&=-x+9\\ y&=-\frac{1}{2}x+\frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

Equation:

$$\begin{aligned} x+2y&=13\\ 2y&=-x+13\\ y&=-\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

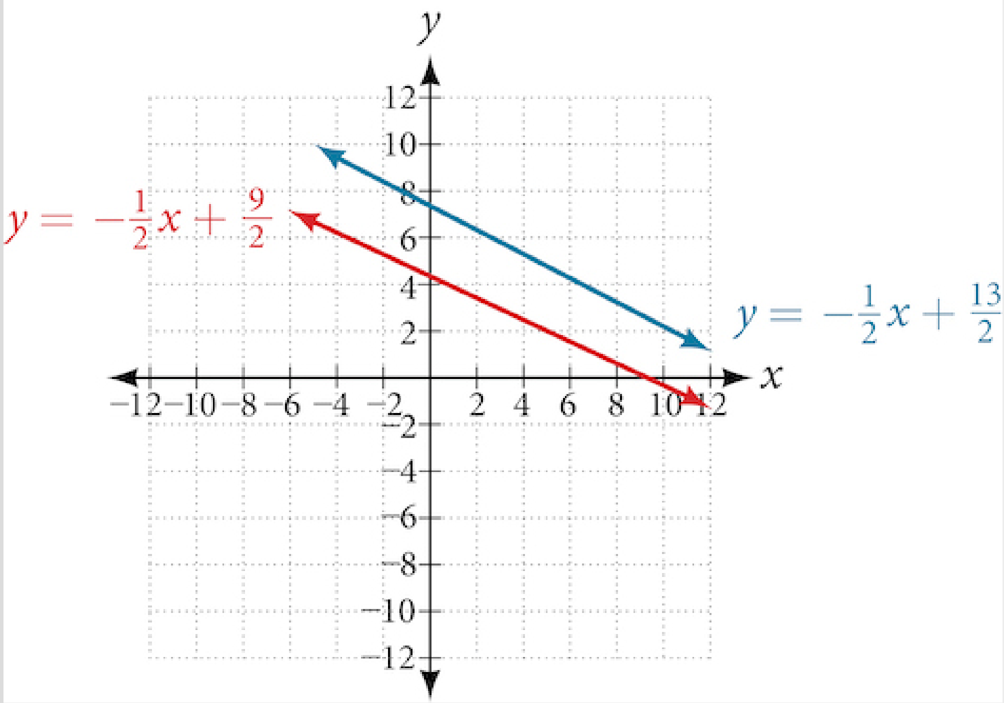
Comparing the equations, we see that they have the same slope but different *y*-intercepts. Therefore, the lines are parallel and do not intersect.

Equation:

$$\begin{aligned} y&=-\frac{1}{2}x+\frac{9}{2}\\ y&=-\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in [link].



Note:
Exercise:

Problem: Solve the following system of equations in two variables.
Equation:

$$\begin{aligned}2y{-}2x &= 2\\2y{-}2x &= 6\end{aligned}$$

Solution:

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0 = 0$.

Example:
Exercise:

Problem:
Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the addition method.
Equation:

$$\begin{aligned} x + 3y &= 2 \\ 3x + 9y &= 6 \end{aligned}$$

Solution:

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let’s focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x -variable.
Equation:

$$\begin{aligned} x + 3y &= 2 \\ (-3)(x + 3y) &= (-3)(2) \\ -3x - 9y &= -6 \end{aligned}$$

Now add the equations.
Equation:

$$\begin{array}{rrl} & -3x-9y & =-6 \\ + & 3x+9y & =6 \\ \hline & 0 & =0 \end{array}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

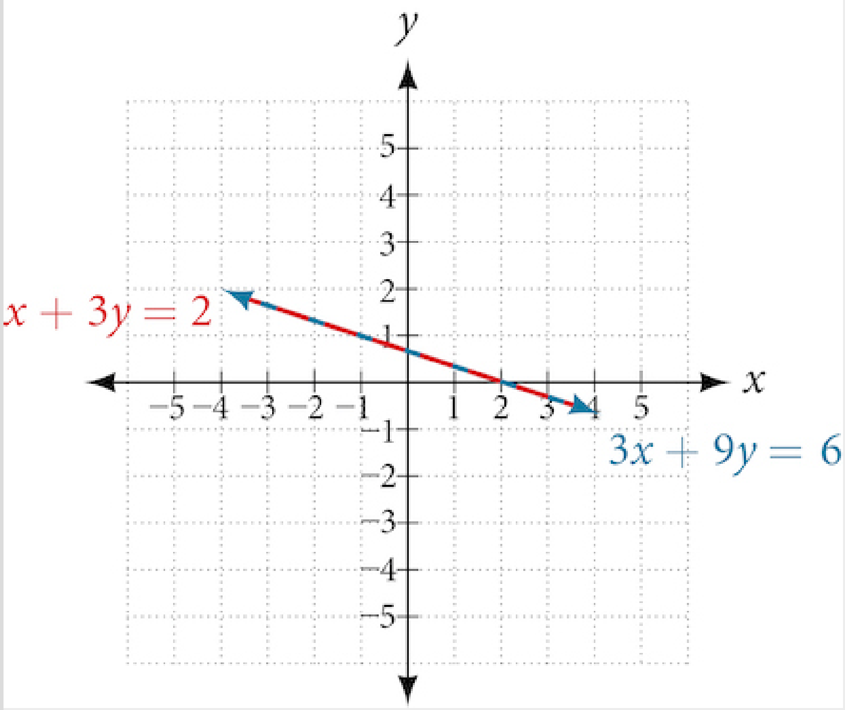
Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let’s look at what happens when we convert the system to slope-intercept form.

Equation:

$$\begin{aligned} x+3y&=2 \\ 3y&=-x+2 \\ y&=-\tfrac{1}{3}x+\tfrac{2}{3} \\ 3x+9y&=6 \\ 9y&=-3x+6 \\ y&=-\tfrac{3}{9}x+\tfrac{6}{9} \\ y&=-\tfrac{1}{3}x+\tfrac{2}{3} \end{aligned}$$

See [link]. Notice the results are the same. The general solution to the system is $\left(x\ ,-\ \frac{1}{3}x+\frac{2}{3}\right)$.



Note:
Exercise:

Problem: Solve the following system of equations in two variables.
Equation:

$$y–2x=5$$
$$−3y+6x=−15$$

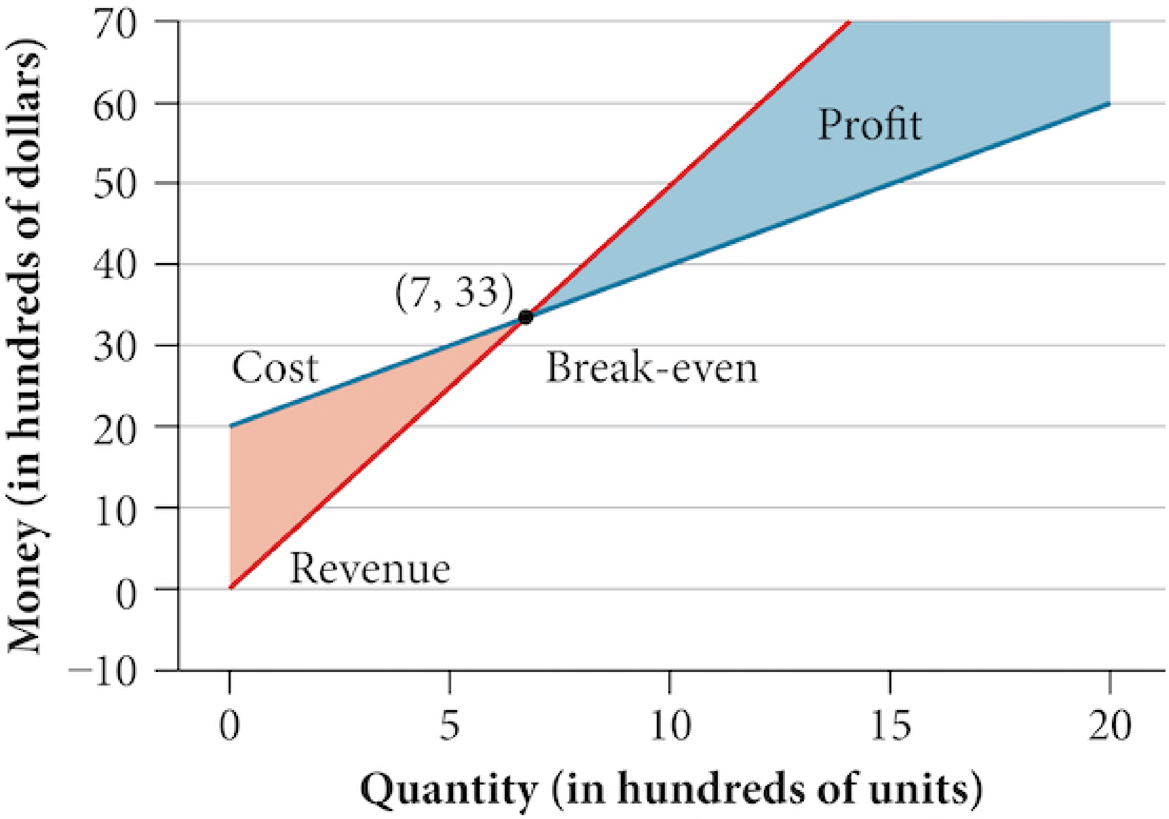
Solution:

The system is dependent so there are infinite solutions of the form $(x,2x+5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's **revenue function** is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where $x =$ quantity and $p =$ price. The revenue function is shown in orange in [link].

The **cost function** is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in [link]. The x -axis represents quantity in hundreds of units. The *y*-axis represents either cost or revenue in hundreds of dollars.



The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is $3,300 and the revenue is also $3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The **profit function** is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example:
Exercise:

Problem:
Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35{,}000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution:

Write the system of equations using y to replace function notation.
Equation:

$$\begin{aligned} y &= 0.85x + 35{,}000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35{,}000$ from the first equation into the second equation and solve for x.
Equation:

$$\begin{aligned} 0.85x + 35{,}000 &= 1.55x \\ 35{,}000 &= 0.7x \\ 50{,}000 &= x \end{aligned}$$

Then, we substitute $x = 50{,}000$ into either the cost function or the revenue function.

Equation:

$$1.55\,(50{,}000) = 77{,}500$$

The break-even point is $(50{,}000, 77{,}500)$.

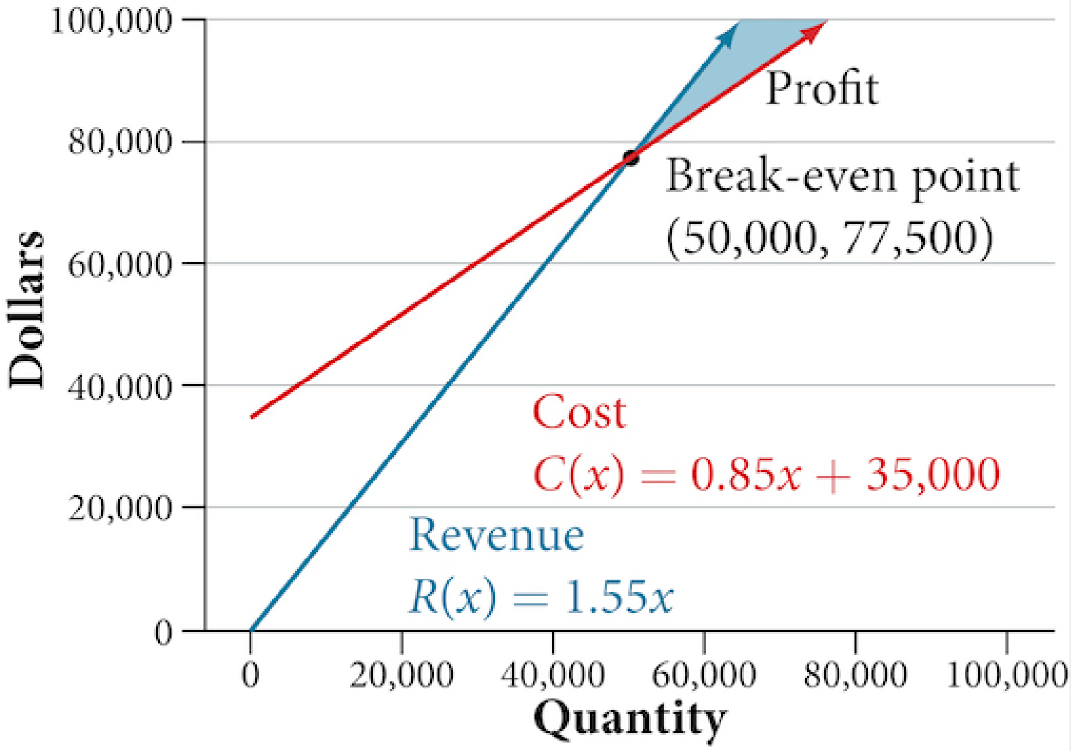
The profit function is found using the formula $P(x) = R(x) - C(x)$.
Equation:

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

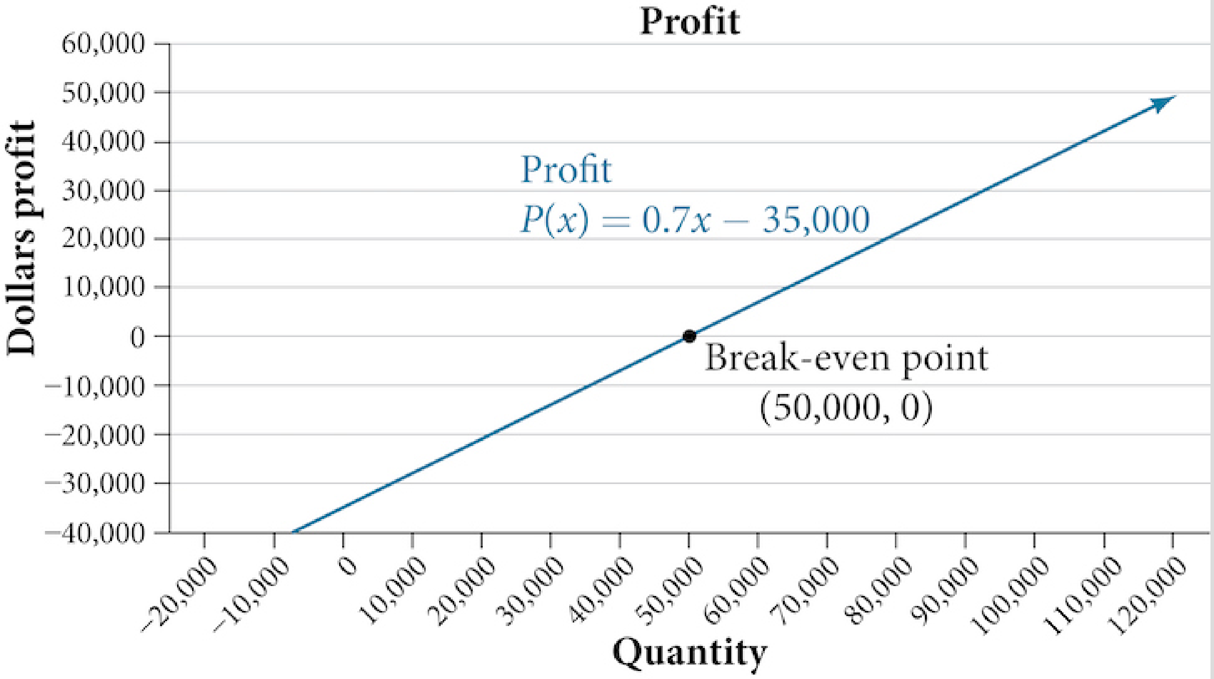
The profit function is $P(x) = 0.7x{-}35{,}000$.

Analysis

The cost to produce 50,000 units is \$77,500, and the revenue from the sales of 50,000 units is also \$77,500. To make a profit, the business must produce and sell more than 50,000 units. See [link].



We see from the graph in [link] that the profit function has a negative value until $x = 50{,}000$, when the graph crosses the *x*-axis. Then, the graph emerges into positive *y*-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.



Example:
Exercise:

Problem:
Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2{,}000$ and the total gate revenue is $\$70{,}000$. How many children and how many adults bought tickets?

Solution:

Let *c* = the number of children and *a* = the number of adults in attendance.

The total number of people is $2{,}000$. We can use this to write an equation for the number of people at the circus that day.
Equation:

$$c + a = 2{,}000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70{,}000$. We can use this to write an equation for the revenue.
Equation:

$$25c + 50a = 70{,}000$$

We now have a system of linear equations in two variables.
Equation:

$$\begin{gathered} c + a = 2{,}000 \\ 25c + 50a = 70{,}000 \end{gathered}$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.
Equation:

$$\begin{gathered} c + a = 2{,}000 \\ a = 2{,}000 - c \end{gathered}$$

Substitute the expression $2{,}000 - c$ in the second equation for a and solve for c.
Equation:

$$\begin{aligned} &25c + 50(2{,}000 - c) = 70{,}000 \\ &25c + 100{,}000 - 50c = 70{,}000 \\ &-25c = -30{,}000 \\ &c = 1{,}200 \end{aligned}$$

Substitute $c = 1{,}200$ into the first equation to solve for a.

Equation:

$$\begin{aligned}1{,}200 + a &= 2{,}000\\ a &= 800\end{aligned}$$

We find that 1,200 children and 800 adults bought tickets to the circus that day.

Note:

Exercise:

Problem:

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If 1,650 meal tickets were bought for a total of $\$14{,}200$, how many children and how many adults bought meal tickets?

Solution:

700 children, 950 adults

Note:

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See [link].
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See [link].
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See [link].
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See [link].
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See [link], [link], and [link].
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See [link].
- The solution to a system of dependent equations will always be true because both equations describe the same line. See [link].
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See [link] and [link].

Section Exercises

Verbal

Exercise:

Problem:

Can a system of linear equations have exactly two solutions? Explain why or why not.

Solution:

No, you can either have zero, one, or infinitely many. Examine graphs.

Exercise:

Problem:

If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

Exercise:

Problem:

If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

Solution:

This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

Exercise:

Problem:

If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

Exercise:

Problem:

Given a system of equations, explain at least two different methods of solving that system.

Solution:

You can solve by substitution (isolating x or y), graphically, or by addition.

Algebraic

For the following exercises, determine whether the given ordered pair is a solution to the system of equations.

Exercise:

Problem: $\begin{aligned} 5x - y &= 4 \\ x + 6y &= 2 \end{aligned}$ and $(4, 0)$

Exercise:

Problem: $\begin{aligned} -3x - 5y &= 13 \\ -x + 4y &= 10 \end{aligned}$ and $(-6, 1)$

Solution:

Yes

Exercise:

Problem: $\begin{aligned} 3x + 7y &= 1 \\ 2x + 4y &= 0 \end{aligned}$ and $(2, 3)$

Exercise:

Problem: $\begin{aligned} -2x + 5y &= 7 \\ 2x + 9y &= 7 \end{aligned}$ and $(-1, 1)$

Solution:

Yes

Exercise:

Problem: $\begin{aligned} x + 8y &= 43 \\ 3x - 2y &= -1 \end{aligned}$ and $(3, 5)$

For the following exercises, solve each system by substitution.

Exercise:

Problem: $$\begin{aligned} x + 3y &= 5 \\ 2x + 3y &= 4 \end{aligned}$$

Solution:

$(-1, 2)$

Exercise:

Problem: $$\begin{aligned} 3x - 2y &= 18 \\ 5x + 10y &= -10 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} 4x + 2y &= -10 \\ 3x + 9y &= 0 \end{aligned}$$

Solution:

$(-3, 1)$

Exercise:

Problem: $$\begin{aligned} 2x + 4y &= -3.8 \\ 9x - 5y &= 1.3 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} -2x + 3y &= 1.2 \\ -3x - 6y &= 1.8 \end{aligned}$$

Solution:

$\left(-\frac{3}{5}, 0\right)$

Exercise:

Problem: $$\begin{aligned} x - 0.2y &= 1 \\ -10x + 2y &= 5 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} 3x + 5y &= 9 \\ 30x + 50y &= -90 \end{aligned}$$

Solution:

No solutions exist.

Exercise:

Problem: $$\begin{aligned} -3x + y &= 2 \\ 12x - 4y &= -8 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y &= 16 \\ \tfrac{1}{6}x + \tfrac{1}{4}y &= 9 \end{aligned}$$

Solution:

$\left(\frac{72}{5}, \frac{132}{5}\right)$

Exercise:

Problem: $$\begin{aligned} -\tfrac{1}{4}x + \tfrac{3}{2}y &= 11 \\ -\tfrac{1}{8}x + \tfrac{1}{3}y &= 3 \end{aligned}$$

For the following exercises, solve each system by addition.

Exercise:

Problem: $$\begin{aligned} -2x + 5y &= -42 \\ 7x + 2y &= 30 \end{aligned}$$

Solution:

$(6, -6)$

Exercise:

Problem: $$\begin{aligned} 6x - 5y &= -34 \\ 2x + 6y &= 4 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} 5x - y &= -2.6 \\ -4x - 6y &= 1.4 \end{aligned}$$

Solution:

$\left(-\frac{1}{2}, \frac{1}{10}\right)$

Exercise:

Problem: $$\begin{aligned} 7x - 2y &= 3 \\ 4x + 5y &= 3.25 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} -\mathrm{x} + 2y &= -1 \\ 5x - 10y &= 6 \end{aligned}$$

Solution:

No solutions exist.

Exercise:

Problem: $$\begin{aligned} 7x + 6y &= 2 \\ -28x - 24y &= -8 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} \frac{5}{6}x + \frac{1}{4}y &= 0 \\ \frac{1}{8}x - \frac{1}{2}y &= -\frac{43}{120} \end{aligned}$$

Solution:

$\left(-\frac{1}{5}, \frac{2}{3}\right)$

Exercise:

Problem: $$\begin{aligned} \frac{1}{3}x + \frac{1}{9}y &= \frac{2}{9} \\ -\frac{1}{2}x + \frac{4}{5}y &= -\frac{1}{3} \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} -0.2x + 0.4y &= 0.6 \\ x - 2y &= -3 \end{aligned}$$

Solution:

$\left(x, \frac{x+3}{2}\right)$

Exercise:

Problem: $$\begin{aligned} -0.1x + 0.2y &= 0.6 \\ 5x - 10y &= 1 \end{aligned}$$

For the following exercises, solve each system by any method.

Exercise:

Problem: $$\begin{aligned} 5x + 9y &= 16 \\ x + 2y &= 4 \end{aligned}$$

Solution:

$(-4, 4)$

Exercise:

Problem: $$\begin{aligned} 6x - 8y &= -0.6 \\ 3x + 2y &= 0.9 \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} 5x - 2y &= 2.25 \\ 7x - 4y &= 3 \end{aligned}$$

Solution:

$\left(\frac{1}{2}, \frac{1}{8}\right)$

Exercise:

Problem: $$\begin{aligned} x - \tfrac{5}{12}y &= -\tfrac{55}{12} \\ -6x + \tfrac{5}{2}y &= \tfrac{55}{2} \end{aligned}$$

Exercise:

Problem: $$\begin{aligned} 7x - 4y &= \tfrac{7}{6} \\ 2x + 4y &= \tfrac{1}{3} \end{aligned}$$

Solution:

$\left(\frac{1}{6}, 0\right)$

Exercise:

Problem: $3x + 6y = 11$
$2x + 4y = 9$

Exercise:

Problem: $\frac{7}{3}x - \frac{1}{6}y = 2$
$-\frac{21}{6}x + \frac{3}{12}y = -3$

Solution:

$(x, 2(7x-6))$

Exercise:

Problem: $\frac{1}{2}x + \frac{1}{3}y = \frac{1}{3}$
$\frac{3}{2}x + \frac{1}{4}y = -\frac{1}{8}$

Exercise:

Problem: $2.2x + 1.3y = -0.1$
$4.2x + 4.2y = 2.1$

Solution:

$\left(-\frac{5}{6}, \frac{4}{3}\right)$

Exercise:

Problem: $0.1x + 0.2y = 2$
$0.35x - 0.3y = 0$

Graphical

For the following exercises, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

Exercise:

Problem:
$$\begin{aligned} 3x - y &= 0.6 \\ x-2y &= 1.3 \end{aligned}$$

Solution:

Consistent with one solution

Exercise:

Problem:
$$\begin{aligned} -x + 2y &= 4 \\ 2x-4y &= 1 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} x + 2y &= 7 \\ 2x + 6y &= 12 \end{aligned}$$

Solution:

Consistent with one solution

Exercise:

Problem:
$$\begin{aligned} 3x-5y &= 7 \\ x-2y &= 3 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 3x-2y &= 5 \\ -9x + 6y &= -15 \end{aligned}$$

Solution:

Dependent with infinitely many solutions

Technology

For the following exercises, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

Exercise:

Problem:
$$\begin{aligned} 0.1x + 0.2y &= 0.3 \\ -0.3x + 0.5y &= 1 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} -0.01x + 0.12y &= 0.62 \\ 0.15x + 0.20y &= 0.52 \end{aligned}$$

Solution:

$(-3.08, 4.91)$

Exercise:

Problem:
$$\begin{aligned} 0.5x + 0.3y &= 4 \\ 0.25x - 0.9y &= 0.46 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 0.15x + 0.27y &= 0.39 \\ -0.34x + 0.56y &= 1.8 \end{aligned}$$

Solution:

$(-1.52, 2.29)$

Exercise:

Problem: $$\begin{aligned}-0.71x + 0.92y &= 0.13\\ 0.83x + 0.05y &= 2.1\end{aligned}$$

Extensions

For the following exercises, solve each system in terms of $A, B, C, D, E,$ and F where A– F are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

Exercise:

Problem: $$\begin{aligned}x + y &= A\\ x - y &= B\end{aligned}$$

Solution:

$\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$

Exercise:

Problem: $$\begin{aligned}x + Ay &= 1\\ x + By &= 1\end{aligned}$$

Exercise:

Problem: $$\begin{aligned}Ax + y &= 0\\ Bx + y &= 1\end{aligned}$$

Solution:

$\left(\frac{-1}{A-B}, \frac{A}{A-B}\right)$

Exercise:

Problem: $$\begin{aligned}Ax + By &= C\\ x + y &= 1\end{aligned}$$

Exercise:

Problem: $$\begin{aligned}Ax + By &= C\\ Dx + Ey &= F\end{aligned}$$

Solution:

$\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE}\right)$

Real-World Applications

For the following exercises, solve for the desired quantity.

Exercise:

Problem:

A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x$. Find the break-even point.

Exercise:

Problem:

A fast-food restaurant has a cost of production $C(x) = 11x + 120$ and a revenue function $R(x) = 5x$. When does the company start to turn a profit?

Solution:

They never turn a profit.

Exercise:

Problem:

A cell phone factory has a cost of production $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x$. What is the break-even point?

Exercise:

Problem:

A musician charges $C(x) = 64x + 20{,}000$ where x is the total number of attendees at the concert. The venue charges $80 per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Solution:

$(1, 250, 100, 000)$

Exercise:

Problem:

A guitar factory has a cost of production $C(x) = 75x + 50{,}000.$ If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the following exercises, use a system of linear equations with two variables and two equations to solve.

Exercise:

Problem: Find two numbers whose sum is 28 and difference is 13.

Solution:

The numbers are 7.5 and 20.5.

Exercise:

Problem:

A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

Exercise:

Problem:

The startup cost for a restaurant is $120,000, and each meal costs $10 for the restaurant to make. If each meal is then sold for $15, after how many meals does the restaurant break even?

Solution:

24,000

Exercise:

Problem:

A moving company charges a flat rate of $150, and an additional $5 for each box. If a taxi service would charge $20 for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

Exercise:

Problem:

A total of 1,595 first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

Solution:

790 sophomores, 805 freshman

Exercise:

Problem:

276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

Exercise:

Problem:

There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

Solution:

56 men, 74 women

Exercise:

Problem:

A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

Exercise:

Problem:

If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

Solution:

10 gallons of 10% solution, 15 gallons of 60% solution

Exercise:

Problem:

An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

Exercise:

Problem:

An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

Solution:

Swan Peak: $750,000, Riverside: $350,000

Exercise:

Problem:

If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $737.50 annual interest, how much was invested in each account?

Exercise:

Problem:

If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

Solution:

$12,500 in the first account, $10,500 in the second account.

Exercise:

Problem:

CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

Exercise:

Problem:

A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

Solution:

High-tops: 45, Low-tops: 15

Exercise:

Problem:

A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

Exercise:

Problem:

Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child’s ticket and the price of the adult ticket?

Solution:

Infinitely many solutions. We need more information.

Glossary

addition method

an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable

break-even point
the point at which a cost function intersects a revenue function; where profit is zero

consistent system
a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system

cost function
the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs

dependent system
a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system

inconsistent system
a system of linear equations with no common solution because they represent parallel lines, which have no point or line in common

independent system
a system of linear equations with exactly one solution pair (x, y)

profit function
the profit function is written as $P(x) = R(x) - C(x)$, revenue minus cost

revenue function
the function that is used to calculate revenue, simply written as $R = xp$, where $x =$ quantity and $p =$ price

substitution method
an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable

system of linear equations

a set of two or more equations in two or more variables that must be considered simultaneously.

Systems of Linear Equations: Three Variables

In this section, you will:

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

LEARNING OBJECTIVES

- Determine whether an ordered triple is a solution of a system of three linear equations with three variables (IA 4.4.1)
- Solve a system of three linear equations with three variables (IA 4.4.2)

Objective 1: Determine whether an ordered triple is a solution of a system of three linear equations with three variables (IA 4.4.1)

A **linear equation with three variables** where *a*, *b*, *c*, and *d* are real numbers and *a*, *b*, and *c* are not all 0, is of the form $ax+by+cz=d$. The graph of a linear equation with three variables is a **plane**.

A system of linear equations with three variables is a set of linear equations with three variables. For example,
Equation:

$$\begin{gathered}3x+y+z=2\\x+2y+z=-3\\3x+y+2z=4\end{gathered}$$

is a system of linear equations with three variables.

Solutions of a system of equations are the values of the variables that make all the equations true. A solution is represented by an **ordered triple** (*x*,*y*,*z*).

Example:
Determine whether the ordered triples are solutions to the given system.
Exercise:

Problem: $\begin{array}{l}3x+y+z=2\\x+2y+z=-3 \text{ at } (1,-3,2) \text{ and at } (4,-1,-5)\\3x+y+2z=4\end{array}$

Solution:

We substitute 1, –3, 2 into all three equations:

$3x+y+z=2$	$x+2y+z=-3$	$3x+y+2z=4$
$3(1)+(-3)+2=2$ $2=2$ True	$1+2(-3)+2=-3$ $-3=-3$ True	$3(1)+(-3)+2(2)=4$ $4=4$ True
Conclusion: Since (1,-3,2) is a solution to all three equations, then it **is a solution** to the system $3x+y+z=2$ $x+2y+z=-3$ $3x+y+2z=4$		

Next we substitute (4, –1, –5) into all three equations:

$3x+y+z=2$	$x+2y+z=-3$	$3x+y+2z=4$
$3(4)+(-1)+(-5)=2$ $6=2$ False	$4+2(-1)+(-5)=-3$ $-3=-3$ True	$3(4)+(-1)+2(-5)=4$ $1=4$ False
Conclusion: Since (4,–1,–5) is not a solution to all three equations, then **it is not a solution** to the system $3x+y+z=2$ $x+2y+z=-3$ $3x+y+2z=4$		

Practice Makes Perfect

Determine whether the ordered pairs are solutions to the given system.
Equation:

$$\begin{aligned}2x-6y+z&=3\\3x-4y-3z&=2\ \text{ at }\ (3,\ 1,\ 3)\ \text{ and at }\ (4,\ 3,\ 7)\\2x+y-2z&=3\end{aligned}$$

Exercise:

Problem: At (3, 1, 3):

$2x-6y+z=3$	$3x-4y-3z=2$	$2x+3y-2z=3$
____________________________	____________________________	____________________________
Conclusion: ____________________________		

Exercise:

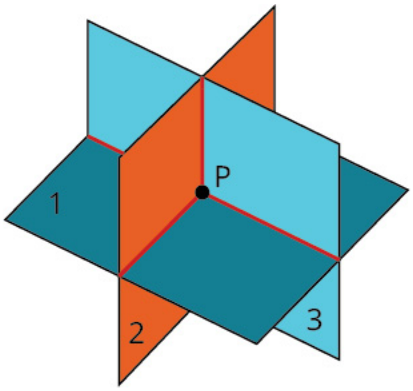
Problem: At (4, 3, 7):

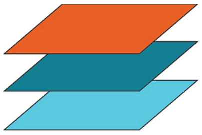
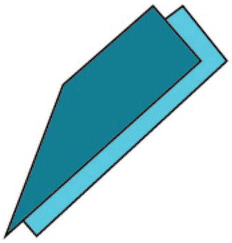
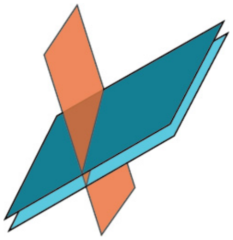
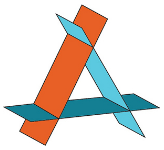
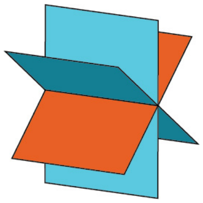
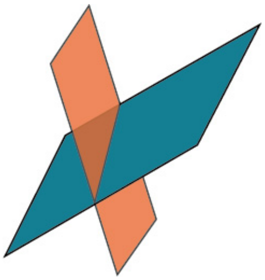
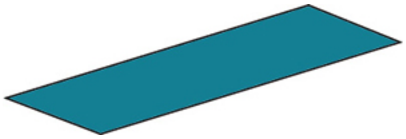
$2x-6y+z=3$	$3x-4y-3z=2$	$2x+3y-2z=3$
____________________________	____________________________	____________________________
Conclusion: ____________________________		

Objective 2: Solve a system of three linear equations with three variables (IA 4.4.2)

When we solve a system of linear equations with three variables, we have many possible solutions.

The solutions are summarized in the table below.

One Solution
The three planes intersect at a common point. 
No Solution

The planes are parallel; they have no points in common. 	Two planes are the same and they are parallel to the third plane. They have no points in common. 	Two planes are parallel and they each intersect the third plane. They have no points in common. 	Each plane intersects the other two but they have no points in common. 
Infinitely many solutions			
The three planes intersect in one line. They have many points on that line in common. 	Two planes are the same and intersect the third one in a line. they have many points on that line in common. 	The three planes are exactly the same. They have many points in common. 	

Note:
Solve a system of linear equations with three variables.

Write the equations in standard form. If any coefficients are fractions, clear them.
Eliminate the same variable from two equations.

a. Decide which variable you will eliminate.
b. Work with a pair of equations to eliminate the chosen variable.
c. Multiply one or both equations so that the coefficients of that variable are opposites.
d. Add the equations resulting from Step 2 to eliminate one variable.

Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
The two new equations form a system of two equations with two variables. Solve this system.
Use the values of the two variables found in Step 4 to find the third variable.
Write the solution as an ordered triple.
Check that the ordered triple is a solution to all three original equations

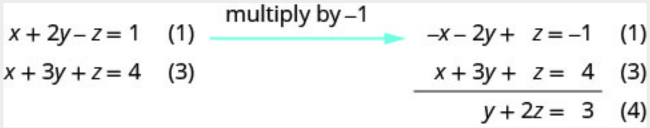
Example:
Exercise:

Problem: Solve the system of equations: $\begin{array}{l} x+2y-z=1 \\ 2x+7y+4z=11. \\ x+3y+z=4 \end{array}$

Solution:
Equation:

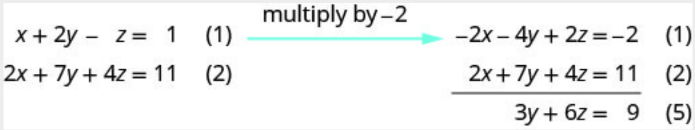
$$\begin{array}{ll} x+2y-z=1 & (1) \\ 2x+7y+4z=11 & (2) \\ x+3y+z=4 & (3) \end{array}$$

Use equation (1) and (3) to eliminate *x*.



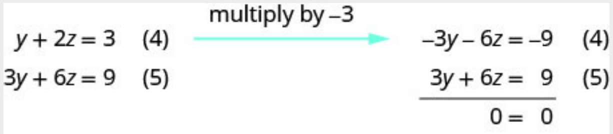
$$\begin{array}{llcll} x+2y-z=1 & (1) & \xrightarrow{\text{multiply by } -1} & -x-2y+z=-1 & (1) \\ x+3y+z=4 & (3) & & x+3y+z=4 & (3) \\ \hline & & & y+2z=3 & (4) \end{array}$$

Use equation (1) and (2) to eliminate *x* again.



$$\begin{array}{llcll} x+2y-z=1 & (1) & \xrightarrow{\text{multiply by } -2} & -2x-4y+2z=-2 & (1) \\ 2x+7y+4z=11 & (2) & & 2x+7y+4z=11 & (2) \\ \hline & & & 3y+6z=9 & (5) \end{array}$$

Use equation (4) and (5) to eliminate y .



$$\begin{array}{llcll} y+2z=3 & (4) & \xrightarrow{\text{multiply by } -3} & -3y-6z=-9 & (4) \\ 3y+6z=9 & (5) & & 3y+6z=9 & (5) \\ \hline & & & 0=0 & \end{array}$$

	There are infinitely many solutions.
Solve equation (4) for *y*.	Represent the solution showing how *x* and *y* are dependent on *z*. $y+2z = 3$ $y = -2z+3$
Use equation (1) to solve for *x*.	$x+2y-z=1$
Substitute $y=-2z+3$.	

	$\begin{aligned} x+2(-2z+3)-z &= 1 \\ x-4z+6-z &= 1 \\ x-5z+6 &= 1 \\ x &= 5z-5 \end{aligned}$

The true statement $0=0$ tells us that this is a dependent system that has infinitely many solutions. The solutions are of the form (x, y, z) where $x=5z-5; y=-2z+3$ and z is any real number.

Note:
Exercise:

Problem: Solve the system of linear equations with three variables.
Equation:

$$\begin{aligned} x+2y-3z=-1 & \quad (1) \\ x-3y+z=1 & \quad (2) \\ 2x-y-2z=2 & \quad (3) \end{aligned}$$

Step 1	Write the equations in standard form. If any coefficients are fractions, clear them. ______________________________
Step 2	Let's use equations (1) and (2) to eliminate x ______________________________
Step 3	Let's use equations (1) and (3) to eliminate x ______________________________
Step 4	Let's now use the new equations (4) and (5) to eliminate y ______________________________
Step 5	______________________________
Step 6	______________________________

Practice Makes Perfect

Exercise:

Determine whether the ordered pair is a solution to the given system

Problem:

$$\begin{array}{l} y - 10z = -8 \\ 2x - y = 2 \quad \text{at } (7,\ 12,\ 2) \text{ and at } (2,\ 2,\ 1) \\ x - 5z = 3 \end{array}$$

Exercise:

Solve the system of linear equations with three variables

Problem:

$$\begin{array}{ll} x + 2y - 3z = -1 & (1) \\ x - 3y + z = 1 & (2) \\ 2x - y - 2z = 2 & (3) \end{array}$$

(credit: “Elembis,” Wikimedia Commons)

John received an inheritance of $12,000 that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $4,000 more in municipal funds than in municipal bonds. He earned $670 in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss.

While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:

Equation:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z, and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

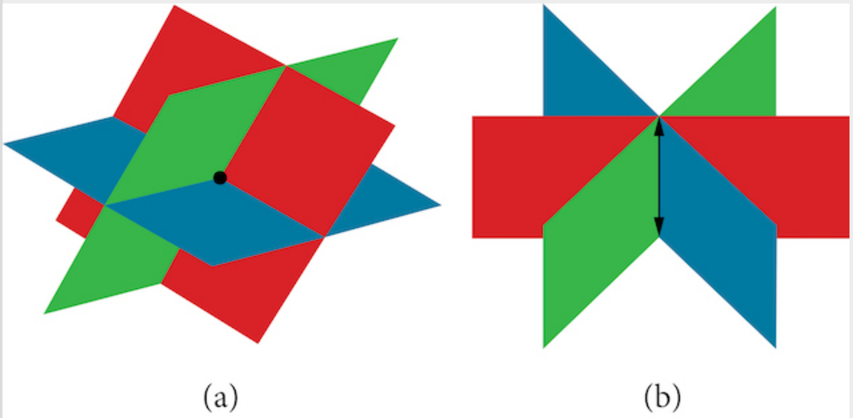
The **solution set** to a three-by-three system is an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

Note:

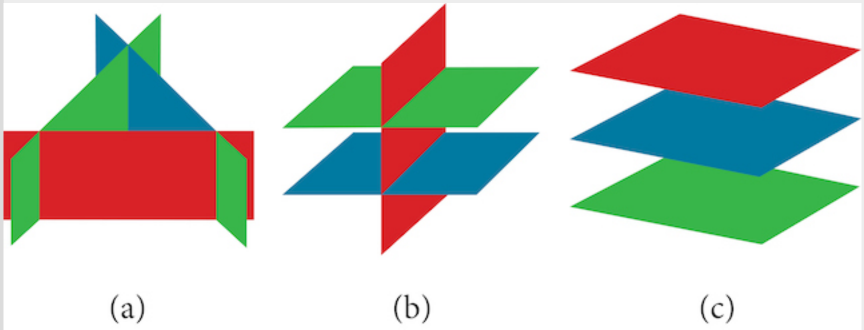
Number of Possible Solutions

[link] and [link] illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a **solution set** consisting of an ordered triple $\{(x, y, z)\}$. Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.



(a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.



All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example:
Exercise:

Problem:
Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3,-2,1)$ is a solution to the system.
Equation:

$$\begin{aligned} x+y+z&=2\\ 6x-4y+5z&=31\\ 5x+2y+2z&=13 \end{aligned}$$

Solution:

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$$\begin{aligned} x+y+z&=2\\ (3)+(-2)+(1)&=2\\ &\text{True} \end{aligned} \qquad \begin{aligned} 6x-4y+5z&=31\\ 6(3)-4(-2)+5(1)&=31\\ 18+8+5&=31\\ &\text{True} \end{aligned} \qquad \begin{aligned} 5x+2y+2z&=13\\ 5(3)+2(-2)+2(1)&=13\\ 15-4+2&=13\\ &\text{True} \end{aligned}$$

The ordered triple $(3,-2,1)$ is indeed a solution to the system.

Note:
Given a linear system of three equations, solve for three unknowns.

1. Pick any pair of equations and solve for one variable.

2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example:
Exercise:

Problem:
Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:
Equation:

$$\begin{aligned} x-2y+3z &= 9 \quad (1) \\ -x+3y-z &= -6 \quad (2) \\ 2x-5y+5z &= 17 \quad (3) \end{aligned}$$

Solution:

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).
Equation:

$$\begin{array}{l} x-2y+3z=9 \qquad (1) \\ \underline{\quad -x+3y-z=-6 \qquad (2)} \\ \qquad y+2z=3 \qquad (3) \end{array}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.
Equation:

$$\begin{array}{l} -2x+4y-6z=-18 \quad (1) \quad \text{multiplied} \quad \text{by} \quad -2 \\ \underline{\ \ 2x-5y+5z=17 \qquad (3) \qquad\qquad\qquad\qquad} \\ \ \ -y-z=-1 \quad (5) \end{array}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.
Equation:

$$\begin{array}{l} y+2z=3 \qquad (4) \\ \underline{-y-z=-1 \quad (5)} \\ \qquad z=2 \qquad (6) \end{array}$$

Choosing one equation from each new system, we obtain the upper triangular form:
Equation:

$$\begin{array}{ll} x - 2y + 3z = 9 & (1) \\ y + 2z = 3 & (4) \\ z = 2 & (6) \end{array}$$

Next, we back-substitute $z = 2$ into equation (4) and solve for y.

Equation:

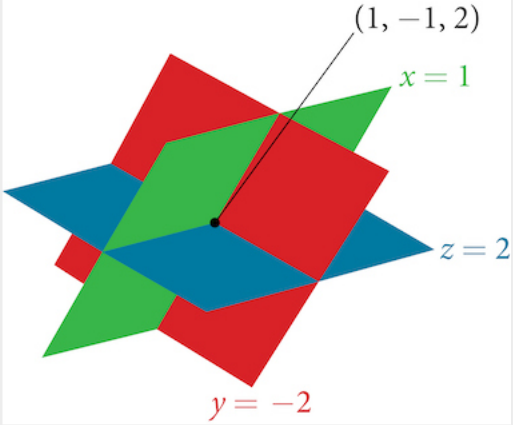
$$\begin{aligned} y + 2(2) &= 3 \\ y + 4 &= 3 \\ y &= -1 \end{aligned}$$

Finally, we can back-substitute $z = 2$ and $y = -1$ into equation (1). This will yield the solution for x.

Equation:

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ x + 2 + 6 &= 9 \\ x &= 1 \end{aligned}$$

The solution is the ordered triple $(1, -1, 2)$. See [link].



Example:
Exercise:

Problem:
Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $12,000 in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $4,000 more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $670. How much did he invest in each type of fund?

Solution:

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

Equation:

$$\begin{aligned} x &= \text{amount invested in money-market fund} \\ y &= \text{amount invested in municipal bonds} \\ z &= \text{amount invested in mutual funds} \end{aligned}$$

The first equation indicates that the sum of the three principal amounts is \$12,000.
Equation:

$$x + y + z = 12{,}000$$

We form the second equation according to the information that John invested \$4,000 more in mutual funds than he invested in municipal bonds.
Equation:

$$z = y + 4{,}000$$

The third equation shows that the total amount of interest earned from each fund equals \$670.
Equation:

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.
Equation:

$$\begin{aligned} x + y + z &= 12{,}000 \\ -y + z &= 4{,}000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,
Equation:

$$\begin{aligned} x + \quad y + z \quad &= 12{,}000 \quad (1) \\ -y + z \quad &= 4{,}000 \qquad (2) \\ 3x + 4y + 7z &= 67{,}000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.
Equation:

$$\begin{aligned} x + \quad y + \quad z &= 12{,}000 \\ 3x + 4y + 7z &= 67{,}000 \\ -y \quad + \quad z &= 4{,}000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.
Equation:

$$\begin{aligned} x + y + z \quad &= 12{,}000 \\ y + 4z &= 31{,}000 \\ -y + z \quad &= 4{,}000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

Equation:

$$\begin{aligned} x + y + \quad z &= 12{,}000 \\ y + 4z &= 31{,}000 \\ 5z \quad &= 35{,}000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

Equation:

$$\begin{aligned} 5z &= 35{,}000 \\ z &= 7{,}000 \end{aligned}$$

$$\begin{aligned} y + 4(7{,}000) &= 31{,}000 \\ y &= 3{,}000 \end{aligned}$$

$$\begin{aligned} x + 3{,}000 + 7{,}000 &= 12{,}000 \\ x &= 2{,}000 \end{aligned}$$

John invested \$2,000 in a money-market fund, \$3,000 in municipal bonds, and \$7,000 in mutual funds.

Note:

Exercise:

Problem: Solve the system of equations in three variables.

Equation:

$$\begin{aligned} 2x + y - 2z &= -1 \\ 3x - 3y - z &= 5 \\ x - 2y + 3z &= 6 \end{aligned}$$

Solution:

$(1, -1, 1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3 = 7$ or some other contradiction.

Example:
Exercise:

Problem:
Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.
Equation:

$$\begin{array}{rl} x - 3y + z = 4 & (1) \\ -x + 2y - 5z = 3 & (2) \\ 5x - 13y + 13z = 8 & (3) \end{array}$$

Solution:

Looking at the coefficients of x, we can see that we can eliminate x by adding equation (1) to equation (2).
Equation:

$$\begin{array}{rl} x - 3y + z = 4 & (1) \\ -x + 2y - 5z = 3 & (2) \\ \hline -y - 4z = 7 & (4) \end{array}$$

Next, we multiply equation (1) by -5 and add it to equation (3).
Equation:

$$\begin{array}{ll} -5x + 15y - 5z = -20 & (1) \quad \text{multiplied} \quad \text{by} \quad -5 \\ 5x - 13y + 13z = 8 & (3) \\ \hline 2y + 8z = -12 & (5) \end{array}$$

Then, we multiply equation (4) by 2 and add it to equation (5).
Equation:

$$\begin{array}{l} -2y - 8z = 14 (4) \quad \text{multiplied} \quad \text{by} \quad 2 \\ 2y + 8z = -12 \quad (5) \\ \hline 0 = 2 \end{array}$$

The final equation $0 = 2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Note:
Exercise:

Problem: Solve the system of three equations in three variables.
Equation:

$$\begin{aligned} x+y+z&=2 \\ y-3z&=1 \\ 2x+y+5z&=0 \end{aligned}$$

Solution:

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

Example:
Exercise:

Problem:
Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.
Equation:

$$\begin{aligned} 2x+y-3z&=0 \quad (1) \\ 4x+2y-6z&=0 \quad (2) \\ x-y+z&=0 \quad (3) \end{aligned}$$

Solution:

First, we can multiply equation (1) by -2 and add it to equation (2).
Equation:

$$\begin{array}{l} -4x-2y+6z=0 \quad \text{equation}(1) \quad \text{multiplied} \quad \text{by} \quad -2 \\ \underline{\quad 4x+2y-6z=0 \qquad (2) \qquad\qquad\qquad\qquad\qquad} \\ \quad 0=0 \end{array}$$

We do not need to proceed any further. The result we get is an identity, $0=0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0=0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have
Equation:

$$\begin{array}{r} 2x + y - 3z = 0 \\ x - y + z = 0 \\ \hline 3x - 2z = 0 \end{array}$$

We then solve the resulting equation for z.

Equation:

$$\begin{aligned} 3x - 2z &= 0 \\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

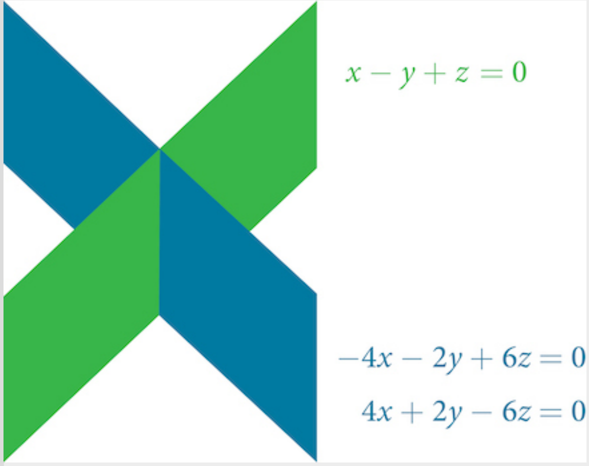
Equation:

$$\begin{aligned} 2x + y - 3\left(\frac{3}{2}x\right) &= 0 \\ 2x + y - \frac{9}{2}x &= 0 \\ y &= \frac{9}{2}x - 2x \\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in [link], two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.



Note:

Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Note:

Exercise:

Problem: Solve the following system.

Equation:

$$\begin{aligned} x+y+z&=7\\ 3x-2y-z&=4\\ x+6y+5z&=24 \end{aligned}$$

Solution:

Infinite number of solutions of the form $(x, 4x–11, -5x+18)$.

Note:
Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple $\{(x, y, z)\}$ that represents the intersection of three planes in space. See [link].
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See [link].
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See [link].
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See [link].
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See [link].
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Section Exercises

Verbal

Exercise:

Problem: Can a linear system of three equations have exactly two solutions? Explain why or why not

Solution:

No, there can be only one, zero, or infinitely many solutions.

Exercise:

Problem:

If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

Exercise:

Problem:

If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

Solution:

Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0,0,0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$\begin{aligned} 2x+3y-6z&=1 \\ -4x-6y+12z&=-2 \\ x+2y+5z&=10 \end{aligned}$$

Exercise:

Problem: Using the method of addition, is there only one way to solve the system?

Exercise:

Problem:

Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Solution:

Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

Algebraic

For the following exercises, determine whether the ordered triple given is the solution to the system of equations.

Exercise:

Problem: $\begin{aligned} 2x-6y+6z&=-12 \\ x+4y+5z&=-1 \\ -x+2y+3z&=-1 \end{aligned}$ and $(0,1,-1)$

Exercise:

Problem: $\begin{aligned} 6x-y+3z&=6 \\ 3x+5y+2z&=0 \\ x+y&=0 \end{aligned}$ and $(3,-3,-5)$

Solution:

No

Exercise:

Problem: $\begin{array}{r} 6x-7y+z=2 \\ -x-y+3z=4 \\ 2x+y-z=1 \end{array}$ and $(4,2,-6)$

Exercise:

Problem: $\begin{array}{r} x-y=0 \\ x-z=5 \\ x-y+z=-1 \end{array}$ and $(4,4,-1)$

Solution:

Yes

Exercise:

Problem: $\begin{array}{r} -x-y+2z=3 \\ 5x+8y-3z=4 \\ -x+3y-5z=-5 \end{array}$ and $(4,1,-7)$

For the following exercises, solve each system by elimination.

Exercise:

Problem: $\begin{array}{r} 3x-4y+2z=-15 \\ 2x+4y+z=16 \\ 2x+3y+5z=20 \end{array}$

Solution:

$(-1,4,2)$

Exercise:

Problem: $\begin{array}{r} 5x-2y+3z=20 \\ 2x-4y-3z=-9 \\ x+6y-8z=21 \end{array}$

Exercise:

Problem: $\begin{array}{r} 5x+2y+4z=9 \\ -3x+2y+z=10 \\ 4x-3y+5z=-3 \end{array}$

Solution:

$\left(-\frac{85}{107},\frac{312}{107},\frac{191}{107}\right)$

Exercise:

Problem:
$$\begin{aligned} 4x-3y+5z&=31\\ -x+2y+4z&=20\\ x+5y-2z&=-29 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 5x-2y+3z&=4\\ -4x+6y-7z&=-1\\ 3x+2y-z&=4 \end{aligned}$$

Solution:

$\left(1,\frac{1}{2},0\right)$

Exercise:

Problem:
$$\begin{aligned} 4x+6y+9z&=0\\ -5x+2y-6z&=3\\ 7x-4y+3z&=-3 \end{aligned}$$

For the following exercises, solve each system by Gaussian elimination.

Exercise:

Problem:
$$\begin{aligned} 2x-y+3z&=17\\ -5x+4y-2z&=-46\\ 2y+5z&=-7 \end{aligned}$$

Solution:

$(4,-6,1)$

Exercise:

Problem:
$$\begin{aligned} 5x-6y+3z&=50\\ -x+4y&=10\\ 2x-z&=10 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 2x+3y-6z&=1\\ -4x-6y+12z&=-2\\ x+2y+5z&=10 \end{aligned}$$

Solution:

$\left(x,\frac{1}{27}(65-16x),\frac{x+28}{27}\right)$

Exercise:

Problem:
$$\begin{aligned} 4x+6y-2z&=8\\ 6x+9y-3z&=12\\ -2x-3y+z&=-4 \end{aligned}$$

Exercise:

Problem:

$$\begin{gathered} 2x+3y-4z=5 \\ -3x+2y+z=11 \\ -x+5y+3z=4 \end{gathered}$$

Solution:

$\left(-\frac{45}{13},\frac{17}{13},-2\right)$

Exercise:

Problem:

$$\begin{gathered} 10x+2y-14z=8 \\ -\mathrm{x}-2y-4z=-1 \\ -12x-6y+6z=-12 \end{gathered}$$

Exercise:

Problem:

$$\begin{gathered} x+y+z=14 \\ 2y+3z=-14 \\ -16y-24z=-112 \end{gathered}$$

Solution:

No solutions exist

Exercise:

Problem:

$$\begin{gathered} 5x-3y+4z=-1 \\ -4x+2y-3z=0 \\ -\mathrm{x}+5y+7z=-11 \end{gathered}$$

Exercise:

Problem:

$$\begin{gathered} x+y+z=0 \\ 2x-y+3z=0 \\ x-z=0 \end{gathered}$$

Solution:

$(0,0,0)$

Exercise:

Problem:

$$\begin{gathered} 3x+2y-5z=6 \\ 5x-4y+3z=-12 \\ 4x+5y-2z=15 \end{gathered}$$

Exercise:

Problem:

$$\begin{gathered} x+y+z=0 \\ 2x-y+3z=0 \\ x-z=1 \end{gathered}$$

Solution:

$\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$

Exercise:

Problem:
$$\begin{array}{l} 3x - \frac{1}{2}y - z = -\frac{1}{2} \\ 4x + z = 3 \\ -x + \frac{3}{2}y = \frac{5}{2} \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} 6x - 5y + 6z = 38 \\ \frac{1}{5}x - \frac{1}{2}y + \frac{3}{5}z = 1 \\ -4x - \frac{3}{2}y - z = -74 \end{array}$$

Solution:

$(7, 20, 16)$

Exercise:

Problem:
$$\begin{array}{l} \frac{1}{2}x - \frac{1}{5}y + \frac{2}{5}z = -\frac{13}{10} \\ \frac{1}{4}x - \frac{2}{5}y - \frac{1}{5}z = -\frac{7}{20} \\ -\frac{1}{2}x - \frac{3}{4}y - \frac{1}{2}z = -\frac{5}{4} \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} -\frac{1}{3}x - \frac{1}{2}y - \frac{1}{4}z = \frac{3}{4} \\ -\frac{1}{2}x - \frac{1}{4}y - \frac{1}{2}z = 2 \\ -\frac{1}{4}x - \frac{3}{4}y - \frac{1}{2}z = -\frac{1}{2} \end{array}$$

Solution:

$(-6, 2, 1)$

Exercise:

Problem:
$$\begin{array}{l} \frac{1}{2}x - \frac{1}{4}y + \frac{3}{4}z = 0 \\ \frac{1}{4}x - \frac{1}{10}y + \frac{2}{5}z = -2 \\ \frac{1}{8}x + \frac{1}{5}y - \frac{1}{8}z = 2 \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} \frac{4}{5}x - \frac{7}{8}y + \frac{1}{2}z = 1 \\ -\frac{4}{5}x - \frac{3}{4}y + \frac{1}{3}z = -8 \\ -\frac{2}{5}x - \frac{7}{8}y + \frac{1}{2}z = -5 \end{array}$$

Solution:

$(5, 12, 15)$

Exercise:

Problem:
$$\begin{array}{l} -\frac{1}{3}x - \frac{1}{8}y + \frac{1}{6}z = -\frac{4}{3} \\ -\frac{2}{3}x - \frac{7}{8}y + \frac{1}{3}z = -\frac{23}{3} \\ -\frac{1}{3}x - \frac{5}{8}y + \frac{5}{6}z = 0 \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} -\frac{1}{4}x - \frac{5}{4}y + \frac{5}{2}z = -5 \\ -\frac{1}{2}x - \frac{5}{3}y + \frac{5}{4}z = \frac{55}{12} \\ -\frac{1}{3}x - \frac{1}{3}y + \frac{1}{3}z = \frac{5}{3} \end{array}$$

Solution:

$(-5, -5, -5)$

Exercise:

Problem:
$$\begin{array}{c} \frac{1}{40}x + \frac{1}{60}y + \frac{1}{80}z = \frac{1}{100} \\ -\frac{1}{2}x - \frac{1}{3}y - \frac{1}{4}z = -\frac{1}{5} \\ \frac{3}{8}x + \frac{3}{12}y + \frac{3}{16}z = \frac{3}{20} \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} 0.1x - 0.2y + 0.3z = 2 \\ 0.5x - 0.1y + 0.4z = 8 \\ 0.7x - 0.2y + 0.3z = 8 \end{array}$$

Solution:

$(10, 10, 10)$

Exercise:

Problem:
$$\begin{array}{l} 0.2x + 0.1y - 0.3z = 0.2 \\ 0.8x + 0.4y - 1.2z = 0.1 \\ 1.6x + 0.8y - 2.4z = 0.2 \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} 1.1x + 0.7y - 3.1z = -1.79 \\ 2.1x + 0.5y - 1.6z = -0.13 \\ 0.5x + 0.4y - 0.5z = -0.07 \end{array}$$

Solution:

$\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$

Exercise:

Problem:
$$\begin{array}{l} 0.5x - 0.5y + 0.5z = 10 \\ 0.2x - 0.2y + 0.2z = 4 \\ 0.1x - 0.1y + 0.1z = 2 \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} 0.1x + 0.2y + 0.3z = 0.37 \\ 0.1x - 0.2y - 0.3z = -0.27 \\ 0.5x - 0.1y - 0.3z = -0.03 \end{array}$$

Solution:

$\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$

Exercise:

Problem:
$$\begin{array}{l} 0.5x - 0.5y - 0.3z = 0.13 \\ 0.4x - 0.1y - 0.3z = 0.11 \\ 0.2x - 0.8y - 0.9z = -0.32 \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} 0.5x + 0.2y - 0.3z = 1 \\ 0.4x - 0.6y + 0.7z = 0.8 \\ 0.3x - 0.1y - 0.9z = 0.6 \end{array}$$

Solution:

$(2, 0, 0)$

Exercise:

Problem:
$$\begin{array}{l} 0.3x + 0.3y + 0.5z = 0.6 \\ 0.4x + 0.4y + 0.4z = 1.8 \\ 0.4x + 0.2y + 0.1z = 1.6 \end{array}$$

Exercise:

Problem:
$$\begin{array}{l} 0.8x + 0.8y + 0.8z = 2.4 \\ 0.3x - 0.5y + 0.2z = 0 \\ 0.1x + 0.2y + 0.3z = 0.6 \end{array}$$

Solution:

$(1, 1, 1)$

Extensions

For the following exercises, solve the system for x, y, and z.

Exercise:

Problem:
$$\begin{aligned} x+y+z&=3 \\ \frac{x-1}{2}+\frac{y-3}{2}+\frac{z+1}{2}&=0 \\ \frac{x-2}{3}+\frac{y+4}{3}+\frac{z-3}{3}&=\frac{2}{3} \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} 5x-3y-\frac{z+1}{2}&=\frac{1}{2} \\ 6x+\frac{y-9}{2}+2z&=-3 \\ \frac{x+8}{2}-4y+z&=4 \end{aligned}$$

Solution:

$\left(\frac{128}{557}, \frac{23}{557}, \frac{28}{557}\right)$

Exercise:

Problem:
$$\begin{aligned} \frac{x+4}{7}-\frac{y-1}{6}+\frac{z+2}{3}&=1 \\ \frac{x-2}{4}+\frac{y+1}{8}-\frac{z+8}{12}&=0 \\ \frac{x+6}{3}-\frac{y+2}{3}+\frac{z+4}{2}&=3 \end{aligned}$$

Exercise:

Problem:
$$\begin{aligned} \frac{x-3}{6}+\frac{y+2}{2}-\frac{z-3}{3}&=2 \\ \frac{x+2}{4}+\frac{y-5}{2}+\frac{z+4}{2}&=1 \\ \frac{x+6}{2}-\frac{y-3}{2}+z+1&=9 \end{aligned}$$

Solution:

$(6, -1, 0)$

Exercise:

Problem:
$$\begin{aligned} \frac{x-1}{3}+\frac{y+3}{4}+\frac{z+2}{6}&=1 \\ 4x+3y-2z&=11 \\ 0.02x+0.015y-0.01z&=0.065 \end{aligned}$$

Real-World Applications

Exercise:

Problem:

Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?

Solution:

24, 36, 48

Exercise:

Problem:

Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

Exercise:

Problem:

At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

Solution:

70 grandparents, 140 parents, 190 children

Exercise:

Problem:

An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

Exercise:

Problem:

Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate's groceries were $2.10 more than your groceries. How much was each of your share of the groceries?

Solution:

Your share was $19.95, Sarah's share was $40, and your other roommate's share was $22.05.

Exercise:

Problem:

Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

Exercise:

Problem:

Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

Solution:

There are infinitely many solutions; we need more information

Exercise:

Problem:

At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

Exercise:

Problem:

A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50. The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

Solution:

500 students, 225 children, and 450 adults

Exercise:

Problem:

In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

Exercise:

Problem:

Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?

Solution:

The BMW was $49,636, the Jeep was $42,636, and the Toyota was $47,727.

Exercise:

Problem:

A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $3\frac{1}{8}\%$ simple interest, and one that paid $2\frac{1}{2}\%$ simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

Exercise:

Problem:

You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $34,000 in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

Solution:

$400,000 in the account that pays 3% interest, $500,000 in the account that pays 4% interest, and $100,000 in the account that pays 2% interest.

Exercise:

Problem:

You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $3,650 in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?

Exercise:

Problem:

The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Solution:

The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world's oil.

Exercise:

Problem:

The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Exercise:

Problem:

The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?[footnote]
"Oil reserves, production and consumption in 2001," accessed April 6, 2014, http://scaruffi.com/politics/oil.html.

Solution:

Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0%

Exercise:

Problem:

The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?[footnote]

"USA: The coming global oil crisis," accessed April 6, 2014, http://www.oilcrisis.com/us/.

Exercise:

Problem:

At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

Solution:

Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

Exercise:

Problem:

Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?[footnote]

"The United States Meat Industry at a Glance," accessed April 6, 2014, http://www.meatami.com/ht/d/sp/i/47465/pid/47465.

Glossary

solution set	the set of all ordered pairs or triples that satisfy all equations in a system of equations

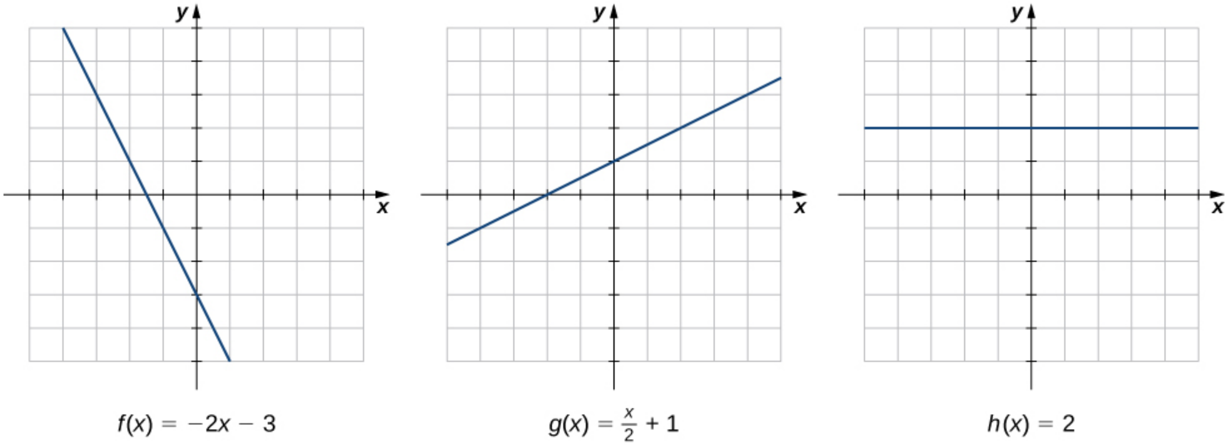
A Preview of Calculus

- Describe the tangent problem and how it led to the idea of a derivative.
- Explain how the idea of a limit is involved in solving the tangent problem.
- Recognize a tangent to a curve at a point as the limit of secant lines.
- Identify instantaneous velocity as the limit of average velocity over a small time interval.
- Describe the area problem and how it was solved by the integral.
- Explain how the idea of a limit is involved in solving the area problem.
- Recognize how the ideas of limit, derivative, and integral led to the studies of infinite series and multivariable calculus.

As we embark on our study of calculus, we shall see how its development arose from common solutions to practical problems in areas such as engineering physics—like the space travel problem posed in the chapter opener. Two key problems led to the initial formulation of calculus: (1) the tangent problem, or how to determine the slope of a line tangent to a curve at a point; and (2) the area problem, or how to determine the area under a curve.

The Tangent Problem and Differential Calculus

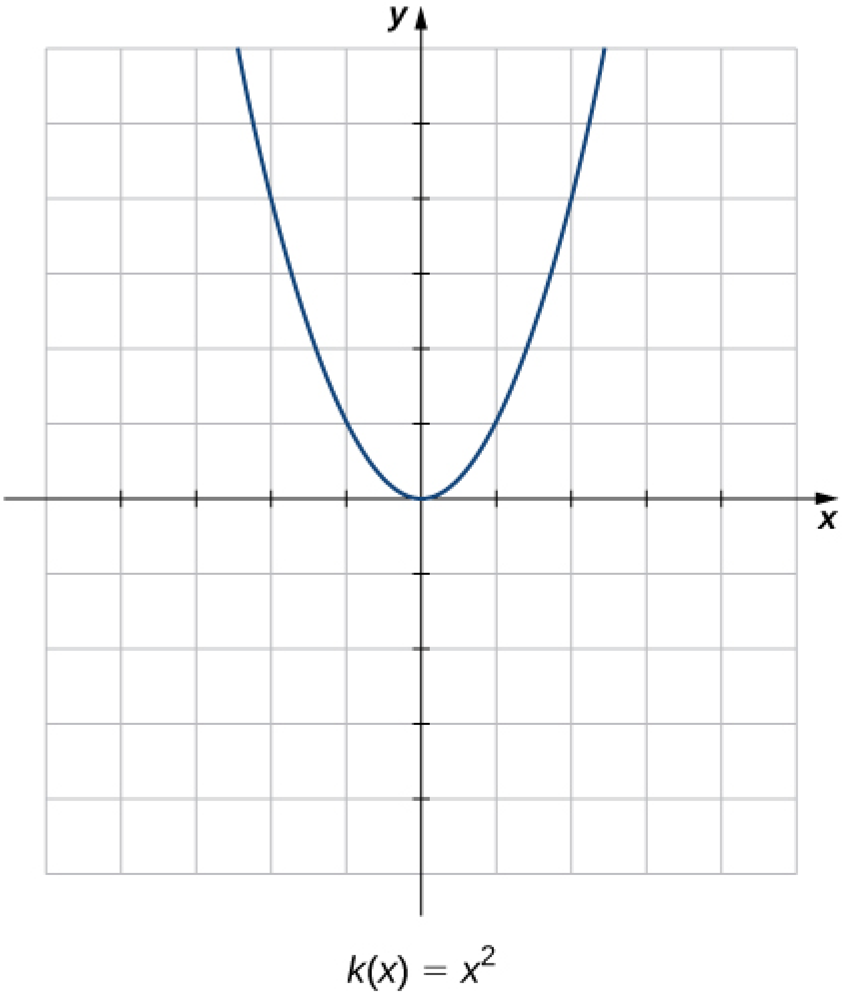
Rate of change is one of the most critical concepts in calculus. We begin our investigation of rates of change by looking at the graphs of the three lines $f\left(x\right)=-2x-3,g\left(x\right)=\frac{1}{2}x+1$, and $h\left(x\right)=2$, shown in [link].



The rate of change of a linear function is constant in each of these three graphs, with the constant determined by the slope.

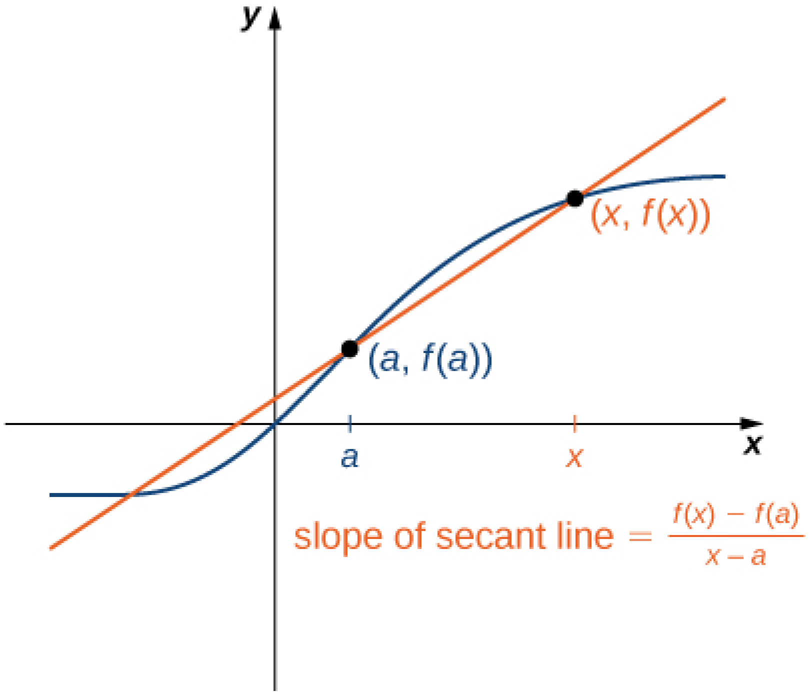
As we move from left to right along the graph of $f\left(x\right)=-2x-3$, we see that the graph decreases at a constant rate. For every 1 unit we move to the right along the *x*-axis, the *y*-coordinate decreases by 2 units. This rate of change is determined by the slope (−2) of the line. Similarly, the slope of 1/2 in the function $g\left(x\right)$ tells us that for every change in *x* of 1 unit there is a corresponding change in *y* of 1/2 unit. The function $h\left(x\right)=2$ has a slope of zero, indicating that the values of the function remain constant. We see that the slope of each linear function indicates the rate of change of the function.

Compare the graphs of these three functions with the graph of $k\left(x\right)={x}^{2}$ ([link]). The graph of $k\left(x\right)={x}^{2}$ starts from the left by decreasing rapidly, then begins to decrease more slowly and level off, and then finally begins to increase—slowly at first, followed by an increasing rate of increase as it moves toward the right. Unlike a linear function, no single number represents the rate of change for this function. We quite naturally ask: How do we measure the rate of change of a nonlinear function?



The function $k(x) = x^2$ does not have a constant rate of change.

We can approximate the rate of change of a function $f(x)$ at a point $(a, f(a))$ on its graph by taking another point $(x, f(x))$ on the graph of $f(x)$, drawing a line through the two points, and calculating the slope of the resulting line. Such a line is called a **secant** line. [link] shows a secant line to a function $f(x)$ at a point $(a, f(a))$.



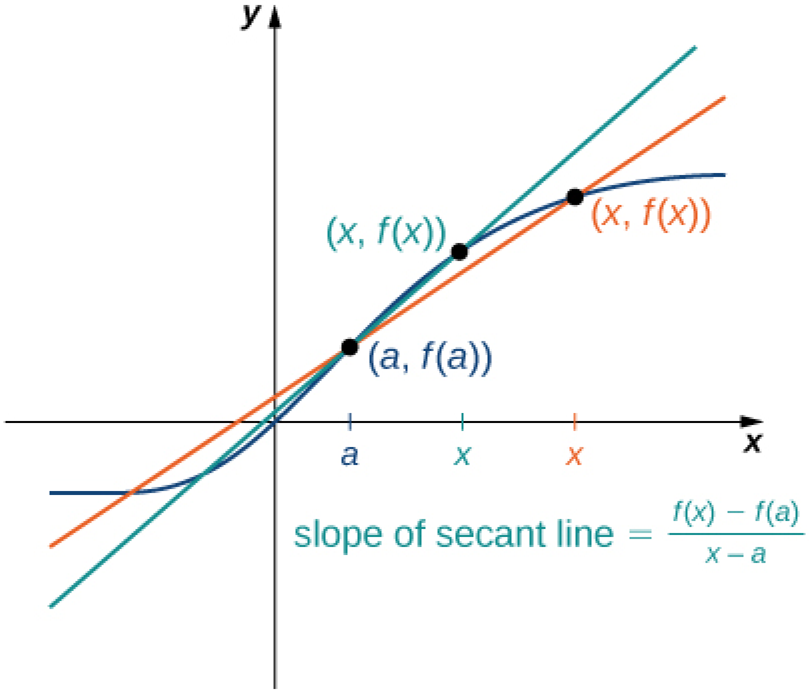
The slope of a secant line through a point $(a, f(a))$ estimates the rate of change of the function at the point $(a, f(a))$.

We formally define a secant line as follows:

Note:
Definition
The **secant** to the function $f(x)$ through the points $(a, f(a))$ and $(x, f(x))$ is the line passing through these points. Its slope is given by
Equation:

$$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}.$$

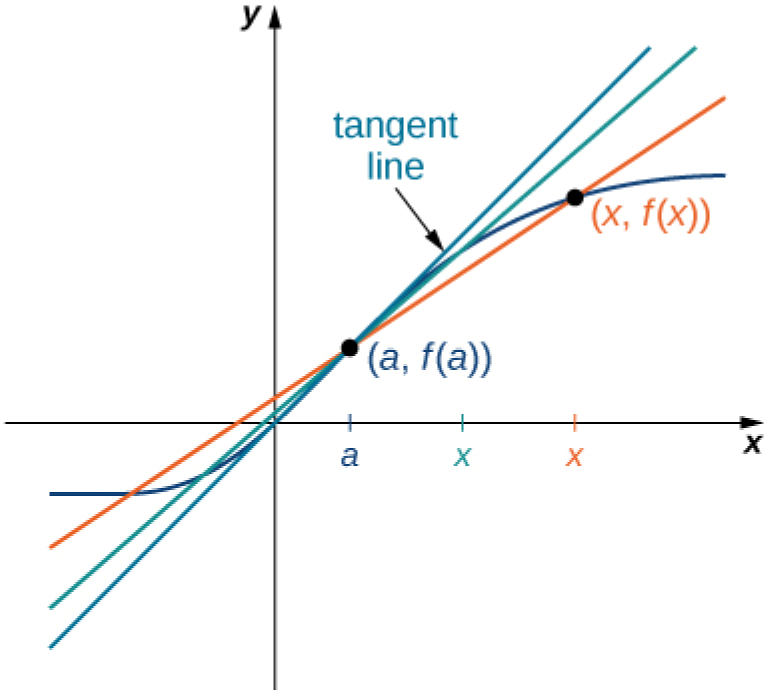
The accuracy of approximating the rate of change of the function with a secant line depends on how close *x* is to a. As we see in [link], if *x* is closer to a, the slope of the secant line is a better measure of the rate of change of $f(x)$ at a.



As *x* gets closer to a, the slope of the secant line becomes a better approximation to the rate of change of the function $f(x)$ at a.

The secant lines themselves approach a line that is called the **tangent** to the function $f(x)$ at *a* ([link]). The slope of the tangent line to the graph at *a* measures the rate of change of the function at a. This value also represents the derivative of the function $f(x)$ at a, or the rate of change of the function at a. This derivative is denoted by $f'(a)$. **Differential calculus** is the field of calculus concerned with the study of derivatives and their applications.

Note:
For an interactive demonstration of the slope of a secant line that you can manipulate yourself, visit this applet (*Note:* this site requires a Java browser plugin): Math Insight.



Solving the Tangent Problem: As *x* approaches *a*, the secant lines approach the tangent line.

[link] illustrates how to find slopes of secant lines. These slopes estimate the slope of the tangent line or, equivalently, the rate of change of the function at the point at which the slopes are calculated.

Example:
Exercise:

Problem:

Finding Slopes of Secant Lines

Estimate the slope of the tangent line (rate of change) to $f\left(x\right)={x}^{2}$ at $x=1$ by finding slopes of secant lines through $\left(1,1\right)$ and each of the following points on the graph of $f\left(x\right)={x}^{2}$.

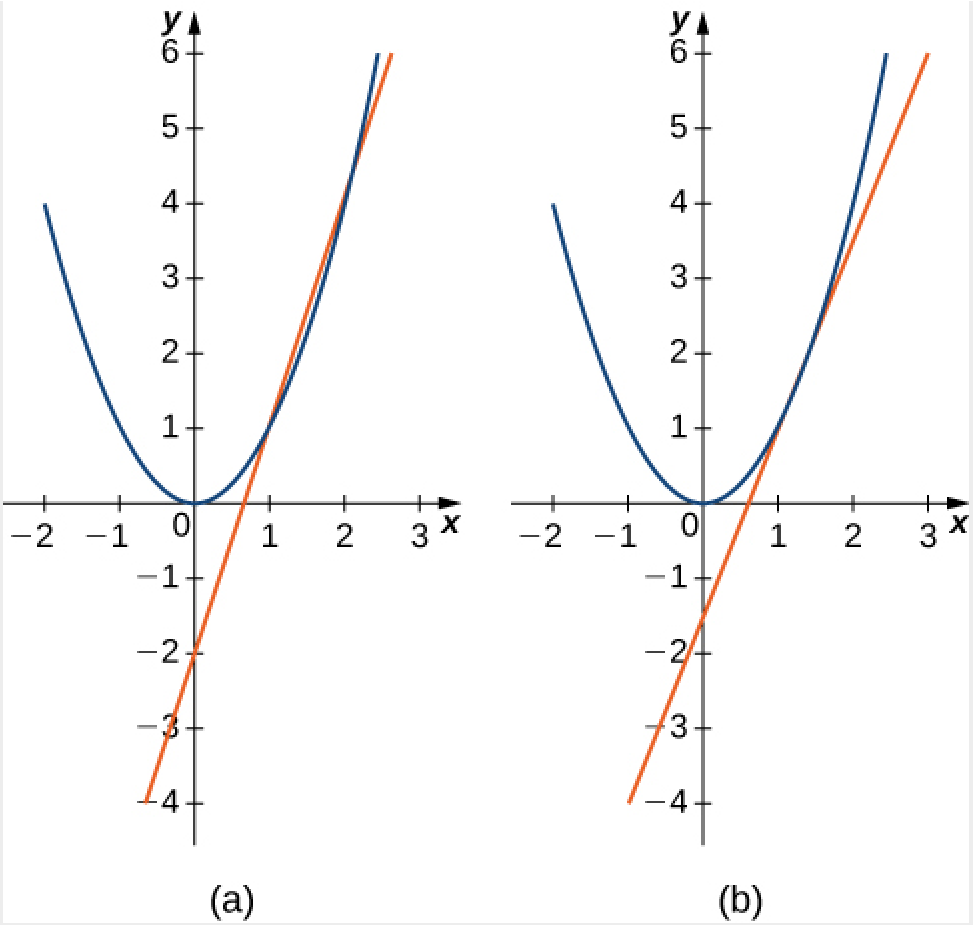
a. $\left(2,4\right)$
b. $\left(\frac{3}{2},\frac{9}{4}\right)$

Solution:

Use the formula for the slope of a secant line from the definition.

a. $m_{\text{sec}}=\frac{4-1}{2-1}=3$
b. $m_{\text{sec}}=\frac{\frac{9}{4}-1}{\frac{3}{2}-1}=\frac{5}{2}=2.5$

The point in part b. is closer to the point $\left(1,1\right)$, so the slope of 2.5 is closer to the slope of the tangent line. A good estimate for the slope of the tangent would be in the range of 2 to 2.5 ([link]).



The secant lines to $f(x) = x^2$ at $(1, 1)$ through (a) $(2, 4)$ and (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ provide successively closer approximations to the tangent line to $f(x) = x^2$ at $(1, 1)$.

Note:
Exercise:

Problem:

Estimate the slope of the tangent line (rate of change) to $f\left(x\right)={x}^{2}$ at $x=1$ by finding slopes of secant lines through $\left(1,1\right)$ and the point $\left(\frac{5}{4},\frac{25}{16}\right)$ on the graph of $f\left(x\right)={x}^{2}$.

Solution:

2.25

Hint

Use [link] and [link] as a solving guide.

We continue our investigation by exploring a related question. Keeping in mind that velocity may be thought of as the rate of change of position, suppose that we have a function, $s\left(t\right)$, that gives the position of an object along a coordinate axis at any given time *t*. Can we use these same ideas to create a reasonable definition of the instantaneous velocity at a given time $t=a$? We start by approximating the instantaneous velocity with an average velocity. First, recall that the speed of an object traveling at a constant rate is the ratio of the distance traveled to the length of time it has traveled. We define the **average velocity** of an object over a time period to be the change in its position divided by the length of the time period.

Note:
Definition
Let $s\left(t\right)$ be the position of an object moving along a coordinate axis at time *t*. The **average velocity** of the object over a time interval $\left[a,t\right]$ where $a<t$ (or $\left[t,a\right]$ if $t<a$) is
Equation:

$$v_{\text{ave}}=\frac{s\left(t\right)-s\left(a\right)}{t-a}.$$

As *t* is chosen closer to *a*, the average velocity becomes closer to the instantaneous velocity. Note that finding the average velocity of a position function over a time interval is essentially the same as finding the slope of a secant line to a function. Furthermore, to find the slope of a tangent line at a point *a*, we let the *x*-values approach *a* in the slope of the secant line. Similarly, to find the instantaneous velocity at time *a*, we let the *t*-values approach *a* in the average velocity. This process of letting *x* or *t* approach *a* in an expression is called taking a **limit**. Thus, we may define the **instantaneous velocity** as follows.

Note:

Definition

For a position function $s\left(t\right)$, the **instantaneous velocity** at a time $t=a$ is the value that the average velocities approach on intervals of the form $\left[a,t\right]$ and $\left[t,a\right]$ as the values of *t* become closer to *a*, provided such a value exists.

[link] illustrates this concept of limits and average velocity.

Example:

Exercise:

Problem:

Finding Average Velocity

A rock is dropped from a height of 64 ft. It is determined that its height (in feet) above ground *t* seconds later (for $0 \le t \le 2$) is given by $s\left(t\right) = -16t^2 + 64$. Find the average velocity of the rock over each of the given time intervals. Use this information to guess the instantaneous velocity of the rock at time $t = 0.5$.

a. $[0.49, 0.5]$
b. $[0.5, 0.51]$

Solution:

Substitute the data into the formula for the definition of average velocity.

a. $v_{\text{ave}} = \frac{s(0.5) - s(0.49)}{0.5 - 0.49} = -15.84$
b. $v_{\text{ave}} = \frac{s(0.51) - s(0.5)}{0.51 - 0.5} = -16.16$

The instantaneous velocity is somewhere between −15.84 and −16.16 ft/sec. A good guess might be −16 ft/sec.

Note:
Exercise:

Problem:

An object moves along a coordinate axis so that its position at time *t* is given by $s\left(t\right) = t^3$. Estimate its instantaneous velocity at time $t = 2$ by computing its average velocity over the time interval $[2, 2.001]$.

Solution:

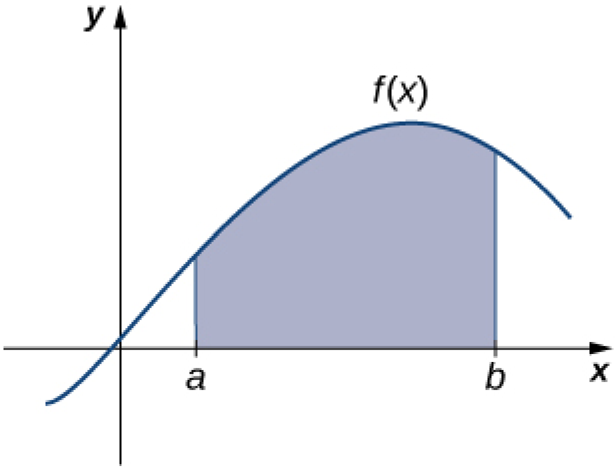
12.006001

Hint

Use $v_{\text{ave}} = \frac{s(2.001) - s(2)}{2.001 - 2}$.

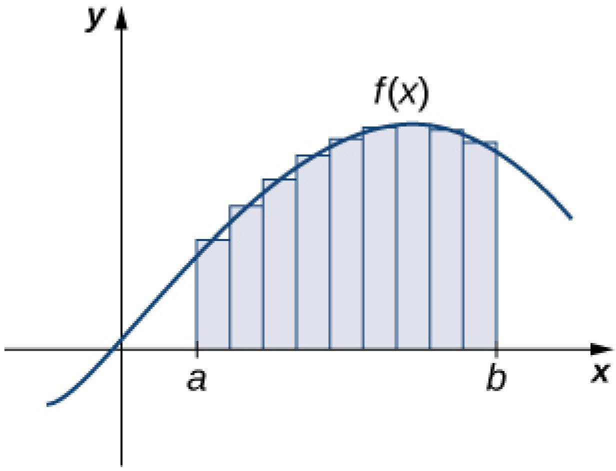
The Area Problem and Integral Calculus

We now turn our attention to a classic question from calculus. Many quantities in physics—for example, quantities of work—may be interpreted as the area under a curve. This leads us to ask the question: How can we find the area between the graph of a function and the *x*-axis over an interval ([link])?



The Area Problem: How do we find the area of the shaded region?

As in the answer to our previous questions on velocity, we first try to approximate the solution. We approximate the area by dividing up the interval $[a, b]$ into smaller intervals in the shape of rectangles. The approximation of the area comes from adding up the areas of these rectangles ([link]).



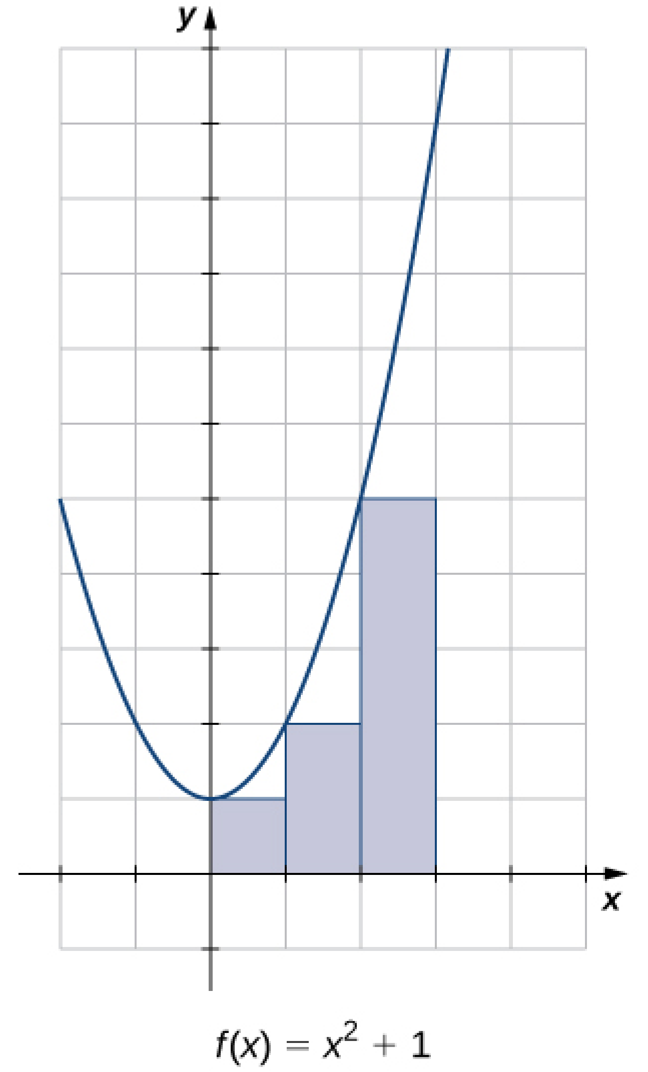
The area of the region under the curve is approximated by summing the areas of thin rectangles.

As the widths of the rectangles become smaller (approach zero), the sums of the areas of the rectangles approach the area between the graph of $f(x)$ and the *x*-axis over the interval $[a, b]$. Once again, we find ourselves taking a limit. Limits of this type serve as a basis for the definition of the definite integral. **Integral calculus** is the study of integrals and their applications.

Example:
Exercise:

Problem:
Estimation Using Rectangles

Estimate the area between the *x*-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown in [link].



The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.

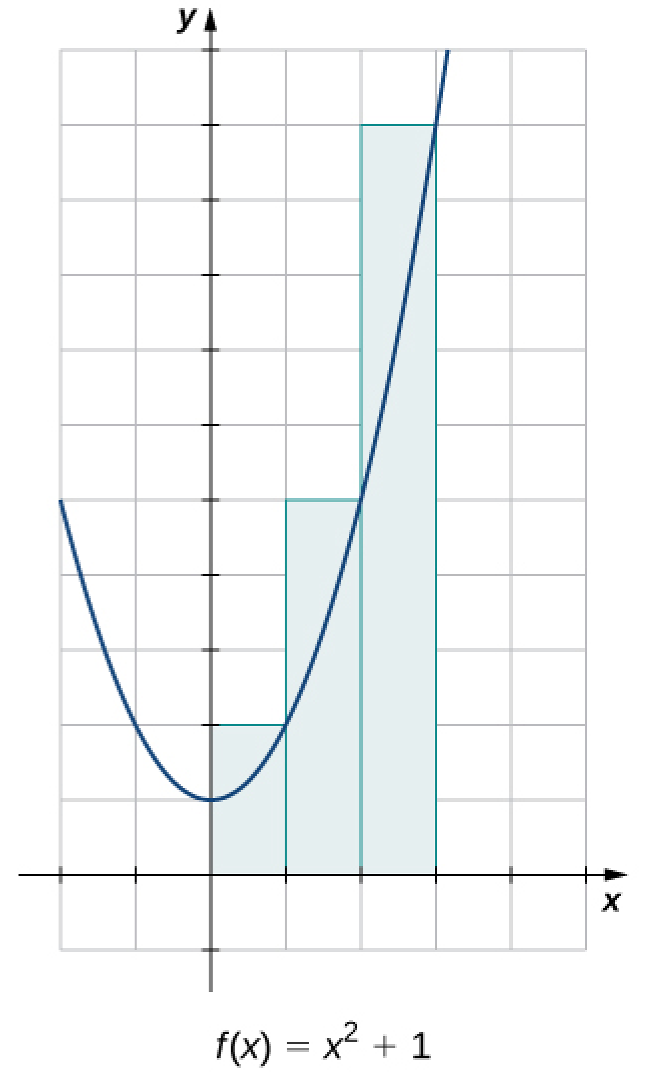
Solution:

The areas of the three rectangles are 1 unit^2, 2 unit^2, and 5 unit^2. Using these rectangles, our area estimate is 8 unit^2.

Note:
Exercise:

Problem:

Estimate the area between the *x*-axis and the graph of $f(x) = x^2 + 1$ over the interval $[0, 3]$ by using the three rectangles shown here:



$f(x) = x^2 + 1$

Solution:

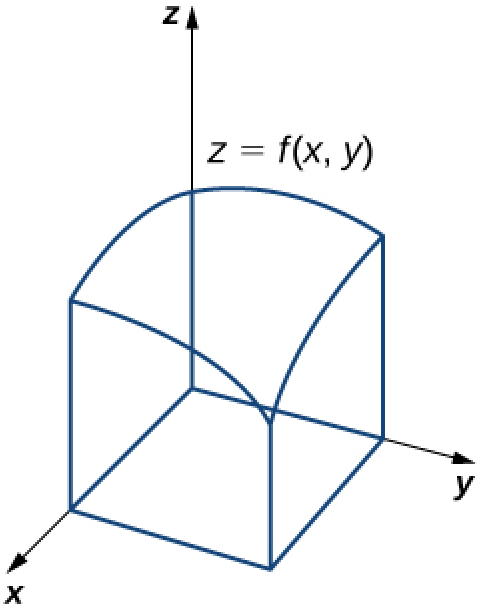
16 unit^2

Hint

Use [link] as a guide.

Other Aspects of Calculus

So far, we have studied functions of one variable only. Such functions can be represented visually using graphs in two dimensions; however, there is no good reason to restrict our investigation to two dimensions. Suppose, for example, that instead of determining the velocity of an object moving along a coordinate axis, we want to determine the velocity of a rock fired from a catapult at a given time, or of an airplane moving in three dimensions. We might want to graph real-value functions of two variables or determine volumes of solids of the type shown in [link]. These are only a few of the types of questions that can be asked and answered using **multivariable calculus**. Informally, multivariable calculus can be characterized as the study of the calculus of functions of two or more variables. However, before exploring these and other ideas, we must first lay a foundation for the study of calculus in one variable by exploring the concept of a limit.



We can use multivariable calculus

to find the volume between a
surface defined by a function of
two variables and a plane.

Key Concepts

- Differential calculus arose from trying to solve the problem of determining the slope of a line tangent to a curve at a point. The slope of the tangent line indicates the rate of change of the function, also called the *derivative*. Calculating a derivative requires finding a limit.
- Integral calculus arose from trying to solve the problem of finding the area of a region between the graph of a function and the *x*-axis. We can approximate the area by dividing it into thin rectangles and summing the areas of these rectangles. This summation leads to the value of a function called the *integral*. The integral is also calculated by finding a limit and, in fact, is related to the derivative of a function.
- Multivariable calculus enables us to solve problems in three-dimensional space, including determining motion in space and finding volumes of solids.

Key Equations

- **Slope of a Secant Line**
  $m_{\text{sec}} = \frac{f(x)-f(a)}{x-a}$
- **Average Velocity over Interval** $[a, t]$
  $v_{\text{ave}} = \frac{s(t)-s(a)}{t-a}$

For the following exercises, points $P\left(1, 2\right)$ and $Q\left(x, y\right)$ are on the graph of the function $f\left(x\right) = x^2 + 1$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q\,(x,y)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	***y***	$Q\,(x,y)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

Solution:

a. 2.2100000; b. 2.0201000; c. 2.0020010; d. 2.0002000; e. (1.1000000, 2.2100000); f. (1.0100000, 2.0201000); g. (1.0010000, 2.0020010); h. (1.0001000, 2.0002000); i. 2.1000000; j. 2.0100000; k. 2.0010000; l. 2.0001000

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to *f* at $x = 1$.

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point *P*. Graph $f\left(x\right)$ and the tangent line.

Solution:

$y=2x$

For the following exercises, points $P\left(1,1\right)$ and $Q\left(x,y\right)$ are on the graph of the function $f\left(x\right)={x}^{3}$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q\left(x,y\right)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	***y***	$Q\left(x,y\right)$	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $x = 1$.

Solution:

3

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point *P*. Graph $f\left(x\right)$ and the tangent line.

For the following exercises, points $P\left(4,2\right)$ and $Q\left(x,y\right)$ are on the graph of the function $f\left(x\right)=\sqrt{x}$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q\left(x,y\right)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	***y***	$Q\left(x,y\right)$	m_{sec}
4.1	a.	e.	i.

x	*y*	$Q(x, y)$	m_{sec}
4.01	b.	f.	j.
4.001	c.	g.	k.
4.0001	d.	h.	l.

Solution:

a. 2.0248457; b. 2.0024984; c. 2.0002500; d. 2.0000250; e. (4.1000000,2.0248457); f. (4.0100000,2.0024984); g. (4.0010000,2.0002500); h. (4.00010000,2.0000250); i. 0.24845673; j. 0.24984395; k. 0.24998438; l. 0.24999844

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $x = 4$.

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point *P*.

Solution:

$y = \frac{x}{4} + 1$

For the following exercises, points $P(1.5, 0)$ and $Q(\phi, y)$ are on the graph of the function $f(\phi) = \cos(\pi\phi)$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: *y*-coordinate of *Q*, the point $Q\left(\varphi,y\right)$, and the slope of the secant line passing through points *P* and *Q*. Round your answer to eight significant digits.

x	***y***	$Q\left(\phi,y\right)$	m_{sec}
1.4	a.	e.	i.
1.49	b.	f.	j.
1.499	c.	g.	k.
1.4999	d.	h.	l.

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the tangent line to *f* at $\varphi=1.5$.

Solution:

π

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point P.

For the following exercises, points $P\left(-1,-1\right)$ and $Q\left(x,y\right)$ are on the graph of the function $f\left(x\right)=\frac{1}{x}$.

Exercise:

Problem:

[T] Complete the following table with the appropriate values: y-coordinate of Q, the point $Q\left(x,y\right)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	$Q\left(x,y\right)$	m_{sec}
−1.05	a.	e.	i.
−1.01	b.	f.	j.
−1.005	c.	g.	k.
−1.001	d.	h.	l.

Solution:

a. −0.95238095; b. −0.99009901; c. −0.99502488; d. −0.99900100; e. (−1;.0500000,−0;.95238095); f. (−1;.0100000,−0;.9909901); g.

(−1;.0050000,−0;.99502488); h. (1.0010000,−0;.99900100); i. −0.95238095; j. −0.99009901; k. −0.99502488; l. −0.99900100

Exercise:

Problem:

Use the values in the right column of the table in the preceding exercise to guess the value of the slope of the line tangent to *f* at $x=-1$.

Exercise:

Problem:

Use the value in the preceding exercise to find the equation of the tangent line at point *P*.

Solution:

$y=-x-2$

For the following exercises, the position function of a ball dropped from the top of a 200-meter tall building is given by $s\left(t\right)=200-4.9t^2$, where position *s* is measured in meters and time *t* is measured in seconds. Round your answer to eight significant digits.

Exercise:

Problem:

[T] Compute the average velocity of the ball over the given time intervals.

- a. $[4.99,5]$
- b. $[5,5.01]$
- c. $[4.999,5]$
- d. $[5,5.001]$

Exercise:

Problem:

Use the preceding exercise to guess the instantaneous velocity of the ball at $t = 5$ sec.

Solution:

−49 m/sec (velocity of the ball is 49 m/sec downward)

For the following exercises, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h\left(t\right) = 15t - 4.9t^2$.

Exercise:

Problem:

[T] Compute the average velocity of the stone over the given time intervals.

a. $[1, 1.05]$
b. $[1, 1.01]$
c. $[1, 1.005]$
d. $[1, 1.001]$

Exercise:

Problem:

Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

Solution:

5.2 m/sec

For the following exercises, consider a rocket shot into the air that then returns to Earth. The height of the rocket in meters is given by

$h(t) = 600 + 78.4t - 4.9t^2$, where *t* is measured in seconds.

Exercise:

Problem:

[T] Compute the average velocity of the rocket over the given time intervals.

a. $[9, 9.01]$
b. $[8.99, 9]$
c. $[9, 9.001]$
d. $[8.999, 9]$

Exercise:

Problem:

Use the preceding exercise to guess the instantaneous velocity of the rocket at $t = 9$ sec.

Solution:

−9.8 m/sec

For the following exercises, consider an athlete running a 40-m dash. The position of the athlete is given by $d(t) = \frac{t^3}{6} + 4t$, where *d* is the position in meters and *t* is the time elapsed, measured in seconds.

Exercise:

Problem:

[T] Compute the average velocity of the runner over the given time intervals.

a. $[1.95, 2.05]$
b. $[1.995, 2.005]$
c. $[1.9995, 2.0005]$
d. $[2, 2.00001]$

Exercise:

Problem:

Use the preceding exercise to guess the instantaneous velocity of the runner at $t = 2$ sec.

Solution:

6 m/sec

For the following exercises, consider the function $f(x) = |x|$.

Exercise:

Problem:

Sketch the graph of *f* over the interval $[-1, 2]$ and shade the region above the *x*-axis.

Exercise:

Problem:

Use the preceding exercise to find the aproximate value of the area between the *x*-axis and the graph of *f* over the interval $[-1, 2]$ using rectangles. For the rectangles, use the square units, and approximate both above and below the lines. Use geometry to find the exact answer.

Solution:

Under, 1 unit^2; over: 4 unit^2. The exact area of the two triangles is $\frac{1}{2}(1)(1) + \frac{1}{2}(2)(2) = 2.5\,\text{units}^2$.

For the following exercises, consider the function $f(x) = \sqrt{1 - x^2}$. (*Hint*: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)

Exercise:

Problem: Sketch the graph of *f* over the interval $[-1, 1]$.

Exercise:

Problem:

Use the preceding exercise to find the aproximate area between the *x*-axis and the graph of *f* over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

Solution:

Under, 0.96 unit^2; over, 1.92 unit^2. The exact area of the semicircle with radius 1 is $\frac{\pi(1)^2}{2} = \frac{\pi}{2}$ unit^2.

For the following exercises, consider the function $f(x) = -x^2 + 1$.

Exercise:

Problem: Sketch the graph of *f* over the interval $[-1, 1]$.

Exercise:

Problem:

Approximate the area of the region between the *x*-axis and the graph of *f* over the interval $[-1, 1]$.

Solution:

Approximately 1.3333333 unit^2

Glossary

average velocity

the change in an object's position divided by the length of a time period; the average velocity of an object over a time interval $[t, a]$ (if $t < a$ or $[a, t]$ if $t > a$), with a position given by $s(t)$, that is $v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}$

differential calculus

the field of calculus concerned with the study of derivatives and their applications

instantaneous velocity

The instantaneous velocity of an object with a position function that is given by $s(t)$ is the value that the average velocities on intervals of the form $[t, a]$ and $[a, t]$ approach as the values of t move closer to a, provided such a value exists

integral calculus

the study of integrals and their applications

limit

the process of letting x or t approach a in an expression; the limit of a function $f(x)$ as x approaches a is the value that $f(x)$ approaches as x approaches a

multivariable calculus

the study of the calculus of functions of two or more variables

secant

A secant line to a function $f(x)$ at a is a line through the point $(a, f(a))$ and another point on the function; the slope of the secant line is given by $m_{\text{sec}} = \frac{f(x)-f(a)}{x-a}$

tangent

A tangent line to the graph of a function at a point $(a, f(a))$ is the line that secant lines through $(a, f(a))$ approach as they are taken through points on the function with x-values that approach a; the slope of the tangent line to a graph at a measures the rate of change of the function at a

The Limit of a Function

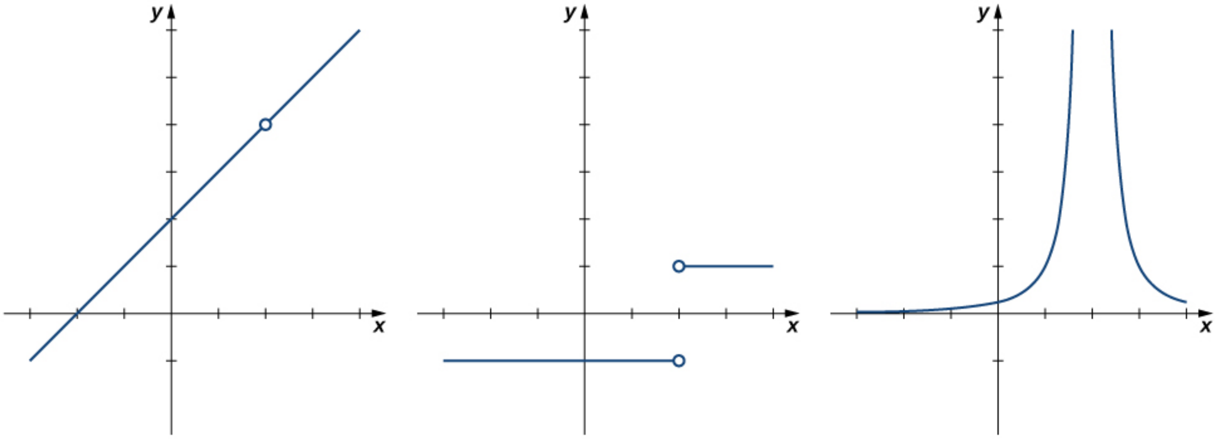
- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.
- Define one-sided limits and provide examples.
- Explain the relationship between one-sided and two-sided limits.
- Using correct notation, describe an infinite limit.
- Define a vertical asymptote.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We therefore begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach. At the end of this chapter, armed with a conceptual understanding of limits, we examine the formal definition of a limit.

We begin our exploration of limits by taking a look at the graphs of the functions
Equation:

$$f\left(x\right)=\frac{x^2-4}{x-2},\ g\left(x\right)=\frac{|x-2|}{x-2},\ \text{and}\ h\left(x\right)=\frac{1}{{\left(x-2\right)}^{2}},$$

which are shown in [link]. In particular, let’s focus our attention on the behavior of each graph at and around $x=2$.



$f(x) = \frac{x^2 - 4}{x - 2}$ (a)

$g(x) = \frac{|x - 2|}{x - 2}$ (b)

$h(x) = \frac{1}{(x - 2)^2}$ (c)

These graphs show the behavior of three different functions around $x = 2$.

Each of the three functions is undefined at $x = 2$, but if we make this statement and no other, we give a very incomplete picture of how each function behaves in the vicinity of $x = 2$. To express the behavior of each graph in the vicinity of 2 more completely, we need to introduce the concept of a limit.

Intuitive Definition of a Limit

Let's first take a closer look at how the function $f\left(x\right) = \left(x^2 - 4\right)/\left(x - 2\right)$ behaves around $x = 2$ in [link]. As the values of *x* approach 2 from either side of 2, the values of $y = f\left(x\right)$ approach 4. Mathematically, we say that the limit of $f\left(x\right)$ as *x* approaches 2 is 4. Symbolically, we express this limit as

Equation:

$$\lim_{x \to 2} f\left(x\right) = 4.$$

From this very brief informal look at one limit, let's start to develop an **intuitive definition of the limit**. We can think of the limit of a function at a number *a* as being the one real number *L* that the functional values approach as the *x*-values approach *a*, provided such a real number *L* exists. Stated more carefully, we have the following definition:

Note:
Definition

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. If *all* values of the function $f(x)$ approach the real number *L* as the values of $x\ (\neq a)$ approach the number *a*, then we say that the limit of $f(x)$ as *x* approaches *a* is *L*. (More succinct, as *x* gets closer to *a*, $f(x)$ gets closer and stays close to *L*.) Symbolically, we express this idea as

Equation:

$$\lim_{x \to a} f(x) = L.$$

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

Note:
Problem-Solving Strategy: Evaluating a Limit Using a Table of Functional Values

1. To evaluate $\lim_{x \to a} f(x)$, we begin by completing a table of functional values. We should choose two sets of *x*-values—one set of values approaching *a* and less than *a*, and another set of values approaching *a* and greater than *a*. [link] demonstrates what your tables might look like.

x	$f(x)$		***x***	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$

x	$f\left(x\right)$		**x**	$f\left(x\right)$
Use additional values as necessary.			Use additional values as necessary.	

Table of Functional Values for $\underset{x\to a}{\lim}f\left(x\right)$

2. Next, let’s look at the values in each of the $f\left(x\right)$ columns and determine whether the values seem to be approaching a single value as we move down each column. In our columns, we look at the sequence $f\left(a-0.1\right),f\left(a-0.01\right),f\left(a-0.001\right).,f\left(a-0.0001\right)$, and so on, and $f\left(a+0.1\right),f\left(a+0.01\right),f\left(a+0.001\right),f\left(a+0.0001\right)$, and so on. (*Note*: Although we have chosen the *x*-values $a\pm 0.1,a\pm 0.01,a\pm 0.001,a\pm 0.0001$, and so forth, and these values will probably work nearly every time, on very rare occasions we may need to modify our choices.)
3. If both columns approach a common *y*-value *L*, we state $\underset{x\to a}{\lim}f\left(x\right)=L$. We can use the following strategy to confirm the result obtained from the table or as an alternative method for estimating a limit.
4. Using a graphing calculator or computer software that allows us graph functions, we can plot the function $f\left(x\right)$, making sure the functional values of $f\left(x\right)$ for *x*-values near *a* are in our window. We can use the trace feature to move along the graph of the function and watch the *y*-value readout as the *x*-values approach *a*. If the *y*-values approach *L* as our *x*-values approach *a* from both directions, then $\underset{x\to a}{\lim}f\left(x\right)=L$. We may need to zoom in on our graph and repeat this process several times.

We apply this Problem-Solving Strategy to compute a limit in [link].

Example:
Exercise:

Problem:
Evaluating a Limit Using a Table of Functional Values 1

Evaluate $\underset{x\to 0}{\lim}\frac{\sin x}{x}$ using a table of functional values.

Solution:

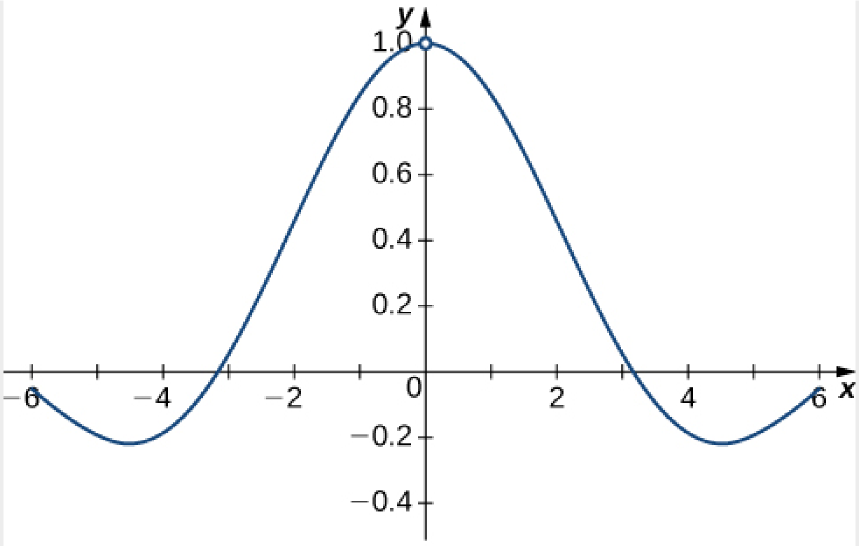
We have calculated the values of $f(x) = (\sin x)/x$ for the values of x listed in [link].

x	$\frac{\sin x}{x}$		x	$\frac{\sin x}{x}$
−0.1	0.998334166468		0.1	0.998334166468
−0.01	0.999983333417		0.01	0.999983333417
−0.001	0.999999833333		0.001	0.999999833333
−0.0001	0.999999998333		0.0001	0.999999998333

Table of Functional Values for $\lim\limits_{x \to 0} \frac{\sin x}{x}$

Note: The values in this table were obtained using a calculator and using all the places given in the calculator output.

As we read down each $\frac{(\sin x)}{x}$ column, we see that the values in each column appear to be approaching one. Thus, it is fairly reasonable to conclude that $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$. A calculator or computer-generated graph of $f(x) = \frac{(\sin x)}{x}$ would be similar to that shown in [link], and it confirms our estimate.



The graph of $f(x) = (\sin x)/x$ confirms the estimate from [link].

Example:

Exercise:

Problem:
Evaluating a Limit Using a Table of Functional Values 2

Evaluate $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Solution:

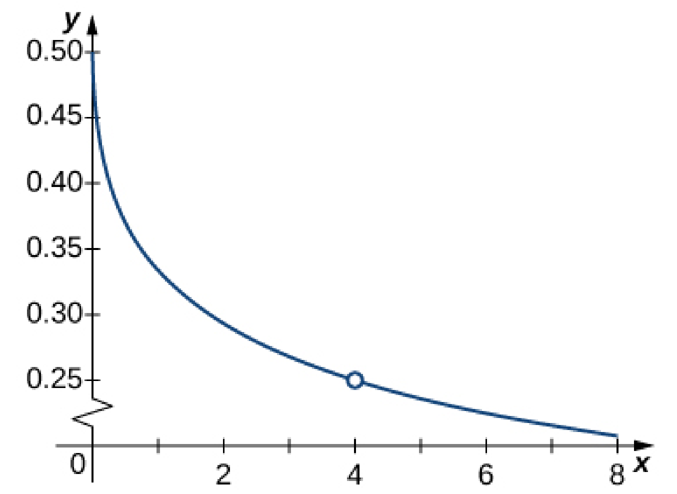
As before, we use a table—in this case, [link]—to list the values of the function for the given values of *x*.

x	$\frac{\sqrt{x}-2}{x-4}$		***x***	$\frac{\sqrt{x}-2}{x-4}$

x	$\frac{\sqrt{x}-2}{x-4}$		**x**	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869		4.1	0.248456731317
3.99	0.25015644562		4.01	0.24984394501
3.999	0.250015627		4.001	0.249984377
3.9999	0.250001563		4.0001	0.249998438
3.99999	0.25000016		4.00001	0.24999984

Table of Functional Values for $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$

After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25. We conclude that $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4} = 0.25$. We confirm this estimate using the graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$ shown in [link].



The graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$ confirms the estimate from [link].

Note:
Exercise:

Problem:

Estimate $\lim\limits_{x\to1}\frac{\frac{1}{x}-1}{x-1}$ using a table of functional values. Use a graph to confirm your estimate.

Solution:

$$\lim\limits_{x\to1}\frac{\frac{1}{x}-1}{x-1}=-1$$

Hint

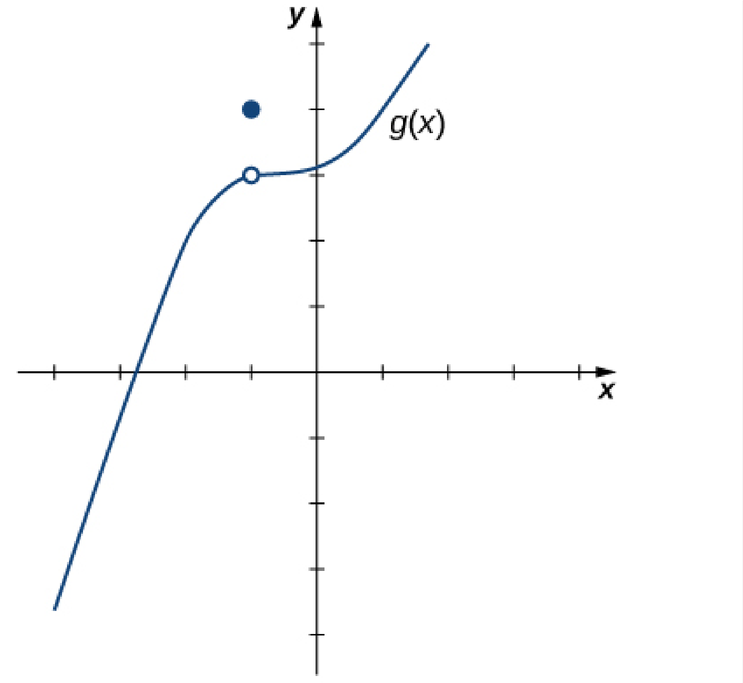
Use 0.9, 0.99, 0.999, 0.9999, 0.99999 and 1.1, 1.01, 1.001, 1.0001, 1.00001 as your table values.

At this point, we see from [link] and [link] that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In [link], we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

Example:
Exercise:

Problem:
Evaluating a Limit Using a Graph

For $g\left(x\right)$ shown in [link], evaluate $\lim\limits_{x\to-1}g\left(x\right)$.



The graph of $g(x)$ includes one value not on a smooth curve.

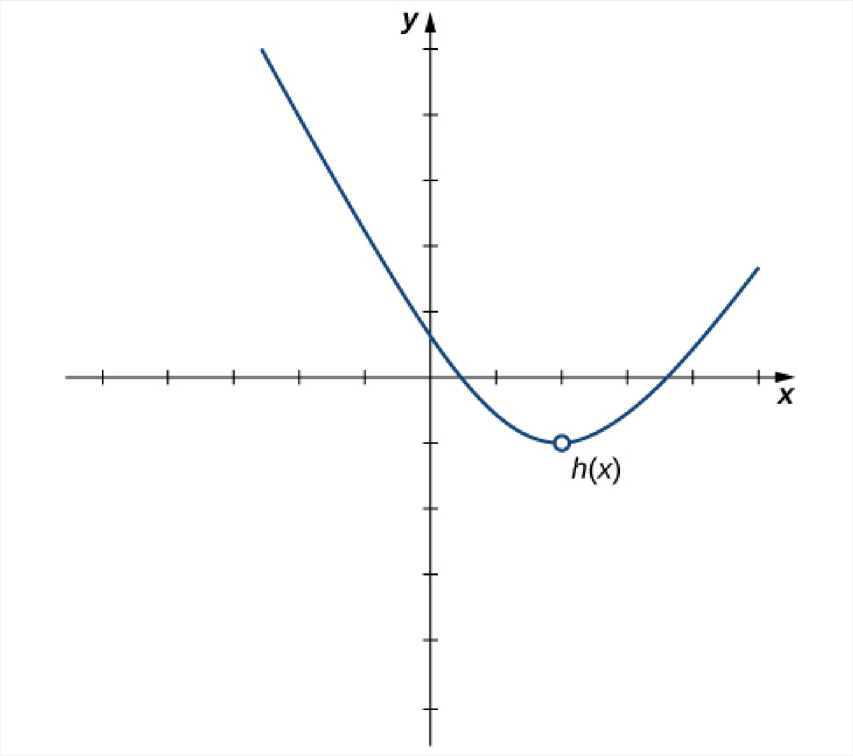
Solution:

Despite the fact that $g(-1)=4$, as the *x*-values approach −1 from either side, the $g(x)$ values approach 3. Therefore, $\lim\limits_{x\to-1} g(x)=3.$ Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on [link], we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

Note:
Exercise:

Problem: Use the graph of $h\left(x\right)$ in [link] to evaluate $\underset{x\to 2}{\lim}h\left(x\right)$, if possible.



Solution:

$\underset{x\to 2}{\lim}h\left(x\right)=-1.$

Hint

What *y*-value does the function approach as the *x*-values approach 2?

Looking at a table of functional values or looking at the graph of a function provides us with useful insight into the value of the limit of a function at a given point. However, these techniques rely too much on guesswork. We eventually need to develop alternative methods of evaluating limits. These new methods are more algebraic in nature and we explore them in the next section; however, at this point we introduce two special limits that are foundational to the techniques to come.

Note:
Two Important Limits

Let *a* be a real number and *c* be a constant.

i. **Equation:**

$$\lim_{x \to a} x = a$$

ii. **Equation:**

$$\lim_{x \to a} c = c$$

We can make the following observations about these two limits.

i. For the first limit, observe that as *x* approaches *a*, so does $f(x)$, because $f(x) = x$. Consequently, $\lim_{x \to a} x = a$.
ii. For the second limit, consider [link].

x	$f(x) = c$		x	$f(x) = c$
$a - 0.1$	c		$a + 0.1$	c
$a - 0.01$	c		$a + 0.01$	c
$a - 0.001$	c		$a + 0.001$	c
$a - 0.0001$	c		$a + 0.0001$	c

Table of Functional Values for $\lim_{x \to a} c = c$

Observe that for all values of *x* (regardless of whether they are approaching *a*), the values $f(x)$ remain constant at *c*. We have no choice but to conclude $\lim_{x \to a} c = c$.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

Example:
Exercise:

Problem:
Evaluating a Limit That Fails to Exist

Evaluate $\lim\limits_{x \to 0} \sin{(1/x)}$ using a table of values.

Solution:

[link] lists values for the function $\sin(1/x)$ for the given values of *x*.

x	$\sin\left(\frac{1}{x}\right)$		*x*	$\sin\left(\frac{1}{x}\right)$
−0.1	0.544021110889		0.1	−0.544021110889
−0.01	0.50636564111		0.01	−0.50636564111
−0.001	−0.8268795405312		0.001	0.826879540532
−0.0001	0.305614388888		0.0001	−0.305614388888
−0.00001	−0.035748797987		0.00001	0.035748797987
−0.000001	0.349993504187		0.000001	−0.349993504187

Table of Functional Values for $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$

After examining the table of functional values, we can see that the *y*-values do not seem to approach any one single value. It appears the limit does not exist.

Before drawing this conclusion, let’s take a more systematic approach. Take the following sequence of *x*-values approaching 0:

Equation:

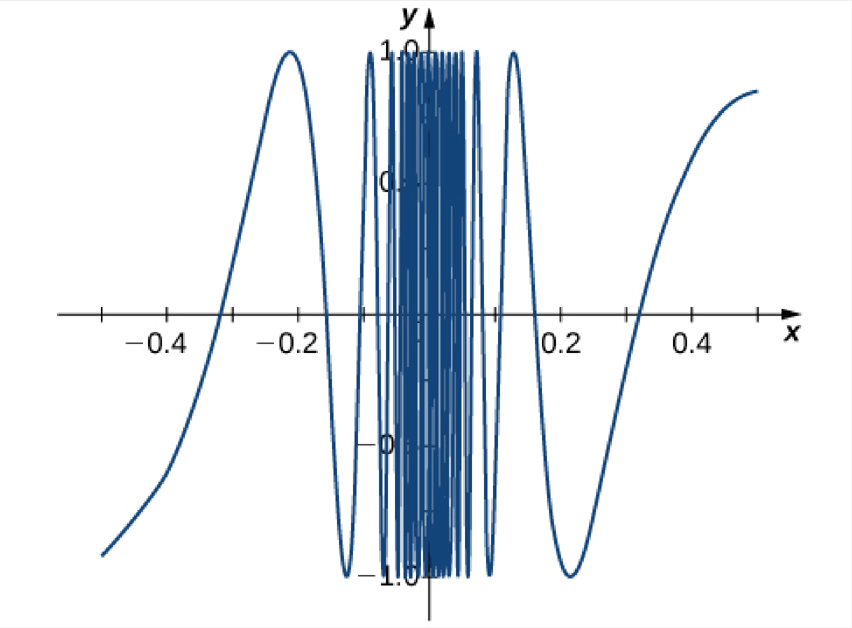
$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots .$$

The corresponding *y*-values are

Equation:

$$1, -1, 1, -1, 1, -1, \ldots .$$

At this point we can indeed conclude that $\lim\limits_{x \to 0} \sin\,(1/x)$ does not exist. (Mathematicians frequently abbreviate “does not exist” as DNE. Thus, we would write $\lim\limits_{x \to 0} \sin\,(1/x)$ DNE.) The graph of $f\,(x) = \sin\,(1/x)$ is shown in [link] and it gives a clearer picture of the behavior of $\sin(1/x)$ as *x* approaches 0. You can see that $\sin(1/x)$ oscillates ever more wildly between −1 and 1 as *x* approaches 0.



The graph of $f\,(x) = \sin(1/x)$ oscillates rapidly between −1 and 1 as *x* approaches 0.

Note:
Exercise:

Problem: Use a table of functional values to evaluate $\lim\limits_{x \to 2} \frac{|x^2-4|}{x-2}$, if possible.

Solution:

$\lim\limits_{x \to 2} \frac{|x^2-4|}{x-2}$ does not exist.

Hint

Use *x*-values 1.9, 1.99, 1.999, 1.9999, 1.9999 and 2.1, 2.01, 2.001, 2.0001, 2.00001 in your table.

One-Sided Limits

Sometimes indicating that the limit of a function fails to exist at a point does not provide us with enough information about the behavior of the function at that particular point. To see this, we now revisit the function $g\left(x\right)=|x-2|/\left(x-2\right)$ introduced at the beginning of the section (see [link](b)). As we pick values of *x* close to 2, $g\left(x\right)$ does not approach a single value, so the limit as *x* approaches 2 does not exist—that is, $\lim\limits_{x \to 2} g\left(x\right)$ DNE. However, this statement alone does not give us a complete picture of the behavior of the function around the *x*-value 2. To provide a more accurate description, we introduce the idea of a **one-sided limit**. For all values to the left of 2 (or *the negative side of* 2), $g\left(x\right)=-1$. Thus, as *x* approaches 2 from the left, $g\left(x\right)$ approaches −1. Mathematically, we say that the limit as *x* approaches 2 from the left is −1. Symbolically, we express this idea as

Equation:

$$\lim_{x \to 2^-} g\left(x\right)=-1.$$

Similarly, as *x* approaches 2 from the right (or *from the positive side*), $g\left(x\right)$ approaches 1. Symbolically, we express this idea as

Equation:

$$\lim_{x \to 2^+} g\left(x\right)=1.$$

We can now present an informal definition of one-sided limits.

Note:
Definition
We define two types of **one-sided limits**.
Limit from the left: Let $f\left(x\right)$ be a function defined at all values in an open interval of the form (*c*, *a*), and let *L* be a real number. If the values of the function $f\left(x\right)$ approach the real number *L* as the values of *x* (where $x<a$) approach the number *a*, then we say that *L* is the limit of $f\left(x\right)$ as *x* approaches a from the left. Symbolically, we express this idea as
Equation:

$$\lim_{x\to a^-} f\left(x\right)=L.$$

Limit from the right: Let $f\left(x\right)$ be a function defined at all values in an open interval of the form $\left(a,c\right)$, and let *L* be a real number. If the values of the function $f\left(x\right)$ approach the real number L as the values of *x* (where $x>a$) approach the number *a*, then we say that *L* is the limit of $f\left(x\right)$ as *x* approaches *a* from the right. Symbolically, we express this idea as
Equation:

$$\lim_{x\to a^+} f\left(x\right)=L.$$

Example:
Exercise:

Problem:
Evaluating One-Sided Limits

For the function $f\left(x\right)=\begin{cases} x+1 & \text{if } x<2 \\ x^2-4 & \text{if } x\ge 2 \end{cases}$, evaluate each of the following limits.

a. $\lim_{x\to 2^-} f\left(x\right)$
b. $\lim_{x\to 2^+} f\left(x\right)$

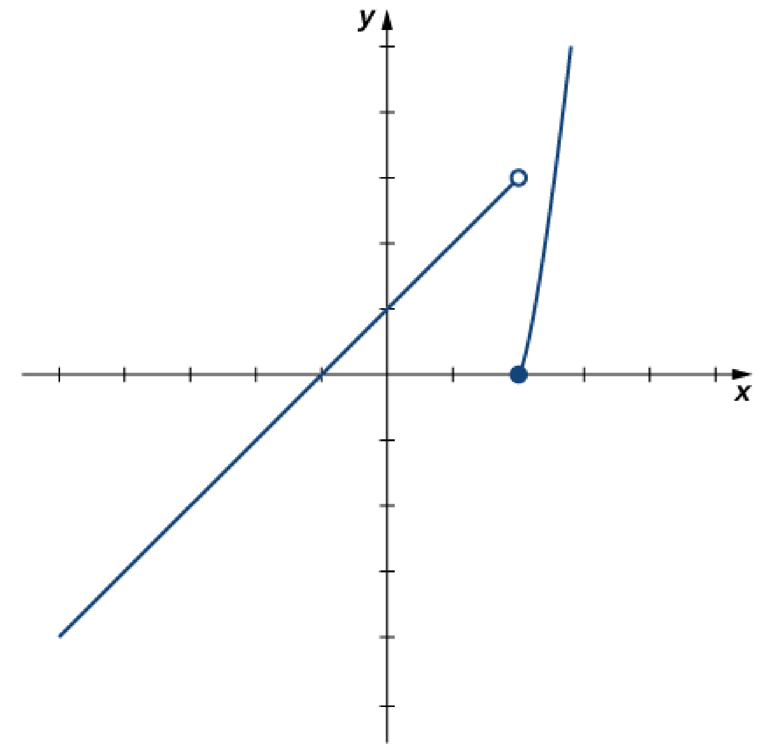
Solution:

We can use tables of functional values again [link]. Observe that for values of *x* less than 2, we use $f\left(x\right)=x+1$ and for values of *x* greater than 2, we use $f\left(x\right)={x}^{2}-4.$

x	$f(x)=x+1$		**x**	$f(x)=x^2{-}4$
1.9	2.9		2.1	0.41
1.99	2.99		2.01	0.0401
1.999	2.999		2.001	0.004001
1.9999	2.9999		2.0001	0.00040001
1.99999	2.99999		2.00001	0.0000400001

Table of Functional Values for $f\left(x\right)=\begin{cases}x+1 \text{ if } x<2\\ x^2-4 \text{ if } x\ge 2\end{cases}$

Based on this table, we can conclude that a. $\underset{x\to {2}^{-}}{\lim}f\left(x\right)=3$ and b. $\underset{x\to {2}^{+}}{\lim}f\left(x\right)=0.$ Therefore, the (two-sided) limit of $f\left(x\right)$ does not exist at $x=2.$ [link] shows a graph of $f\left(x\right)$ and reinforces our conclusion about these limits.



The graph of $f(x) = \begin{cases} x+1 \text{ if } x<2 \\ x^2-4 \text{ if } x \geq 2 \end{cases}$ has a break at $x = 2$.

Note:
Exercise:

Problem:

Use a table of functional values to estimate the following limits, if possible.

a. $\lim\limits_{x \to 2^-} \frac{|x^2-4|}{x-2}$
b. $\lim\limits_{x \to 2^+} \frac{|x^2-4|}{x-2}$

Solution:

a. $\lim\limits_{x \to 2^-} \frac{|x^2-4|}{x-2} = -4$; b. $\lim\limits_{x \to 2^+} \frac{|x^2-4|}{x-2} = 4$

Hint

a. Use *x*-values 1.9, 1.99, 1.999, 1.9999, 1.9999 to estimate $\lim\limits_{x \to 2^-} \frac{|x^2-4|}{x-2}$.

b. Use *x*-values 2.1, 2.01, 2.001, 2.0001, 2.00001 to estimate $\lim\limits_{x \to 2^+} \frac{|x^2-4|}{x-2}$.

(These tables are available from a previous Checkpoint problem.)

Let us now consider the relationship between the limit of a function at a point and the limits from the right and left at that point. It seems clear that if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. Similarly, if the limit from the left and the limit from the right take on different values, the limit of the function does not exist. These conclusions are summarized in [link].

Note:
Relating One-Sided and Two-Sided Limits
Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. Then,
Equation:

$$\lim_{x \to a} f(x) = L. \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Infinite Limits

Evaluating the limit of a function at a point or evaluating the limit of a function from the right and left at a point helps us to characterize the behavior of a function around a given value. As we shall see, we can also describe the behavior of functions that do not have finite limits.

We now turn our attention to $h(x) = 1/(x-2)^2$, the third and final function introduced at the beginning of this section (see [link](c)). From its graph we see that as the values of *x* approach 2, the values of $h(x) = 1/(x-2)^2$ become larger and larger and, in fact, become infinite. Mathematically, we say that the limit of $h(x)$ as *x* approaches 2 is positive infinity. Symbolically, we express this idea as

Equation:

$$\lim_{x \to 2} h(x) = +\infty.$$

More generally, we define **infinite limits** as follows:

Note:

Definition

We define three types of **infinite limits**.

Infinite limits from the left: Let $f(x)$ be a function defined at all values in an open interval of the form (b, a).

i. If the values of $f(x)$ increase without bound as the values of *x* (where $x < a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the left is positive infinity and we write

   Equation:

$$\lim_{x \to a^-} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of *x* (where $x < a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the left is negative infinity and we write

   Equation:

$$\lim_{x \to a^-} f(x) = -\infty.$$

Infinite limits from the right: Let $f(x)$ be a function defined at all values in an open interval of the form (a, c).

i. If the values of $f(x)$ increase without bound as the values of *x* (where $x > a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the right is positive infinity and we write

   Equation:

$$\lim_{x \to a^+} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of *x* (where $x > a$) approach the number *a*, then we say that the limit as *x* approaches *a* from the right is negative infinity and we write
Equation:

$$\lim_{x \to a^+} f(x) = -\infty.$$

Two-sided infinite limit: Let $f(x)$ be defined for all $x \neq a$ in an open interval containing *a*.

i. If the values of $f(x)$ increase without bound as the values of *x* (where $x \neq a$) approach the number *a*, then we say that the limit as *x* approaches *a* is positive infinity and we write
Equation:

$$\lim_{x \to a} f(x) = +\infty.$$

ii. If the values of $f(x)$ decrease without bound as the values of *x* (where $x \neq a$) approach the number *a*, then we say that the limit as *x* approaches *a* is negative infinity and we write
Equation:

$$\lim_{x \to a} f(x) = -\infty.$$

It is important to understand that when we write statements such as $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$ we are describing the behavior of the function, as we have just defined it. We are not asserting that a limit exists. For the limit of a function $f(x)$ to exist at *a*, it must approach a real number *L* as *x* approaches *a*. That said, if, for example, $\lim_{x \to a} f(x) = +\infty$, we always write $\lim_{x \to a} f(x) = +\infty$ rather than $\lim_{x \to a} f(x)$ DNE.

Example:
Exercise:

Problem:

Recognizing an Infinite Limit

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f(x) = 1/x$ to confirm your conclusion.

a. $\lim\limits_{x \to 0^-} \frac{1}{x}$
b. $\lim\limits_{x \to 0^+} \frac{1}{x}$
c. $\lim\limits_{x \to 0} \frac{1}{x}$

Solution:

Begin by constructing a table of functional values.

x	$\frac{1}{x}$		***x***	$\frac{1}{x}$
−0.1	−10		0.1	10
−0.01	−100		0.01	100
−0.001	−1000		0.001	1000
−0.0001	−10,000		0.0001	10,000
−0.00001	−100,000		0.00001	100,000
−0.000001	−1,000,000		0.000001	1,000,000

Table of Functional Values for $f(x) = \frac{1}{x}$

a. The values of $1/x$ decrease without bound as *x* approaches 0 from the left. We conclude that
Equation:

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

b. The values of $1/x$ increase without bound as *x* approaches 0 from the right. We conclude that

Equation:

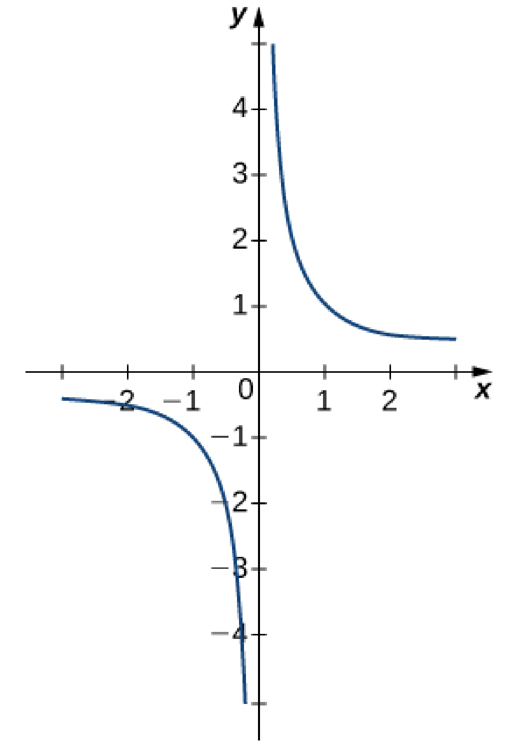
$$\lim_{x \to 0^+} \frac{1}{x} = +\infty.$$

c. Since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ have different values, we conclude that

Equation:

$$\lim_{x \to 0} \frac{1}{x} \text{ DNE.}$$

The graph of $f(x) = 1/x$ in [link] confirms these conclusions.



The graph of $f(x) = 1/x$ confirms that the limit as *x* approaches 0 does not exist.

Note:
Exercise:

Problem:

Evaluate each of the following limits, if possible. Use a table of functional values and graph $f\left(x\right)=1/x^2$ to confirm your conclusion.

a. $\underset{x\to 0^-}{\lim}\frac{1}{x^2}$
b. $\underset{x\to 0^+}{\lim}\frac{1}{x^2}$
c. $\underset{x\to 0}{\lim}\frac{1}{x^2}$

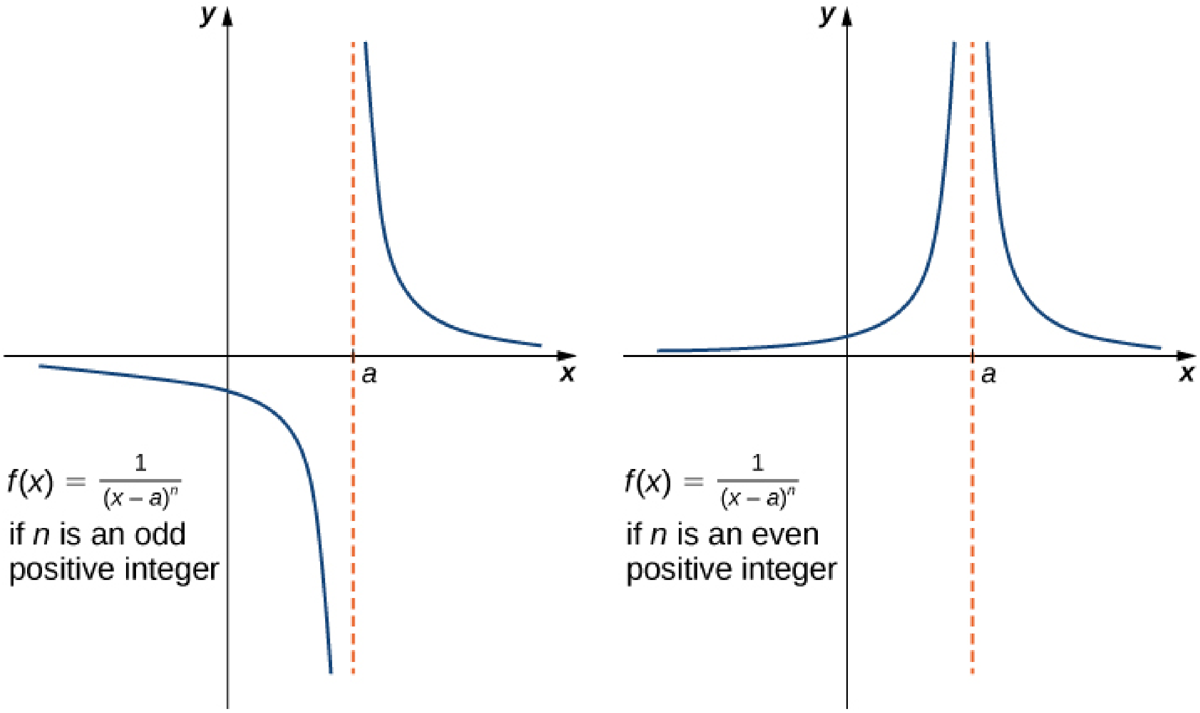
Solution:

a. $\underset{x\to 0^-}{\lim}\frac{1}{x^2}=+\infty$; b. $\underset{x\to 0^+}{\lim}\frac{1}{x^2}=+\infty$; c. $\underset{x\to 0}{\lim}\frac{1}{x^2}=+\infty$

Hint

Follow the procedures from [link].

It is useful to point out that functions of the form $f\left(x\right)=1/{\left(x-a\right)}^{n}$, where *n* is a positive integer, have infinite limits as *x* approaches *a* from either the left or right ([link]). These limits are summarized in [link].



The function $f(x) = 1/(x-a)^n$ has infinite limits at a.

Note:

Infinite Limits from Positive Integers

If n is a positive even integer, then

Equation:

$$\lim_{x \to a} \frac{1}{(x-a)^n} = +\infty.$$

If n is a positive odd integer, then

Equation:

$$\lim_{x \to a^+} \frac{1}{(x-a)^n} = +\infty$$

and

Equation:

$$\lim_{x \to a^-} \frac{1}{(x-a)^n} = -\infty.$$

We should also point out that in the graphs of $f(x) = 1/(x-a)^n$, points on the graph having *x*-coordinates very near to *a* are very close to the vertical line $x = a$. That is, as *x* approaches *a*, the points on the graph of $f(x)$ are closer to the line $x = a$. The line $x = a$ is called a **vertical asymptote** of the graph. We formally define a vertical asymptote as follows:

Note:
Definition
Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical asymptote** of $f(x)$.
Equation:

$$\begin{array}{rcl} \lim_{x \to a^-} f(x) & = & +\infty \text{ or } -\infty \\ \lim_{x \to a^+} f(x) & = & +\infty \text{ or } -\infty \\ & \text{or} & \\ \lim_{x \to a} f(x) & = & +\infty \text{ or } -\infty \end{array}$$

Example:
Exercise:

Problem:
Finding a Vertical Asymptote

Evaluate each of the following limits using [link]. Identify any vertical asymptotes of the function $f(x) = 1/(x+3)^4$.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4}$
b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4}$
c. $\lim_{x \to -3} \frac{1}{(x+3)^4}$

Solution:

We can use [link] directly.

a. $\lim_{x \to -3^-} \frac{1}{(x+3)^4} = +\infty$
b. $\lim_{x \to -3^+} \frac{1}{(x+3)^4} = +\infty$
c. $\lim_{x \to -3} \frac{1}{(x+3)^4} = +\infty$

The function $f(x) = 1/(x+3)^4$ has a vertical asymptote of $x = -3$.

Note:
Exercise:

Problem:

Evaluate each of the following limits. Identify any vertical asymptotes of the function $f(x) = \frac{1}{(x-2)^3}$.

a. $\lim_{x \to 2^-} \frac{1}{(x-2)^3}$
b. $\lim_{x \to 2^+} \frac{1}{(x-2)^3}$
c. $\lim_{x \to 2} \frac{1}{(x-2)^3}$

Solution:

a. $\lim_{x \to 2^-} \frac{1}{(x-2)^3} = -\infty$; b. $\lim_{x \to 2^+} \frac{1}{(x-2)^3} = +\infty$; c. $\lim_{x \to 2} \frac{1}{(x-2)^3}$ DNE. The line $x = 2$ is the vertical asymptote of $f(x) = 1/(x-2)^3$.

Hint

Use [link].

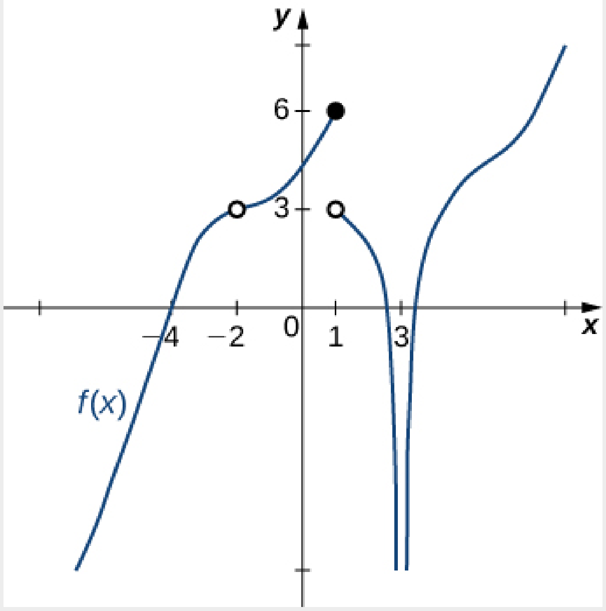
In the next example we put our knowledge of various types of limits to use to analyze the behavior of a function at several different points.

Example:
Exercise:

Problem:
Behavior of a Function at Different Points

Use the graph of $f(x)$ in [link] to determine each of the following values:

a. $\lim_{x\to -4^-} f(x)$; $\lim_{x\to -4^+} f(x)$; $\lim_{x\to -4} f(x)$; $f(-4)$
b. $\lim_{x\to -2^-} f(x)$; $\lim_{x\to -2^+} f(x)$; $\lim_{x\to -2} f(x)$; $f(-2)$
c. $\lim_{x\to 1^-} f(x)$; $\lim_{x\to 1^+} f(x)$; $\lim_{x\to 1} f(x)$; $f(1)$
d. $\lim_{x\to 3^-} f(x)$; $\lim_{x\to 3^+} f(x)$; $\lim_{x\to 3} f(x)$; $f(3)$



The graph shows $f(x)$.

Solution:

Using [link] and the graph for reference, we arrive at the following values:

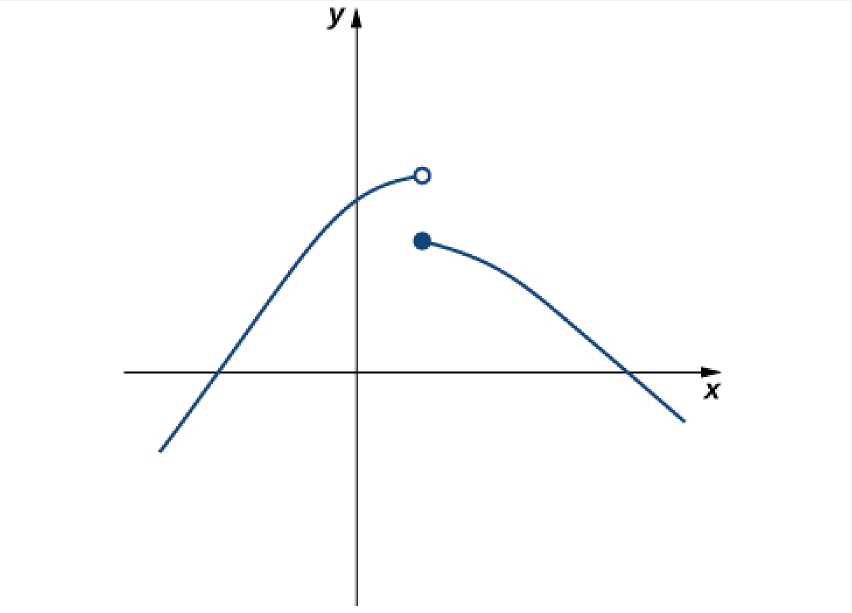
a. $\lim_{x\to -4^-} f(x) = 0$; $\lim_{x\to -4^+} f(x) = 0$; $\lim_{x\to -4} f(x) = 0$; $f(-4) = 0$
b. $\lim_{x\to -2^-} f(x) = 3.$; $\lim_{x\to -2^+} f(x) = 3$; $\lim_{x\to -2} f(x) = 3$; $f(-2)$ is undefined

c. $\lim_{x\to 1^-} f(x) = 6$; $\lim_{x\to 1^+} f(x) = 3$; $\lim_{x\to 1} f(x)$ DNE; $f(1) = 6$

d. $\lim_{x\to 3^-} f(x) = -\infty$; $\lim_{x\to 3^+} f(x) = -\infty$; $\lim_{x\to 3} f(x) = -\infty$; $f(3)$ is undefined

Note:
Exercise:

Problem: Evaluate $\lim_{x\to 1} f(x)$ for $f(x)$ shown here:



Solution:

Does not exist.

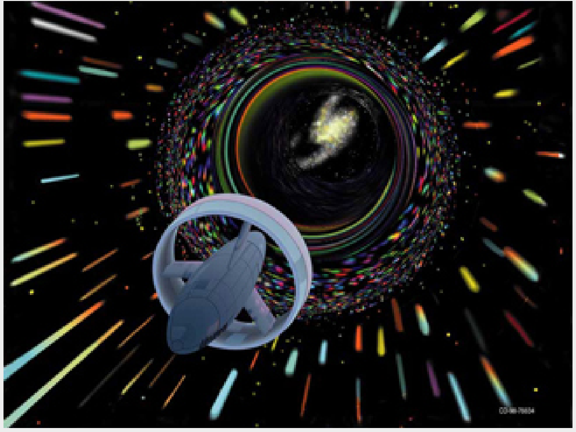
Hint

Compare the limit from the right with the limit from the left.

Example:
Exercise:

Problem:

Chapter Opener: Einstein’s Equation



(credit: NASA)

In the chapter opener we mentioned briefly how Albert Einstein showed that a limit exists to how fast any object can travel. Given Einstein’s equation for the mass of a moving object, what is the value of this bound?

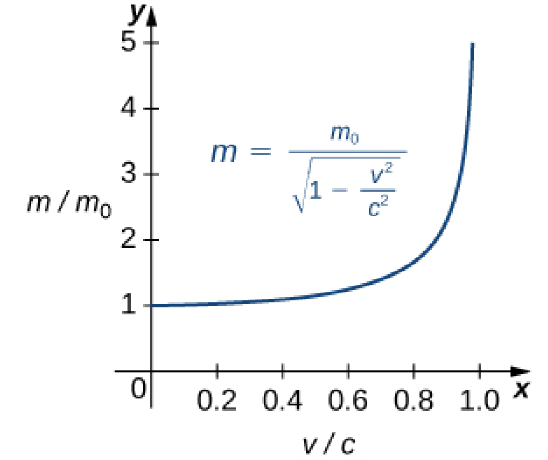
Solution:

Our starting point is Einstein’s equation for the mass of a moving object,

Equation:

$$m=\frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}},$$

where m_0 is the object’s mass at rest, v is its speed, and c is the speed of light. To see how the mass changes at high speeds, we can graph the ratio of masses m/m_0 as a function of the ratio of speeds, v/c ([link]).



This graph shows the ratio of masses as a function of the ratio of speeds in Einstein’s equation for the mass of a moving object.

We can see that as the ratio of speeds approaches 1—that is, as the speed of the object approaches the speed of light—the ratio of masses increases without bound. In other words, the function has a vertical asymptote at $v/c = 1$. We can try a few values of this ratio to test this idea.

$\frac{v}{c}$	$\sqrt{1 - \frac{v^2}{c^2}}$	$\frac{m}{m_0}$
0.99	0.1411	7.089
0.999	0.0447	22.37
0.9999	0.0141	70.71

Ratio of Masses and Speeds for a Moving Object

Thus, according to [link], if an object with mass 100 kg is traveling at 0.9999*c*, its mass becomes 7071 kg. Since no object can have an infinite mass, we conclude that no object can travel at or more than the speed of light.

Key Concepts

- A table of values or graph may be used to estimate a limit.
- If the limit of a function at a point does not exist, it is still possible that the limits from the left and right at that point may exist.
- If the limits of a function from the left and right exist and are equal, then the limit of the function is that common value.
- We may use limits to describe infinite behavior of a function at a point.

Key Equations

- **Intuitive Definition of the Limit**
  $\lim\limits_{x \to a} f(x) = L$
- **Two Important Limits**
  $\lim\limits_{x \to a} x = a \quad \lim\limits_{x \to a} c = c$
- **One-Sided Limits**
  $\lim\limits_{x \to a^-} f(x) = L \quad \lim\limits_{x \to a^+} f(x) = L$
- **Infinite Limits from the Left**
  $\lim\limits_{x \to a^-} f(x) = +\infty \quad \lim\limits_{x \to a^-} f(x) = -\infty$
- **Infinite Limits from the Right**
  $\lim\limits_{x \to a^+} f(x) = +\infty \quad \lim\limits_{x \to a^+} f(x) = -\infty$
- **Two-Sided Infinite Limits**
  $\lim\limits_{x \to a} f(x) = +\infty : \lim\limits_{x \to a^-} f(x) = +\infty$ and $\lim\limits_{x \to a^+} f(x) = +\infty$
  $\lim\limits_{x \to a} f(x) = -\infty : \lim\limits_{x \to a^-} f(x) = -\infty$ and $\lim\limits_{x \to a^+} f(x) = -\infty$

For the following exercises, consider the function $f(x) = \frac{x^2-1}{|x-1|}$.

Exercise:

Problem:

[T] Complete the following table for the function. Round your solutions to four decimal places.

x	$f(x)$		x	$f(x)$
0.9	a.		1.1	e.
0.99	b.		1.01	f.
0.999	c.		1.001	g.
0.9999	d.		1.0001	h.

Exercise:

Problem:

What do your results in the preceding exercise indicate about the two-sided limit $\lim_{x\to 1} f(x)$? Explain your response.

Solution:

$\lim_{x\to 1} f(x)$ does not exist because $\lim_{x\to 1^-} f(x) = -2 \neq \lim_{x\to 1^+} f(x) = 2.$

For the following exercises, consider the function $f(x) = (1+x)^{1/x}$.

Exercise:

Problem:

[T] Make a table showing the values of *f* for $x = -0.01, -0.001, -0.0001, -0.00001$ and for $x = 0.01, 0.001, 0.0001, 0.00001$. Round your solutions to five decimal places.

x	$f(x)$		x	$f(x)$
−0.01	a.		0.01	e.
−0.001	b.		0.001	f.

x	$f(x)$		***x***	$f(x)$
−0.0001	c.		0.0001	g.
−0.00001	d.		0.00001	h.

Exercise:

Problem:

What does the table of values in the preceding exercise indicate about the function $f(x) = (1+x)^{1/x}$?

Solution:

$\lim\limits_{x \to 0} (1+x)^{1/x} = 2.7183$

Exercise:

Problem:

To which mathematical constant does the limit in the preceding exercise appear to be getting closer?

In the following exercises, use the given values to set up a table to evaluate the limits. Round your solutions to eight decimal places.

Exercise:

Problem: **[T]** $\lim\limits_{x \to 0} \frac{\sin 2x}{x};\ \pm 0.1, \pm 0.01, \pm 0.001, \pm .0001$

x	$\frac{\sin 2x}{x}$		***x***	$\frac{\sin 2x}{x}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.

x	$\frac{\sin 2x}{x}$		***x***	$\frac{\sin 2x}{x}$
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

Solution:

a. 1.98669331; b. 1.99986667; c. 1.99999867; d. 1.99999999; e. 1.98669331; f. 1.99986667; g. 1.99999867; h. 1.99999999; $\lim_{x \to 0} \frac{\sin 2x}{x} = 2$

Exercise:

Problem: [T] $\lim_{x \to 0} \frac{\sin 3x}{x}$ ±0.1, ±0.01, ±0.001, ±0.0001

X	$\frac{\sin 3x}{x}$		***x***	$\frac{\sin 3x}{x}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

Exercise:

Problem:

Use the preceding two exercises to conjecture (guess) the value of the following limit: $\lim_{x \to 0} \frac{\sin ax}{x}$ for *a*, a positive real value.

Solution:

$$\lim_{x \to 0} \frac{\sin ax}{x} = a$$

[T] In the following exercises, set up a table of values to find the indicated limit. Round to eight digits.
Exercise:

Problem: $\lim_{x \to 2} \frac{x^2-4}{x^2+x-6}$

x	$\frac{x^2-4}{x^2+x-6}$		*x*	$\frac{x^2-4}{x^2+x-6}$
1.9	a.		2.1	e.
1.99	b.		2.01	f.
1.999	c.		2.001	g.
1.9999	d.		2.0001	h.

Exercise:

Problem: $\lim_{x \to 1} (1-2x)$

x	$1-2x$		*x*	$1-2x$
0.9	a.		1.1	e.
0.99	b.		1.01	f.
0.999	c.		1.001	g.

x	$1-2x$		**x**	$1-2x$
0.9999	d.		1.0001	h.

Solution:

a. −0.80000000; b. −0.98000000; c. −0.99800000; d. −0.99980000; e. −1.2000000; f. −1.0200000; g. −1.0020000; h. −1.0002000; $\lim\limits_{x\to 1}(1-2x)=-1$

Exercise:

Problem: $\lim\limits_{x\to 0}\frac{5}{1-e^{1/x}}$

x	$\frac{5}{1-e^{1/x}}$		**x**	$\frac{5}{1-e^{1/x}}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

Exercise:

Problem: $\lim\limits_{z\to 0}\frac{z-1}{z^2(z+3)}$

z	$\frac{z-1}{z^2(z+3)}$		z	$\frac{z-1}{z^2(z+3)}$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

Solution:

a. −37.931934; b. −3377.9264; c. −333,777.93; d. −33,337,778; e. −29.032258; f. −3289.0365; g. −332,889.04; h. −33,328,889 $\lim\limits_{x \to 0} \frac{z-1}{z^2(z+3)} = -\infty$

Exercise:

Problem: $\lim\limits_{t \to 0^+} \frac{\cos t}{t}$

t	$\frac{\cos t}{t}$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

Exercise:

Problem: $\lim\limits_{x \to 2} \frac{1-\frac{2}{x}}{x^2-4}$

x	$\frac{1-\frac{2}{x}}{x^2-4}$		x	$\frac{1-\frac{2}{x}}{x^2-4}$
1.9	a.		2.1	e.
1.99	b.		2.01	f.
1.999	c.		2.001	g.
1.9999	d.		2.0001	h.

Solution:

a. 0.13495277; b. 0.12594300; c. 0.12509381; d. 0.12500938; e. 0.11614402; f. 0.12406794; g. 0.12490631; h. 0.12499063; $\therefore \lim_{x\to 2}\frac{1-\frac{2}{x}}{x^2-4}=0.1250=\frac{1}{8}$

[T] In the following exercises, set up a table of values and round to eight significant digits. Based on the table of values, make a guess about what the limit is. Then, use a calculator to graph the function and determine the limit. Was the conjecture correct? If not, why does the method of tables fail?

Exercise:

Problem: $\lim_{\theta\to 0}\sin\left(\frac{\pi}{\theta}\right)$

θ	$\sin\left(\frac{\pi}{\theta}\right)$		θ	$\sin\left(\frac{\pi}{\theta}\right)$
−0.1	a.		0.1	e.
−0.01	b.		0.01	f.
−0.001	c.		0.001	g.
−0.0001	d.		0.0001	h.

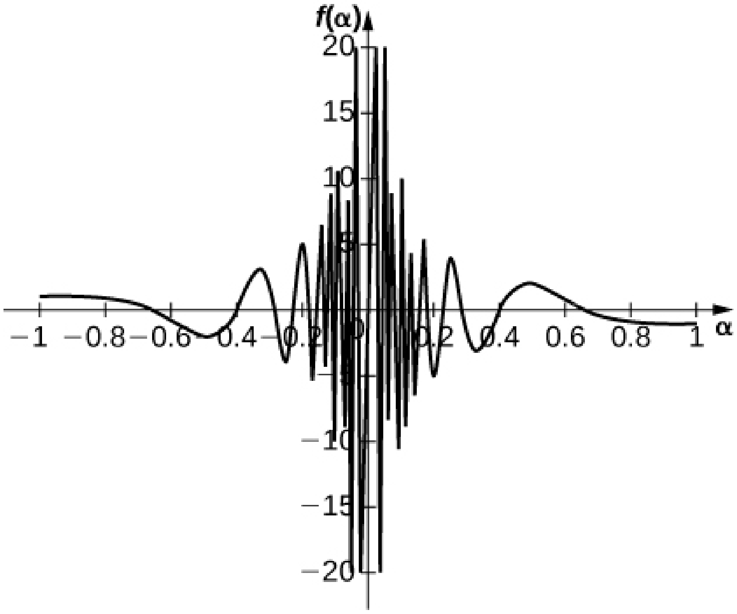
Exercise:

Problem: $\lim\limits_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$

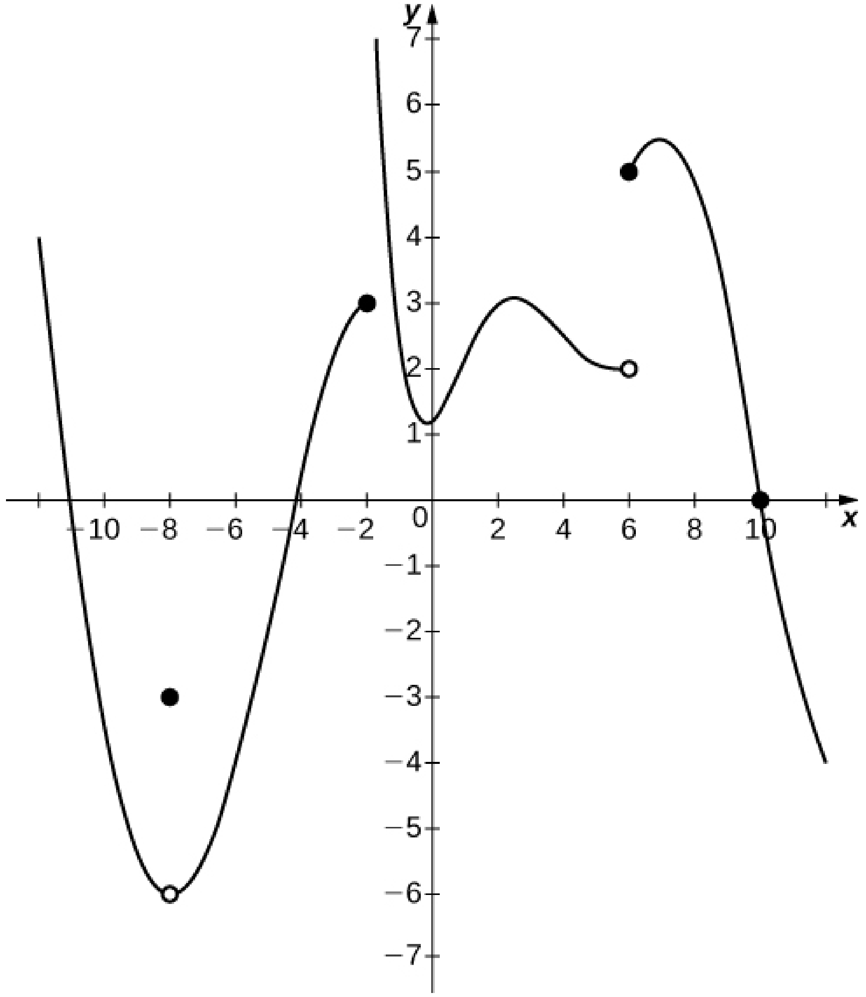
a	$\frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right)$
0.1	a.
0.01	b.
0.001	c.
0.0001	d.

Solution:

a. −10.00000; b. −100.00000; c. −1000.0000; d. −10,000.000; Guess: $\lim\limits_{\alpha \to 0^+} \frac{1}{\alpha} \cos\left(\frac{\pi}{\alpha}\right) = \infty$, actual: DNE



In the following exercises, consider the graph of the function $y = f(x)$ shown here. Which of the statements about $y = f(x)$ are true and which are false? Explain why a statement is false.



Exercise:

Problem: $\lim_{x \to 10} f(x) = 0$

Exercise:

Problem: $\lim_{x \to -2^+} f(x) = 3$

Solution:

False; $\lim\limits_{x \to -2^+} f(x) = +\infty$

Exercise:

Problem: $\lim\limits_{x \to -8} f(x) = f(-8)$

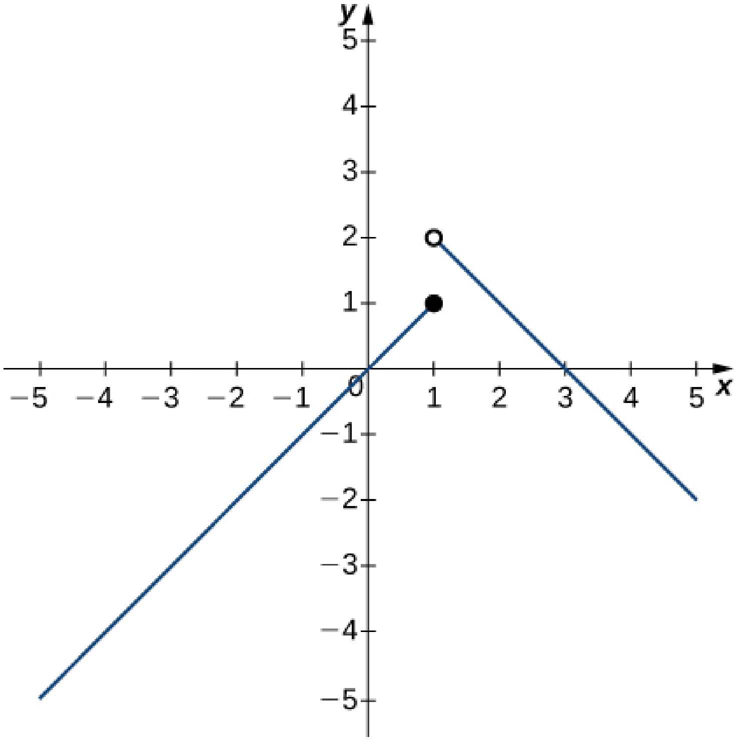
Exercise:

Problem: $\lim\limits_{x \to 6} f(x) = 5$

Solution:

False; $\lim\limits_{x \to 6} f(x)$ DNE since $\lim\limits_{x \to 6^-} f(x) = 2$ and $\lim\limits_{x \to 6^+} f(x) = 5$.

In the following exercises, use the following graph of the function $y = f(x)$ to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim\limits_{x \to 1^-} f(x)$

Exercise:

Problem: $\lim_{x \to 1^+} f(x)$

Solution:

2

Exercise:

Problem: $\lim_{x \to 1} f(x)$

Exercise:

Problem: $\lim_{x \to 2} f(x)$

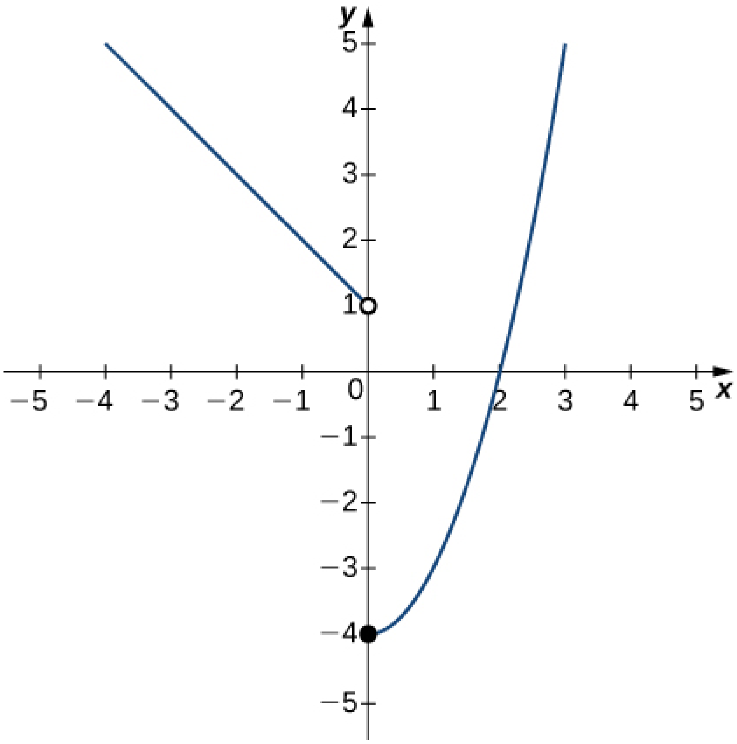
Solution:

1

Exercise:

Problem: $f(1)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \to 0^-} f(x)$

Solution:

1

Exercise:

Problem: $\lim_{x \to 0^+} f(x)$

Exercise:

Problem: $\lim_{x \to 0} f(x)$

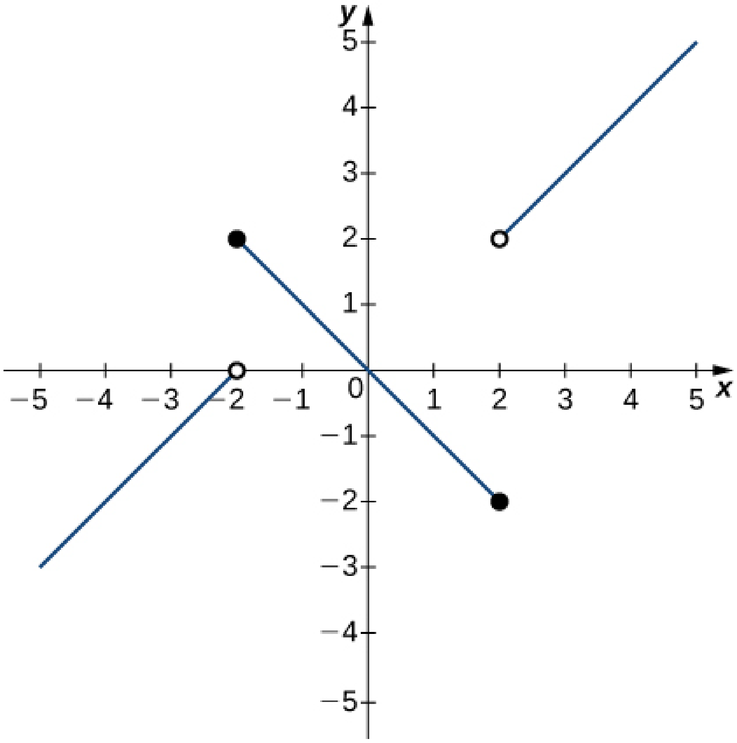
Solution:

DNE

Exercise:

Problem: $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \to -2^-} f(x)$

Solution:

0

Exercise:

Problem: $\lim_{x \to -2^+} f(x)$

Exercise:

Problem: $\lim_{x \to -2} f(x)$

Solution:

DNE

Exercise:

Problem: $\lim_{x \to 2^-} f(x)$

Exercise:

Problem: $\lim_{x \to 2^+} f(x)$

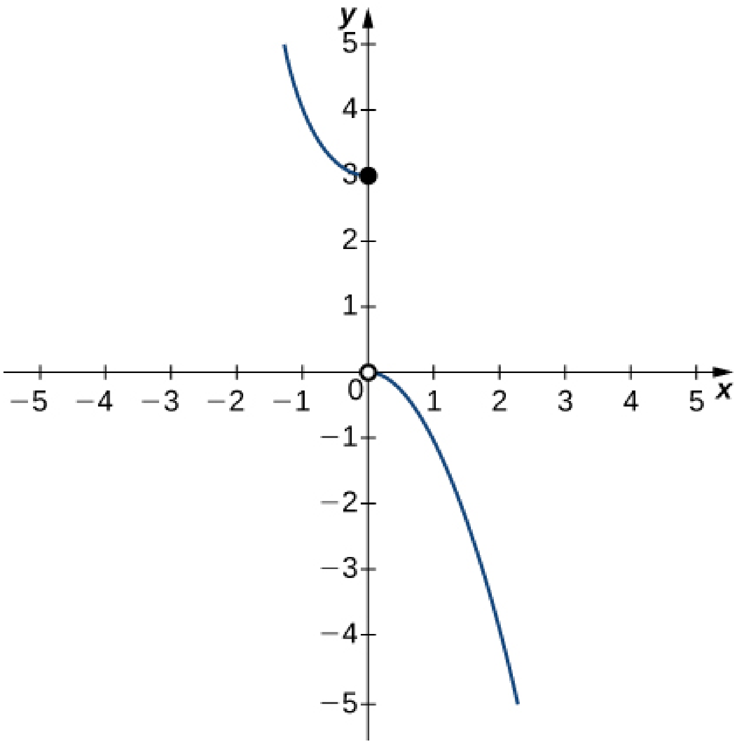
Solution:

2

Exercise:

Problem: $\lim_{x \to 2} f(x)$

In the following exercises, use the graph of the function $y = g(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim\limits_{x\to 0^-} g\left(x\right)$

Solution:

3

Exercise:

Problem: $\lim\limits_{x\to 0^+} g\left(x\right)$

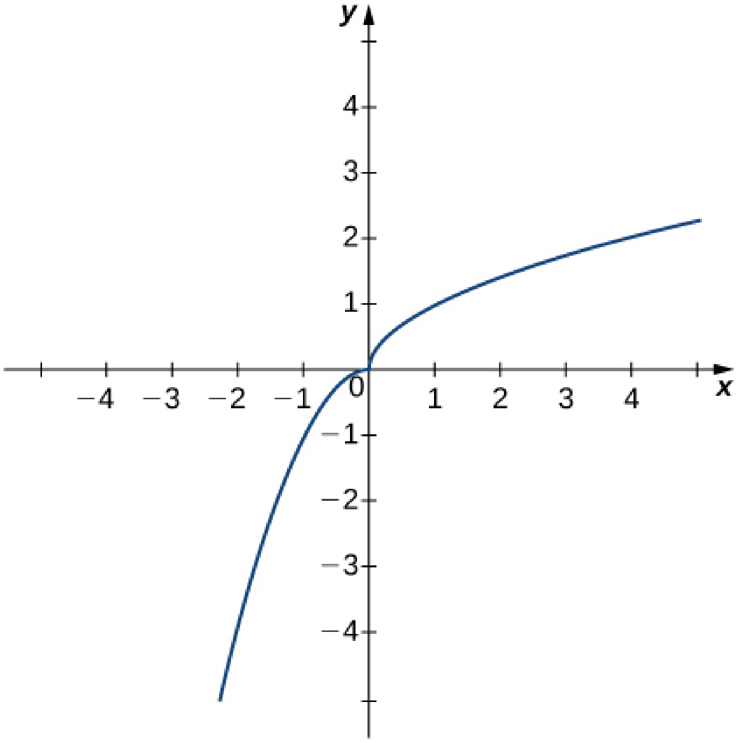
Exercise:

Problem: $\lim\limits_{x\to 0} g\left(x\right)$

Solution:

DNE

In the following exercises, use the graph of the function $y=h\left(x\right)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim\limits_{x \to 0^-} h(x)$

Exercise:

Problem: $\lim\limits_{x \to 0^+} h(x)$

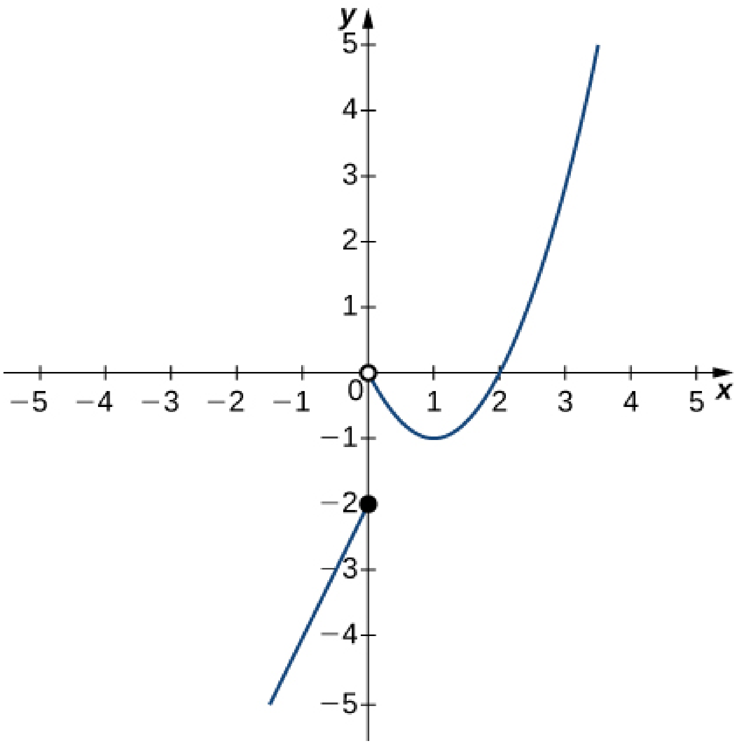
Solution:

0

Exercise:

Problem: $\lim\limits_{x \to 0} h(x)$

In the following exercises, use the graph of the function $y = f(x)$ shown here to find the values, if possible. Estimate when necessary.



Exercise:

Problem: $\lim_{x \to 0^-} f\left(x\right)$

Solution:

-2

Exercise:

Problem: $\lim_{x \to 0^+} f\left(x\right)$

Exercise:

Problem: $\lim_{x \to 0} f\left(x\right)$

Solution:

DNE

Exercise:

Problem: $\lim_{x \to 1} f\left(x\right)$

Exercise:

Problem: $\lim_{x \to 2} f\left(x\right)$

Solution:

0

In the following exercises, sketch the graph of a function with the given properties.

Exercise:

Problem: $\lim_{x \to 2} f\left(x\right) = 1$, $\lim_{x \to 4^-} f\left(x\right) = 3$, $\lim_{x \to 4^+} f\left(x\right) = 6$, $f\left(4\right)$ is not defined.

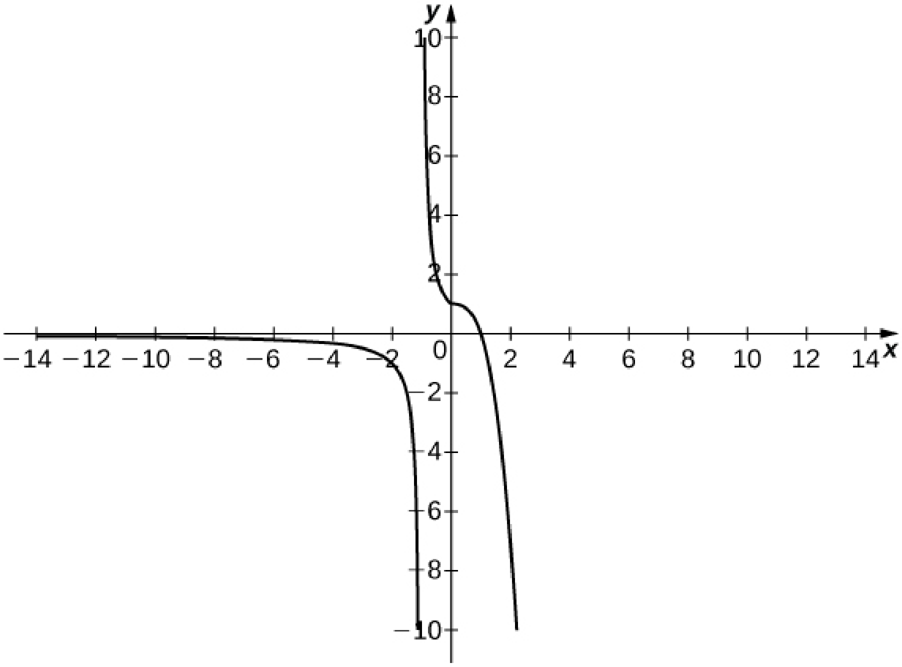
Exercise:

Problem:

$As\ x \to -\infty\ ,\ f(x) \to 0,\ \lim\limits_{x \to -1^-} f(x) = -\infty,$
$\lim\limits_{x \to -1^+} f(x) = \infty, \lim\limits_{x \to 0} f(x) = f(0), f(0) = 1,\ As\ x \to \infty,\ f(x) \to -\infty$

Solution:

Answers may vary.



Exercise:

Problem:

$As\ x \to -\infty,\ f(x) \to 2, \lim\limits_{x \to 3^-} f(x) = -\infty,$
$\lim\limits_{x \to 3^+} f(x) = \infty,\ As\ x \to \infty,\ f(x) \to 2, f(0) = \frac{-1}{3}$

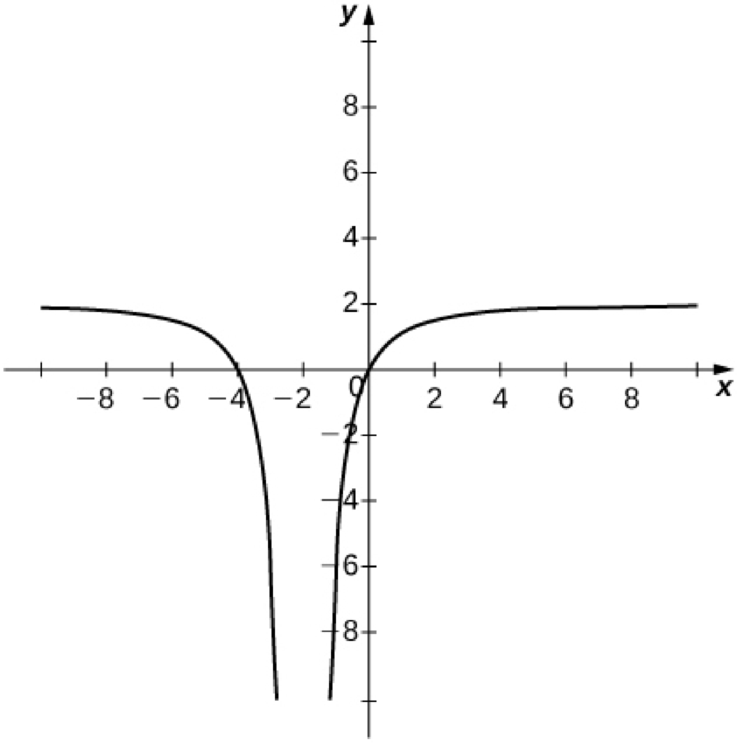
Exercise:

Problem:

$As\ x \to -\infty,\ f(x) \to 2, \lim\limits_{x \to -2} f(x) = -\infty,$
$As\ x \to \infty,\ f(x) \to 2, f(0) = 0$

Solution:

Answers may vary.



Exercise:

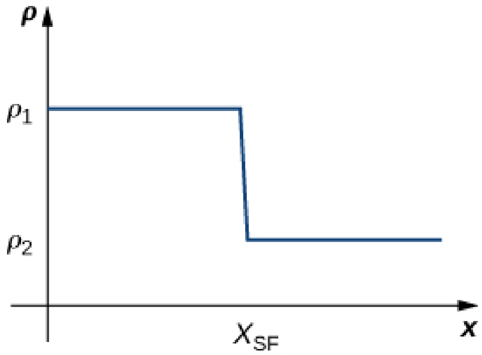
Problem:

$As\ x \rightarrow -\infty,\ f(x) \rightarrow 0, \lim_{x \rightarrow -1^-} f(x) = \infty, \lim_{x \rightarrow -1^+} f(x) = -\infty,$
$f(0) = -1, \lim_{x \rightarrow 1^-} f(x) = -\infty, \lim_{x \rightarrow 1^+} f(x) = \infty,\ As\ x \rightarrow \infty,\ f(x) \rightarrow 0$

Exercise:

Problem:

Shock waves arise in many physical applications, ranging from supernovas to detonation waves. A graph of the density of a shock wave with respect to distance, *x*, is shown here. We are mainly interested in the location of the front of the shock, labeled x_{SF} in the diagram.



a. Evaluate $\lim\limits_{x \to x_{SF}^{+}} \rho\left(x\right)$.

b. Evaluate $\lim\limits_{x \to x_{SF}^{-}} \rho\left(x\right)$.

c. Evaluate $\lim\limits_{x \to x_{SF}} \rho\left(x\right)$. Explain the physical meanings behind your answers.

Solution:

a. ρ_2 b. ρ_1 c. DNE unless $\rho_1 = \rho_2$. As you approach x_{SF} from the right, you are in the high-density area of the shock. When you approach from the left, you have not experienced the “shock” yet and are at a lower density.

Exercise:

Problem:

A track coach uses a camera with a fast shutter to estimate the position of a runner with respect to time. A table of the values of position of the athlete versus time is given here, where *x* is the position in meters of the runner and *t* is time in seconds. What is $\lim\limits_{t \to 2} x\left(t\right)$? What does it mean physically?

***t* (sec)**	***x* (m)**
1.75	4.5
1.95	6.1
1.99	6.42

***t* (sec)**	***x* (m)**
2.01	6.58
2.05	6.9
2.25	8.5

Glossary

infinite limit
: A function has an infinite limit at a point a if it either increases or decreases without bound as it approaches a

intuitive definition of the limit
: If all values of the function $f\left(x\right)$ approach the real number L as the values of $x\left(\ne a\right)$ approach a, $f\left(x\right)$ approaches L

one-sided limit
: A one-sided limit of a function is a limit taken from either the left or the right

vertical asymptote
: A function has a vertical asymptote at $x=a$ if the limit as *x* approaches a from the right or left is infinite

The Limit Laws

- Recognize the basic limit laws.
- Use the limit laws to evaluate the limit of a function.
- Evaluate the limit of a function by factoring.
- Use the limit laws to evaluate the limit of a polynomial or rational function.
- Evaluate the limit of a function by factoring or by using conjugates.
- Evaluate the limit of a function by using the squeeze theorem.

In the previous section, we evaluated limits by looking at graphs or by constructing a table of values. In this section, we establish laws for calculating limits and learn how to apply these laws. In the Student Project at the end of this section, you have the opportunity to apply these limit laws to derive the formula for the area of a circle by adapting a method devised by the Greek mathematician Archimedes. We begin by restating two useful limit results from the previous section. These two results, together with the limit laws, serve as a foundation for calculating many limits.

Evaluating Limits with the Limit Laws

The first two limit laws were stated in [link] and we repeat them here. These basic results, together with the other limit laws, allow us to evaluate limits of many algebraic functions.

Note:
Basic Limit Results
For any real number *a* and any constant *c*,

i. **Equation:**

$$\lim_{x \to a} x = a$$

ii. **Equation:**

$$\lim_{x \to a} c = c$$

Example:
Exercise:

Problem:
Evaluating a Basic Limit

Evaluate each of the following limits using [link].

a. $\lim_{x \to 2} x$
b. $\lim_{x \to 2} 5$

Solution:

a. The limit of x as x approaches a is a: $\lim\limits_{x \to 2} x = 2$.
b. The limit of a constant is that constant: $\lim\limits_{x \to 2} 5 = 5$.

We now take a look at the **limit laws**, the individual properties of limits. The proofs that these laws hold are omitted here.

Note:
Limit Laws
Let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a. Assume that L and M are real numbers such that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Let c be a constant. Then, each of the following statements holds:
Sum law for limits: $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$
Difference law for limits: $\lim\limits_{x \to a} (f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = L - M$
Constant multiple law for limits: $\lim\limits_{x \to a} cf(x) = c \cdot \lim\limits_{x \to a} f(x) = cL$
Product law for limits: $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = L \cdot M$
Quotient law for limits: $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$
Power law for limits: $\lim\limits_{x \to a} (f(x))^n = \left(\lim\limits_{x \to a} f(x)\right)^n = L^n$ for every positive integer n.
Root law for limits: $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even and $f(x) \geq 0$.

We now practice applying these limit laws to evaluate a limit.

Example:
Exercise:

Problem:
Evaluating a Limit Using Limit Laws

Use the limit laws to evaluate $\lim\limits_{x \to -3} (4x + 2)$.

Solution:

Let's apply the limit laws one step at a time to be sure we understand how they work. We need to keep in mind the requirement that, at each application of a limit law, the new limits must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to -3}(4x+2) &= \lim_{x \to -3} 4x + \lim_{x \to -3} 2 && \text{Apply the sum law.} \\ &= 4 \cdot \lim_{x \to -3} x + \lim_{x \to -3} 2 && \text{Apply the constant multiple law.} \\ &= 4 \cdot (-3) + 2 = -10. && \text{Apply the basic limit results and simplify.} \end{aligned}$$

Example:
Exercise:

Problem:
Using Limit Laws Repeatedly

Use the limit laws to evaluate $\lim\limits_{x \to 2} \frac{2x^2-3x+1}{x^3+4}$.

Solution:

To find this limit, we need to apply the limit laws several times. Again, we need to keep in mind that as we rewrite the limit in terms of other limits, each new limit must exist for the limit law to be applied.

$$\begin{aligned} \lim_{x \to 2} \frac{2x^2-3x+1}{x^3+4} &= \frac{\lim_{x \to 2}(2x^2-3x+1)}{\lim_{x \to 2}(x^3+4)} && \text{Apply the quotient law, making sure that. } (2)^3+4 \neq 0 \\ &= \frac{2 \cdot \lim_{x \to 2} x^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\lim_{x \to 2} x^3 + \lim_{x \to 2} 4} && \text{Apply the sum law and constant multiple law.} \\ &= \frac{2 \cdot \left(\lim_{x \to 2} x\right)^2 - 3 \cdot \lim_{x \to 2} x + \lim_{x \to 2} 1}{\left(\lim_{x \to 2} x\right)^3 + \lim_{x \to 2} 4} && \text{Apply the power law.} \\ &= \frac{2(4)-3(2)+1}{(2)^3+4} = \frac{1}{4}. && \text{Apply the basic limit laws and simplify.} \end{aligned}$$

Note:
Exercise:

Problem:

Use the limit laws to evaluate $\lim\limits_{x \to 6}(2x-1)\sqrt{x+4}$. In each step, indicate the limit law applied.

Solution:

$11\sqrt{10}$

Hint

Begin by applying the product law.

Limits of Polynomial and Rational Functions

By now you have probably noticed that, in each of the previous examples, it has been the case that $\lim\limits_{x \to a} f(x) = f(a)$. This is not always true, but it does hold for all polynomials for any choice of *a* and for all rational functions at all values of *a* for which the rational function is defined.

Note:
Limits of Polynomial and Rational Functions
Let $p(x)$ and $q(x)$ be polynomial functions. Let *a* be a real number. Then,
Equation:

$$\lim_{x \to a} p(x) = p(a)$$

Equation:

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ when } q(a) \neq 0.$$

To see that this theorem holds, consider the polynomial $p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$. By applying the sum, constant multiple, and power laws, we end up with
Equation:

$$\begin{aligned}
\lim_{x \to a} p(x) &= \lim_{x \to a} \left(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0\right) \\
&= c_n \left(\lim_{x \to a} x\right)^n + c_{n-1} \left(\lim_{x \to a} x\right)^{n-1} + \cdots + c_1 \left(\lim_{x \to a} x\right) + \lim_{x \to a} c_0 \\
&= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 \\
&= p(a).
\end{aligned}$$

It now follows from the quotient law that if $p(x)$ and $q(x)$ are polynomials for which $q(a) \neq 0$, then
Equation:

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}.$$

[link] applies this result.

Example:
Exercise:

Problem:
Evaluating a Limit of a Rational Function

Evaluate the $\lim\limits_{x \to 3} \frac{2x^2 - 3x + 1}{5x + 4}$.

Solution:

Since 3 is in the domain of the rational function $f\left(x\right)=\frac{2x^2-3x+1}{5x+4}$, we can calculate the limit by substituting 3 for *x* into the function. Thus,

Equation:

$$\lim_{x\to 3}\frac{2x^2-3x+1}{5x+4}=\frac{10}{19}.$$

Note:

Exercise:

Problem: Evaluate $\lim_{x\to -2}\left(3x^3-2x+7\right)$.

Solution:

-13;

Hint

Use [link]

Additional Limit Evaluation Techniques

As we have seen, we may evaluate easily the limits of polynomials and limits of some (but not all) rational functions by direct substitution. However, as we saw in the introductory section on limits, it is certainly possible for $\lim_{x\to a}f\left(x\right)$ to exist when $f\left(a\right)$ is undefined. The following observation allows us to evaluate many limits of this type:

If for all $x\ne a$, $f\left(x\right)=g\left(x\right)$ over some open interval containing *a*, then $\lim_{x\to a}f\left(x\right)=\lim_{x\to a}g\left(x\right)$.

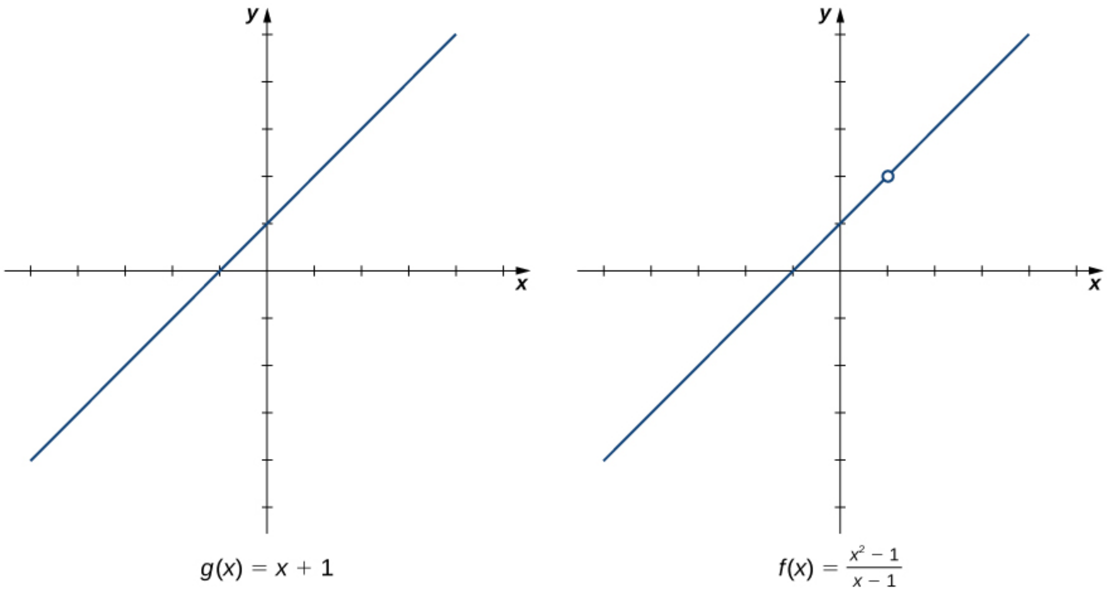
To understand this idea better, consider the limit $\lim_{x\to 1}\frac{x^2-1}{x-1}$.

The function

Equation:

$$\begin{aligned}f\left(x\right)&=\frac{x^2-1}{x-1}\\&=\frac{(x-1)(x+1)}{x-1}\end{aligned}$$

and the function $g\left(x\right)=x+1$ are identical for all values of $x\ne 1$. The graphs of these two functions are shown in [link].



The graphs of $f\left(x\right)$ and $g\left(x\right)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

Equation:

$$\begin{aligned}\lim_{x \to 1} \frac{x^2-1}{x-1} &= \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x \to 1} \left(x+1\right) \\ &= 2.\end{aligned}$$

The limit has the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f\left(x\right) = 0$ and $\lim_{x \to a} g\left(x\right) = 0$. (In this case, we say that $f(x)/g(x)$ has the indeterminate form $0/0.$) The following Problem-Solving Strategy provides a general outline for evaluating limits of this type.

Note:

Problem-Solving Strategy: Calculating a Limit When $f\left(x\right)/g\left(x\right)$ has the Indeterminate Form 0/0

1. First, we need to make sure that our function has the appropriate form and cannot be evaluated immediately using the limit laws.
2. We then need to find a function that is equal to $h\left(x\right) = f(x)/g(x)$ for all $x \neq a$ over some interval containing a. To do this, we may need to try one or more of the following steps:
   a. If $f\left(x\right)$ and $g\left(x\right)$ are polynomials, we should factor each function and cancel out any common factors.
   b. If the numerator or denominator contains a difference involving a square root, we should try multiplying the numerator and denominator by the conjugate of the expression involving the square root.
   c. If $f(x)/g(x)$ is a complex fraction, we begin by simplifying it.

3. Last, we apply the limit laws.

The next examples demonstrate the use of this Problem-Solving Strategy. [link] illustrates the factor-and-cancel technique; [link] shows multiplying by a conjugate. In [link], we look at simplifying a complex fraction.

Example:
Exercise:

Problem:
Evaluating a Limit by Factoring and Canceling

Evaluate $\lim_{x \to 3} \frac{x^2-3x}{2x^2-5x-3}$.

Solution:

Step 1. The function $f(x) = \frac{x^2-3x}{2x^2-5x-3}$ is undefined for $x = 3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:
Equation:

$$\lim_{x \to 3} \frac{x^2-3x}{2x^2-5x-3} = \lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)}$$

Step 2. For all $x \neq 3$, $\frac{x^2-3x}{2x^2-5x-3} = \frac{x}{2x+1}$. Therefore,
Equation:

$$\lim_{x \to 3} \frac{x(x-3)}{(x-3)(2x+1)} = \lim_{x \to 3} \frac{x}{2x+1}.$$

Step 3. Evaluate using the limit laws:
Equation:

$$\lim_{x \to 3} \frac{x}{2x+1} = \frac{3}{7}.$$

Note:
Exercise:

Problem: Evaluate $\lim_{x \to -3} \frac{x^2+4x+3}{x^2-9}$.

Solution:

$\frac{1}{3}$

Hint

Follow the steps in the Problem-Solving Strategy and [link].

Example:
Exercise:

Problem:
Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim\limits_{x \to -1} \frac{\sqrt{x+2}-1}{x+1}$.

Solution:

Step 1. $\frac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at -1. Let’s begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:
Equation:

$$\lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1} = \lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1} \cdot \frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}.$$

Step 2. We then multiply out the numerator. We don’t multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:
Equation:

$$= \lim_{x \to -1} \frac{x+1}{(x+1)\left(\sqrt{x+2}+1\right)}.$$

Step 3. Then we cancel:
Equation:

$$= \lim_{x \to -1} \frac{1}{\sqrt{x+2}+1}.$$

Step 4. Last, we apply the limit laws:
Equation:

$$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}.$$

Note:
Exercise:

Problem: Evaluate $\lim_{x\to 5}\frac{\sqrt{x-1}-2}{x-5}$.

Solution:

$\frac{1}{4}$

Hint

Follow the steps in the Problem-Solving Strategy and [link].

Example:
Exercise:

Problem:
Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$.

Solution:

Step 1. $\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2\left(x+1\right)/2\left(x+1\right)$:
Equation:

$$\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}\cdot\frac{2\left(x+1\right)}{2\left(x+1\right)}.$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $\left(x-1\right)$:
Equation:

$$=\lim_{x\to 1}\frac{2-\left(x+1\right)}{2\left(x-1\right)\left(x+1\right)}.$$

Step 3. Then, we simplify the numerator:
Equation:

$$=\lim_{x\to 1}\frac{-x+1}{2\left(x-1\right)\left(x+1\right)}.$$

Step 4. Now we factor out −1 from the numerator:
Equation:

$$=\lim_{x\to 1}\frac{-\left(x-1\right)}{2\left(x-1\right)\left(x+1\right)}.$$

Step 5. Then, we cancel the common factors of $(x-1)$:
Equation:

$$=\lim_{x\to 1}\frac{-1}{2\left(x+1\right)}.$$

Step 6. Last, we evaluate using the limit laws:
Equation:

$$\lim_{x\to 1}\frac{-1}{2(x+1)}=-\frac{1}{4}.$$

Note:
Exercise:

Problem: Evaluate $\lim_{x\to -3}\frac{\frac{1}{x+2}+1}{x+3}$.

Solution:

-1;

Hint

Follow the steps in the Problem-Solving Strategy and [link].

[link] does not fall neatly into any of the patterns established in the previous examples. However, with a little creativity, we can still use these same techniques.

Example:
Exercise:

Problem:
Evaluating a Limit When the Limit Laws Do Not Apply

Evaluate $\lim_{x\to 0}\left(\frac{1}{x}+\frac{5}{x(x-5)}\right)$.

Solution:

Both $1/x$ and $5/x\left(x-5\right)$ fail to have a limit at zero. Since neither of the two functions has a limit at zero, we cannot apply the sum law for limits; we must use a different strategy. In this case, we find the limit by performing addition and then applying one of our previous strategies. Observe that
Equation:

$$\begin{aligned} \frac{1}{x} + \frac{5}{x(x-5)} &= \frac{x-5+5}{x(x-5)} \\ &= \frac{x}{x(x-5)}. \end{aligned}$$

Thus,

Equation:

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{x} + \frac{5}{x(x-5)} \right) &= \lim_{x \to 0} \frac{x}{x(x-5)} \\ &= \lim_{x \to 0} \frac{1}{x-5} \\ &= -\frac{1}{5}. \end{aligned}$$

Note:

Exercise:

Problem: Evaluate $\lim_{x \to 3} \left(\frac{1}{x-3} - \frac{4}{x^2-2x-3} \right)$.

Solution:

$\frac{1}{4}$

Hint

Use the same technique as [link]. Don’t forget to factor $x^2 - 2x - 3$ before getting a common denominator.

Let’s now revisit one-sided limits. Simple modifications in the limit laws allow us to apply them to one-sided limits. For example, to apply the limit laws to a limit of the form $\lim_{x \to a^-} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (b, a); for a limit of the form $\lim_{x \to a^+} h(x)$, we require the function $h(x)$ to be defined over an open interval of the form (a, c). [link] illustrates this point.

Example:

Exercise:

Problem:

Evaluating a One-Sided Limit Using the Limit Laws

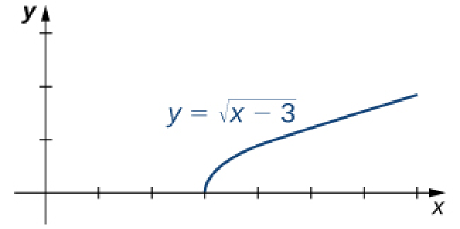
Evaluate each of the following limits, if possible.

a. $\lim_{x \to 3^-} \sqrt{x-3}$

b. $\lim_{x \to 3^+} \sqrt{x-3}$

Solution:

[link] illustrates the function $f(x) = \sqrt{x-3}$ and aids in our understanding of these limits.



The graph shows the function $f(x) = \sqrt{x-3}$.

a. The function $f(x) = \sqrt{x-3}$ is defined over the interval $[3, +\infty)$. Since this function is not defined to the left of 3, we cannot apply the limit laws to compute $\lim\limits_{x \to 3^-} \sqrt{x-3}$. In fact, since $f(x) = \sqrt{x-3}$ is undefined to the left of 3, $\lim\limits_{x \to 3^-} \sqrt{x-3}$ does not exist.
b. Since $f(x) = \sqrt{x-3}$ is defined to the right of 3, the limit laws do apply to $\lim\limits_{x \to 3^+} \sqrt{x-3}$. By applying these limit laws we obtain $\lim\limits_{x \to 3^+} \sqrt{x-3} = 0$.

In [link] we look at one-sided limits of a piecewise-defined function and use these limits to draw a conclusion about a two-sided limit of the same function.

Example:
Exercise:

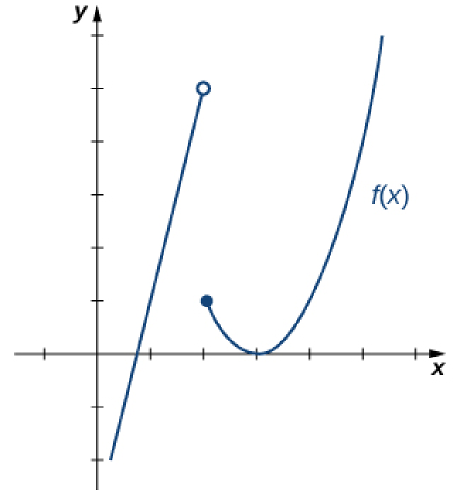
Problem:
Evaluating a Two-Sided Limit Using the Limit Laws

For $f(x) = \begin{cases} 4x - 3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$, evaluate each of the following limits:

a. $\lim\limits_{x \to 2^-} f(x)$
b. $\lim\limits_{x \to 2^+} f(x)$
c. $\lim\limits_{x \to 2} f(x)$

Solution:

[link] illustrates the function $f(x)$ and aids in our understanding of these limits.



This graph shows a function $f\left(x\right)$.

a. Since $f\left(x\right)=4x-3$ for all *x* in $\left(-\infty,2\right)$, replace $f\left(x\right)$ in the limit with $4x-3$ and apply the limit laws:
Equation:

$$\lim_{x\to2^-}f\left(x\right)=\lim_{x\to2^-}\left(4x-3\right)=5.$$

b. Since $f\left(x\right)=\left(x-3\right)^2$ for all *x* in $\left(2,+\infty\right)$, replace $f\left(x\right)$ in the limit with $\left(x-3\right)^2$ and apply the limit laws:
Equation:

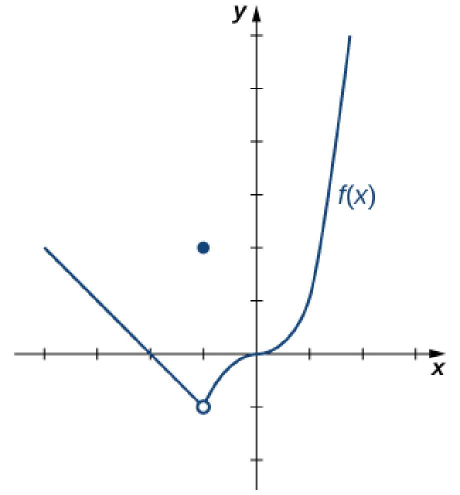
$$\lim_{x\to2^+}f\left(x\right)=\lim_{x\to2^+}\left(x-3\right)^2=1.$$

c. Since $\lim_{x\to2^-}f\left(x\right)=5$ and $\lim_{x\to2^+}f\left(x\right)=1$, we conclude that $\lim_{x\to2}f\left(x\right)$ does not exist.

Note:
Exercise:

Problem: Graph $f\left(x\right)=\begin{matrix}-x-2\text{ if }x<-1\\ 2\text{ if }x=-1\\ x^3\text{ if }x>-1\end{matrix}$ and evaluate $\lim_{x\to-1^-}f\left(x\right)$.

Solution:



$$\lim_{x \to -1^-} f(x) = -1$$

Hint

Use the method in [link] to evaluate the limit.

We now turn our attention to evaluating a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $\lim_{x \to a} f(x) = K$, where $K \neq 0$ and $\lim_{x \to a} g(x) = 0$. That is, $f(x)/g(x)$ has the form $K/0, K \neq 0$ at *a*.

Example:
Exercise:

Problem:
Evaluating a Limit of the Form $K/0, K \neq 0$ Using the Limit Laws

Evaluate $\lim_{x \to 2^-} \frac{x-3}{x^2-2x}$.

Solution:

Step 1. After substituting in $x = 2$, we see that this limit has the form $-1/0$. That is, as *x* approaches 2 from the left, the numerator approaches −1; and the denominator approaches 0. Consequently, the magnitude of $\frac{x-3}{x(x-2)}$ becomes infinite. To get a better idea of what the limit is, we need to factor the denominator:

Equation:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = \lim_{x \to 2^-} \frac{x-3}{x(x-2)}.$$

Step 2. Since $x - 2$ is the only part of the denominator that is zero when 2 is substituted, we then separate $1/(x-2)$ from the rest of the function:

Equation:

$$= \lim_{x \to 2^-} \frac{x-3}{x} \cdot \frac{1}{x-2}.$$

Step 3. $\lim_{x \to 2^-} \frac{x-3}{x} = -\frac{1}{2}$ and $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$. Therefore, the product of $(x-3)/x$ and $1/(x-2)$ has a limit of $+\infty$:

Equation:

$$\lim_{x \to 2^-} \frac{x-3}{x^2-2x} = +\infty.$$

Note:
Exercise:

Problem: Evaluate $\lim_{x \to 1} \frac{x+2}{(x-1)^2}$.

Solution:

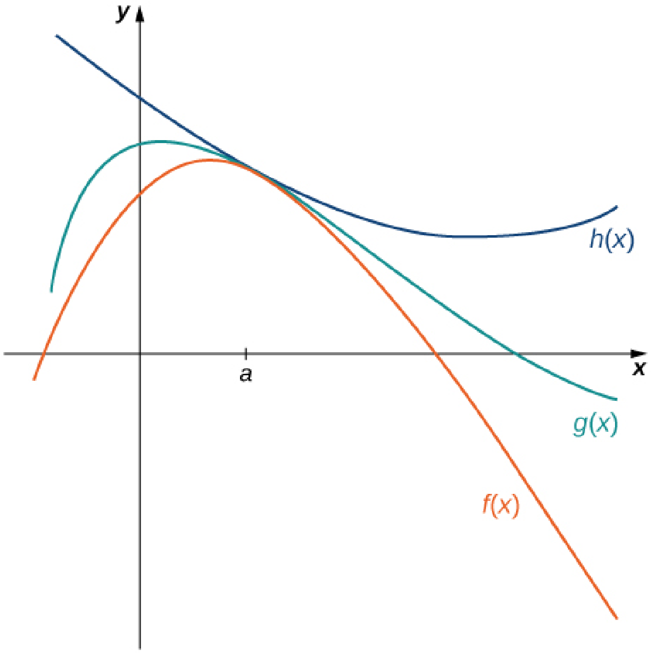
$+\infty$

Hint

Use the methods from [link].

The Squeeze Theorem

The techniques we have developed thus far work very well for algebraic functions, but we are still unable to evaluate limits of very basic trigonometric functions. The next theorem, called the **squeeze theorem**, proves very useful for establishing basic trigonometric limits. This theorem allows us to calculate limits by “squeezing” a function, with a limit at a point *a* that is unknown, between two functions having a common known limit at *a*. [link] illustrates this idea.



The Squeeze Theorem applies when $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Note:
The Squeeze Theorem
Let $f(x)$, $g(x)$, and $h(x)$ be defined for all $x \ne a$ over an open interval containing a. If
Equation:

$$f(x) \le g(x) \le h(x)$$

for all $x \ne a$ in an open interval containing a and
Equation:

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

where L is a real number, then $\lim_{x \to a} g(x) = L$.

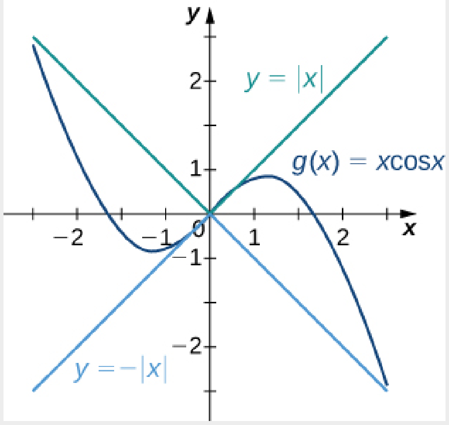
Example:
Exercise:

Problem:
Applying the Squeeze Theorem

Apply the squeeze theorem to evaluate $\lim_{x \to 0} x \cos x$.

Solution:

Because $-1 \le \cos x \le 1$ for all *x*, we have $-|x| \le x\cos x \le |x|$. Since $\lim_{x \to 0}(-|x|) = 0 = \lim_{x \to 0}|x|$, from the squeeze theorem, we obtain $\lim_{x \to 0} x\cos x = 0$. The graphs of $f(x) = -|x|, g(x) = x\cos x$, and $h(x) = |x|$ are shown in [link].



The graphs of $f(x), g(x)$, and $h(x)$ are shown around the point $x = 0$.

Note:
Exercise:

Problem: Use the squeeze theorem to evaluate $\lim_{x \to 0} x^2 \sin \frac{1}{x}$.

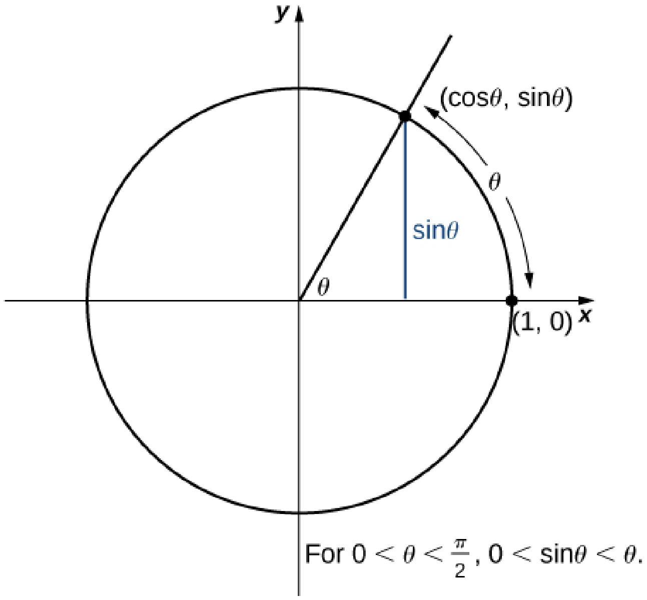
Solution:

0

Hint

Use the fact that $-x^2 \le x^2 \sin(1/x) \le x^2$ to help you find two functions such that $x^2 \sin(1/x)$ is squeezed between them.

We now use the squeeze theorem to tackle several very important limits. Although this discussion is somewhat lengthy, these limits prove invaluable for the development of the material in both the next section and the next chapter. The first of these limits is $\lim_{\theta \to 0} \sin\theta$. Consider the unit circle shown in [link]. In the figure, we see that $\sin\theta$ is the *y*-coordinate on the unit circle and it corresponds to the line segment shown in blue. The radian measure of angle θ is the length of the arc it subtends on the unit circle. Therefore, we see that for $0 < \theta < \frac{\pi}{2}, 0 < \sin\theta < \theta$.



The sine function is shown as a line on the unit circle.

Because $\lim\limits_{\theta \to 0^+} 0 = 0$ and $\lim\limits_{\theta \to 0^+} \theta = 0$, by using the squeeze theorem we conclude that

Equation:

$$\lim_{\theta \to 0^+} \sin \theta = 0.$$

To see that $\lim\limits_{\theta \to 0^-} \sin \theta = 0$ as well, observe that for $-\frac{\pi}{2} < \theta < 0, 0 < -\theta < \frac{\pi}{2}$ and hence, $0 < \sin\left(-\theta\right) < -\theta$. Consequently, $0 < -\sin \theta < -\theta$. It follows that $0 > \sin \theta > \theta$. An application of the squeeze theorem produces the desired limit. Thus, since $\lim\limits_{\theta \to 0^+} \sin \theta = 0$ and $\lim\limits_{\theta \to 0^-} \sin \theta = 0$,

Equation:

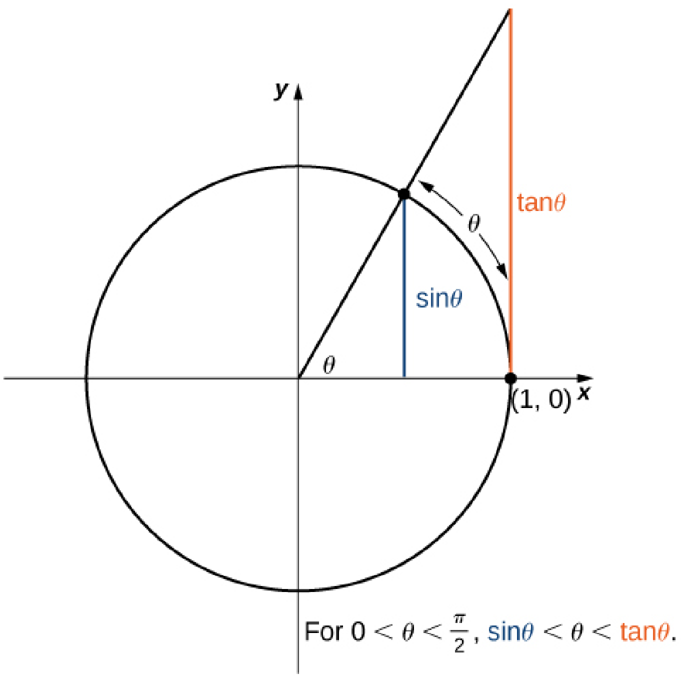
$$\lim_{\theta \to 0} \sin \theta = 0.$$

Next, using the identity $\cos \theta = \sqrt{1 - \sin^2 \theta}$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we see that

Equation:

$$\lim_{\theta \to 0} \cos \theta = \lim_{\theta \to 0} \sqrt{1 - \sin^2 \theta} = 1.$$

We now take a look at a limit that plays an important role in later chapters—namely, $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta}$. To evaluate this limit, we use the unit circle in [link]. Notice that this figure adds one additional triangle to [link]. We see that the length of the side opposite angle θ in this new triangle is $\tan \theta$. Thus, we see that for $0 < \theta < \frac{\pi}{2}, \sin \theta < \theta < \tan \theta$.


y
x
tanθ
θ
sinθ
θ
(1, 0)
For $0 < \theta < \frac{\pi}{2}$, $\sin\theta < \theta < \tan\theta$.

The sine and tangent functions are shown as lines on the unit circle.

By dividing by $\sin\theta$ in all parts of the inequality, we obtain

Equation:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Equivalently, we have

Equation:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim\limits_{\theta\to 0^+} 1 = 1 = \lim\limits_{\theta\to 0^+} \cos\theta$, we conclude that $\lim\limits_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$. By applying a manipulation similar to that used in demonstrating that $\lim\limits_{\theta\to 0^-} \sin\theta = 0$, we can show that $\lim\limits_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$. Thus,

Equation:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

In [link] we use this limit to establish $\lim\limits_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$. This limit also proves useful in later chapters.

Example:
Exercise:

Problem:
Evaluating an Important Trigonometric Limit

Evaluate $\lim\limits_{\theta \to 0} \frac{1-\cos\theta}{\theta}$.

Solution:

In the first step, we multiply by the conjugate so that we can use a trigonometric identity to convert the cosine in the numerator to a sine:
Equation:

$$\begin{aligned} \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} &= \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta} \\ &= \lim_{\theta \to 0} \frac{1-\cos^2\theta}{\theta(1+\cos\theta)} \\ &= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1+\cos\theta)} \\ &= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{1+\cos\theta} \\ &= 1 \cdot \frac{0}{2} = 0. \end{aligned}$$

Therefore,
Equation:

$$\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0.$$

Note:
Exercise:

Problem: Evaluate $\lim\limits_{\theta \to 0} \frac{1-\cos\theta}{\sin\theta}$.

Solution:

0

Hint

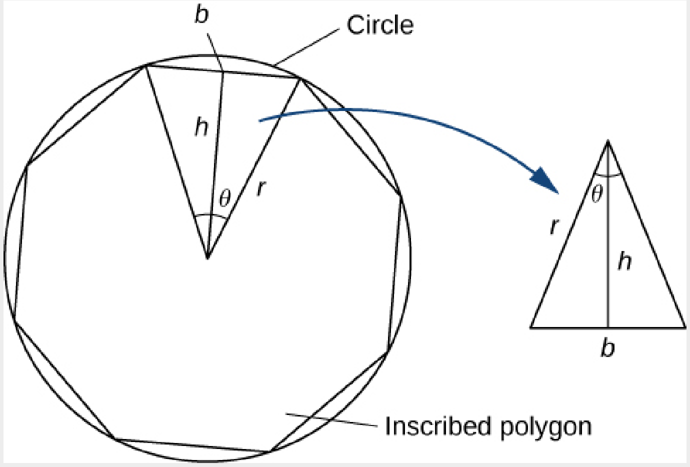
Multiply numerator and denominator by $1+\cos\theta$.

Note:
Deriving the Formula for the Area of a Circle
Some of the geometric formulas we take for granted today were first derived by methods that anticipate some of the methods of calculus. The Greek mathematician Archimedes (ca. 287−212; BCE) was

particularly inventive, using polygons inscribed within circles to approximate the area of the circle as the number of sides of the polygon increased. He never came up with the idea of a limit, but we can use this idea to see what his geometric constructions could have predicted about the limit.

We can estimate the area of a circle by computing the area of an inscribed regular polygon. Think of the regular polygon as being made up of *n* triangles. By taking the limit as the vertex angle of these triangles goes to zero, you can obtain the area of the circle. To see this, carry out the following steps:

1. Express the height *h* and the base *b* of the isosceles triangle in [link] in terms of θ and *r*.



2. Using the expressions that you obtained in step 1, express the area of the isosceles triangle in terms of θ and *r*.
   (Substitute $(1/2)\sin\theta$ for $\sin(\theta/2)\cos(\theta/2)$ in your expression.)
3. If an *n*-sided regular polygon is inscribed in a circle of radius *r*, find a relationship between θ and *n*. Solve this for *n*. Keep in mind there are 2π radians in a circle. (Use radians, not degrees.)
4. Find an expression for the area of the *n*-sided polygon in terms of *r* and θ.
5. To find a formula for the area of the circle, find the limit of the expression in step 4 as θ goes to zero. (*Hint:* $\lim\limits_{\theta\to 0}\frac{(\sin\theta)}{\theta}=1$).

The technique of estimating areas of regions by using polygons is revisited in Introduction to Integration.

Key Concepts

- The limit laws allow us to evaluate limits of functions without having to go through step-by-step processes each time.
- For polynomials and rational functions, $\lim\limits_{x\to a} f(x)=f(a)$.
- You can evaluate the limit of a function by factoring and canceling, by multiplying by a conjugate, or by simplifying a complex fraction.
- The squeeze theorem allows you to find the limit of a function if the function is always greater than one function and less than another function with limits that are known.

Key Equations

- **Basic Limit Results**
  $\lim\limits_{x\to a} x=a \quad \lim\limits_{x\to a} c=c$

- **Important Limits**
  $\lim\limits_{\theta \to 0} \sin\theta = 0$
  $\lim\limits_{\theta \to 0} \cos\theta = 1$
  $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$
  $\lim\limits_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$

In the following exercises, use the limit laws to evaluate each limit. Justify each step by indicating the appropriate limit law(s).
Exercise:

Problem: $\lim\limits_{x \to 0} \left(4x^2 - 2x + 3\right)$

Solution:

Use constant multiple law and difference law: $\lim\limits_{x \to 0} \left(4x^2 - 2x + 3\right) = 4\lim\limits_{x \to 0} x^2 - 2\lim\limits_{x \to 0} x + \lim\limits_{x \to 0} 3 = 3$

Exercise:

Problem: $\lim\limits_{x \to 1} \frac{x^3+3x^2+5}{4-7x}$

Exercise:

Problem: $\lim\limits_{x \to -2} \sqrt{x^2 - 6x + 3}$

Solution:

Use root law: $\lim\limits_{x \to -2} \sqrt{x^2 - 6x + 3} = \sqrt{\lim\limits_{x \to -2} (x^2 - 6x + 3)} = \sqrt{19}$

Exercise:

Problem: $\lim\limits_{x \to -1} (9x + 1)^2$

In the following exercises, use direct substitution to evaluate each limit.
Exercise:

Problem: $\lim\limits_{x \to 7} x^2$

Solution:

49

Exercise:

Problem: $\lim\limits_{x \to -2} \left(4x^2 - 1\right)$

Exercise:

Problem: $\lim\limits_{x \to 0} \frac{1}{1+\sin x}$

Solution:

1

Exercise:

Problem: $\lim\limits_{x \to 2} e^{2x-x^2}$

Exercise:

Problem: $\lim\limits_{x \to 1} \frac{2-7x}{x+6}$

Solution:

$-\frac{5}{7}$

Exercise:

Problem: $\lim\limits_{x \to 3} \ln e^{3x}$

In the following exercises, use direct substitution to show that each limit leads to the indeterminate form $0/0$. Then, evaluate the limit.

Exercise:

Problem: $\lim\limits_{x \to 4} \frac{x^2-16}{x-4}$

Solution:

$\lim\limits_{x \to 4} \frac{x^2-16}{x-4} = \frac{16-16}{4-4} = \frac{0}{0}$; then, $\lim\limits_{x \to 4} \frac{x^2-16}{x-4} = \lim\limits_{x \to 4} \frac{(x+4)(x-4)}{x-4} = 8$

Exercise:

Problem: $\lim\limits_{x \to 2} \frac{x-2}{x^2-2x}$

Exercise:

Problem: $\lim\limits_{x \to 6} \frac{3x-18}{2x-12}$

Solution:

$\lim\limits_{x \to 6} \frac{3x-18}{2x-12} = \frac{18-18}{12-12} = \frac{0}{0}$; then, $\lim\limits_{x \to 6} \frac{3x-18}{2x-12} = \lim\limits_{x \to 6} \frac{3(x-6)}{2(x-6)} = \frac{3}{2}$

Exercise:

Problem: $\lim\limits_{h \to 0} \frac{(1+h)^2-1}{h}$

Exercise:

Problem: $\lim\limits_{t \to 9} \frac{t-9}{\sqrt{t}-3}$

Solution:

$$\lim\limits_{x \to 9} \frac{t-9}{\sqrt{t}-3} = \frac{9-9}{3-3} = \frac{0}{0}\text{; then, } \lim\limits_{t \to 9} \frac{t-9}{\sqrt{t}-3} = \lim\limits_{t \to 9} \frac{t-9}{\sqrt{t}-3} \frac{\sqrt{t}+3}{\sqrt{t}+3} = \lim\limits_{t \to 9} \left(\sqrt{t}+3\right) = 6$$

Exercise:

Problem: $\lim\limits_{h \to 0} \frac{\frac{1}{a+h}-\frac{1}{a}}{h}$, where *a* is a non-zero real-valued constant

Exercise:

Problem: $\lim\limits_{\theta \to \pi} \frac{\sin\theta}{\tan\theta}$

Solution:

$$\lim\limits_{\theta \to \pi} \frac{\sin\theta}{\tan\theta} = \frac{\sin\pi}{\tan\pi} = \frac{0}{0}\text{; then, } \lim\limits_{\theta \to \pi} \frac{\sin\theta}{\tan\theta} = \lim\limits_{\theta \to \pi} \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} = \lim\limits_{\theta \to \pi} \cos\theta = -1$$

Exercise:

Problem: $\lim\limits_{x \to 1} \frac{x^3-1}{x^2-1}$

Exercise:

Problem: $\lim\limits_{x \to 1/2} \frac{2x^2+3x-2}{2x-1}$

Solution:

$$\lim\limits_{x \to 1/2} \frac{2x^2+3x-2}{2x-1} = \frac{\frac{1}{2}+\frac{3}{2}-2}{1-1} = \frac{0}{0}\text{; then, } \lim\limits_{x \to 1/2} \frac{2x^2+3x-2}{2x-1} = \lim\limits_{x \to 1/2} \frac{(2x-1)(x+2)}{2x-1} = \frac{5}{2}$$

Exercise:

Problem: $\lim\limits_{x \to -3} \frac{\sqrt{x+4}-1}{x+3}$

In the following exercises, use direct substitution to obtain an undefined expression. Then, use the method of [link] to simplify the function to help determine the limit.

Exercise:

Problem: $\lim\limits_{x \to -2^-} \frac{2x^2+7x-4}{x^2+x-2}$

Solution:

$-\infty$

Exercise:

Problem: $\lim_{x \to -2^+} \frac{2x^2+7x-4}{x^2+x-2}$

Exercise:

Problem: $\lim_{x \to 1^-} \frac{2x^2+7x-4}{x^2+x-2}$

Solution:

$-\infty$

Exercise:

Problem: $\lim_{x \to 1^+} \frac{2x^2+7x-4}{x^2+x-2}$

In the following exercises, assume that $\lim_{x \to 6} f(x) = 4$, $\lim_{x \to 6} g(x) = 9$, and $\lim_{x \to 6} h(x) = 6$. Use these three facts and the limit laws to evaluate each limit.

Exercise:

Problem: $\lim_{x \to 6} 2f(x)g(x)$

Solution:

$\lim_{x \to 6} 2f(x)g(x) = 2\lim_{x \to 6} f(x) \lim_{x \to 6} g(x) = 72$

Exercise:

Problem: $\lim_{x \to 6} \frac{g(x)-1}{f(x)}$

Exercise:

Problem: $\lim_{x \to 6} \left(f(x) + \frac{1}{3}g(x)\right)$

Solution:

$\lim_{x \to 6} \left(f(x) + \frac{1}{3}g(x)\right) = \lim_{x \to 6} f(x) + \frac{1}{3}\lim_{x \to 6} g(x) = 7$

Exercise:

Problem: $\lim_{x \to 6} \frac{(h(x))^3}{2}$

Exercise:

Problem: $\lim_{x \to 6} \sqrt{g(x) - f(x)}$

Solution:

$$\lim_{x\to 6}\sqrt{g(x)-f(x)}=\sqrt{\lim_{x\to 6}g(x)-\lim_{x\to 6}f(x)}=\sqrt{5}$$

Exercise:

Problem: $\lim_{x\to 6} x\cdot h(x)$

Exercise:

Problem: $\lim_{x\to 6}\left[(x+1)\cdot f(x)\right]$

Solution:

$$\lim_{x\to 6}\left[(x+1)f(x)\right]=\left(\lim_{x\to 6}(x+1)\right)\left(\lim_{x\to 6}f(x)\right)=28$$

Exercise:

Problem: $\lim_{x\to 6}\left(f(x)\cdot g(x)-h(x)\right)$

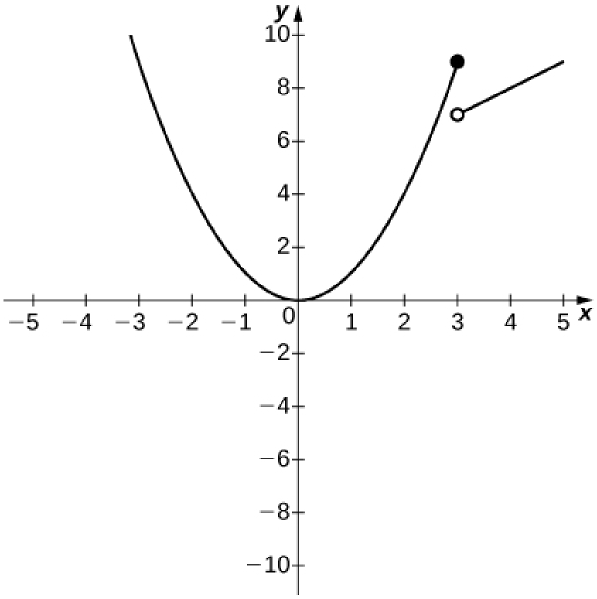
[T] In the following exercises, use a calculator to draw the graph of each piecewise-defined function and study the graph to evaluate the given limits.

Exercise:

Problem: $f(x)=\begin{cases}x^2, & x\le 3\\ x+4, & x>3\end{cases}$

a. $\lim_{x\to 3^-}f(x)$
b. $\lim_{x\to 3^+}f(x)$

Solution:



a. 9; b. 7

Exercise:

Problem: $g(x) = \begin{cases} x^3 - 1, & x \leq 0 \\ 1, & x > 0 \end{cases}$

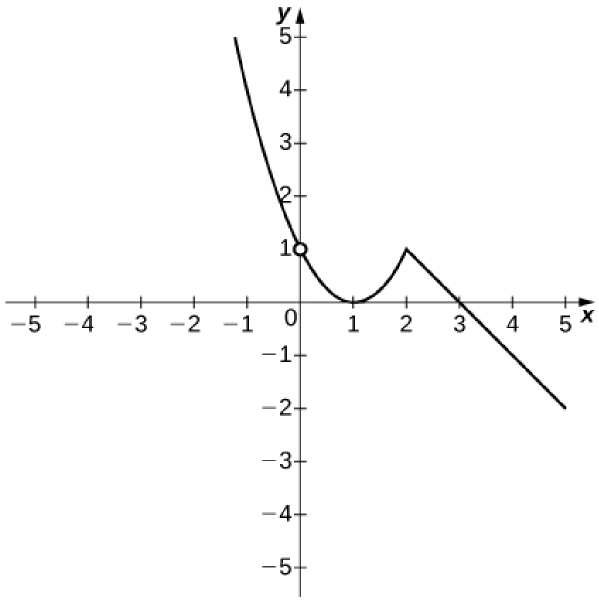
a. $\lim\limits_{x \to 0^-} g(x)$
b. $\lim\limits_{x \to 0^+} g(x)$

Exercise:

Problem: $h(x) = \begin{cases} x^2 - 2x + 1, & x < 2 \\ 3 - x, & x \geq 2 \end{cases}$

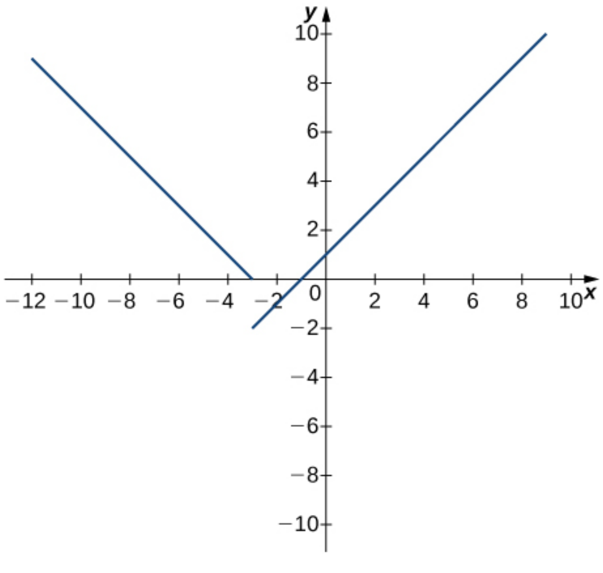
a. $\lim\limits_{x \to 2^-} h(x)$
b. $\lim\limits_{x \to 2^+} h(x)$

Solution:

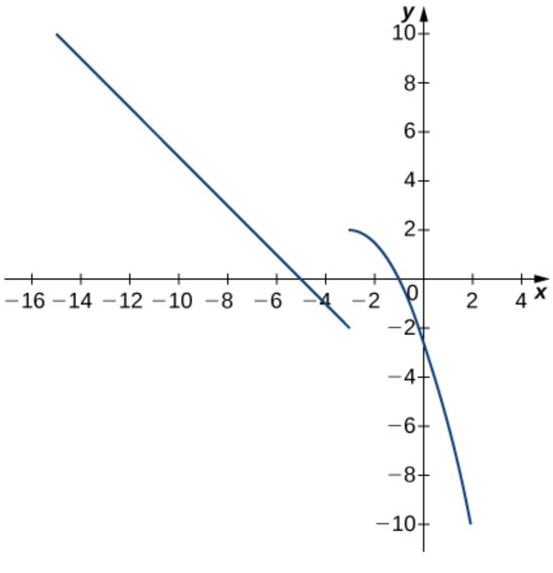


a. 1; b. 1

In the following exercises, use the following graphs and the limit laws to evaluate each limit.



$y = f(x)$



$y = g(x)$

Exercise:

Problem: $\lim\limits_{x \to -3^{+}} \left(f\left(x\right) + g\left(x\right)\right)$

Exercise:

Problem: $\lim\limits_{x \to -3^{-}} \left(f\left(x\right) - 3g\left(x\right)\right)$

Solution:

$$\lim_{x \to -3^{-}} \left(f\left(x\right) - 3g\left(x\right)\right) = \lim_{x \to -3^{-}} f\left(x\right) - 3 \lim_{x \to -3^{-}} g\left(x\right) = 0 + 6 = 6$$

Exercise:

Problem: $\lim\limits_{x\to 0}\frac{f(x)g(x)}{3}$

Exercise:

Problem: $\lim\limits_{x\to -5}\frac{2+g(x)}{f(x)}$

Solution:

$$\lim_{x\to -5}\frac{2+g(x)}{f(x)} = \frac{2+\left(\lim\limits_{x\to -5}g(x)\right)}{\lim\limits_{x\to -5}f(x)} = \frac{2+0}{2} = 1$$

Exercise:

Problem: $\lim\limits_{x\to 1}\left(f\left(x\right)\right)^2$

Exercise:

Problem: $\lim\limits_{x\to 1}\sqrt[3]{f\left(x\right)-g\left(x\right)}$

Solution:

$$\lim_{x\to 1}\sqrt[3]{f\left(x\right)-g\left(x\right)} = \sqrt[3]{\lim_{x\to 1}f\left(x\right)-\lim_{x\to 1}g\left(x\right)} = \sqrt[3]{2+5} = \sqrt[3]{7}$$

Exercise:

Problem: $\lim\limits_{x\to -7}\left(x\cdot g\left(x\right)\right)$

Exercise:

Problem: $\lim\limits_{x\to -9}\left[x\cdot f\left(x\right)+2\cdot g\left(x\right)\right]$

Solution:

$$\lim_{x\to -9}\left(xf\left(x\right)+2g\left(x\right)\right) = \left(\lim_{x\to -9}x\right)\left(\lim_{x\to -9}f\left(x\right)\right)+2\lim_{x\to -9}\left(g\left(x\right)\right) = \left(-9\right)\left(6\right)+2\left(4\right) = -46$$

For the following problems, evaluate the limit using the squeeze theorem. Use a calculator to graph the functions $f\left(x\right)$, $g\left(x\right)$, and $h\left(x\right)$ when possible.

Exercise:

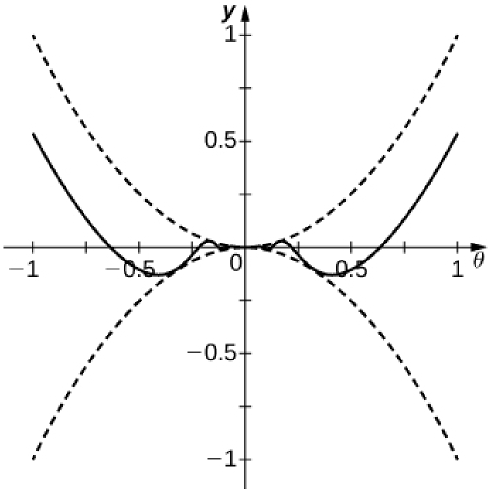
Problem: **[T]** True or False? If $2x-1\le g\left(x\right)\le x^2-2x+3$, then $\lim\limits_{x\to 2}g\left(x\right)=0$.

Exercise:

Problem: **[T]** $\lim\limits_{\theta\to 0}\theta^2\cos\left(\frac{1}{\theta}\right)$

Solution:

The limit is zero.



Exercise:

Problem: $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} 0, & x \text{ rational} \\ x^2, & x \text{ irrational} \end{cases}$

Exercise:

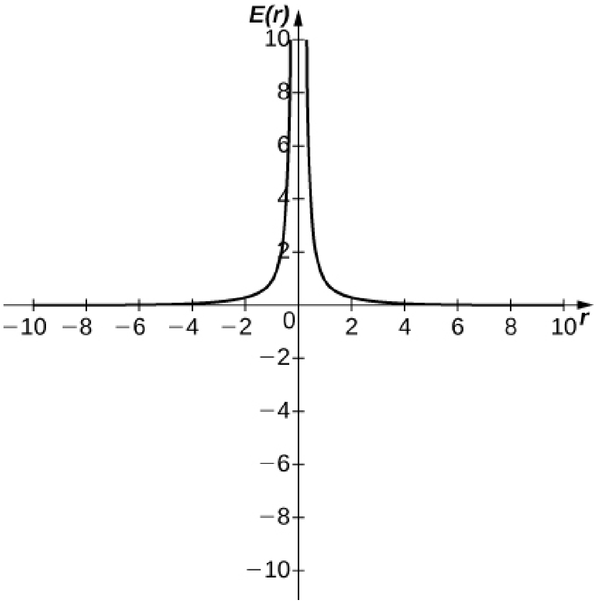
Problem:

[T] In physics, the magnitude of an electric field generated by a point charge at a distance *r* in vacuum is governed by Coulomb's law: $E(r) = \frac{q}{4\pi\varepsilon_0 r^2}$, where *E* represents the magnitude of the electric field, *q* is the charge of the particle, *r* is the distance between the particle and where the strength of the field is measured, and $\frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant: $8.988 \times 10^9\,\mathrm{N}\cdot\mathrm{m}^2/\mathrm{C}^2$.

a. Use a graphing calculator to graph $E(r)$ given that the charge of the particle is $q = 10^{-10}$.
b. Evaluate $\lim\limits_{r \to 0^+} E(r)$. What is the physical meaning of this quantity? Is it physically relevant? Why are you evaluating from the right?

Solution:

a.



b. ∞. The magnitude of the electric field as you approach the particle q becomes infinite. It does not make physical sense to evaluate negative distance.

Exercise:

Problem: **[T]** The density of an object is given by its mass divided by its volume: $\rho = m/V$.

a. Use a calculator to plot the volume as a function of density $(V = m/\rho)$, assuming you are examining something of mass 8 kg $(m = 8)$.
b. Evaluate $\lim_{\rho \to 0^+} V(\rho)$ and explain the physical meaning.

Glossary

constant multiple law for limits
the limit law $\lim_{x \to a} cf(x) = c \cdot \lim_{x \to a} f(x) = cL$

difference law for limits
the limit law $\lim_{x \to a} (f(x) - g(x)) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$

limit laws
the individual properties of limits; for each of the individual laws, let $f(x)$ and $g(x)$ be defined for all $x \neq a$ over some open interval containing a; assume that L and M are real numbers so that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$; let c be a constant

power law for limits
the limit law $\lim_{x \to a} (f(x))^n = \left(\lim_{x \to a} f(x)\right)^n = L^n$ for every positive integer n

product law for limits

the limit law $\lim\limits_{x \to a} \left(f\left(x\right) \cdot g\left(x\right)\right) = \lim\limits_{x \to a} f\left(x\right) \cdot \lim\limits_{x \to a} g\left(x\right) = L \cdot M$

quotient law for limits
the limit law $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \frac{L}{M}$ for $M \neq 0$

root law for limits
the limit law $\lim\limits_{x \to a} \sqrt[n]{f\left(x\right)} = \sqrt[n]{\lim\limits_{x \to a} f\left(x\right)} = \sqrt[n]{L}$ for all L if n is odd and for $L \geq 0$ if n is even

squeeze theorem
states that if $f\left(x\right) \leq g\left(x\right) \leq h\left(x\right)$ for all $x \neq a$ over an open interval containing a and $\lim\limits_{x \to a} f\left(x\right) = L = \lim\limits_{x \to a} h\left(x\right)$ where L is a real number, then $\lim\limits_{x \to a} g\left(x\right) = L$

sum law for limits
The limit law $\lim\limits_{x \to a} \left(f\left(x\right) + g\left(x\right)\right) = \lim\limits_{x \to a} f\left(x\right) + \lim\limits_{x \to a} g\left(x\right) = L + M$

Continuity

- Explain the three conditions for continuity at a point.
- Describe three kinds of discontinuities.
- Define continuity on an interval.
- State the theorem for limits of composite functions.
- Provide an example of the intermediate value theorem.

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called *continuous*. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a *discontinuity at a point* where a break occurs.

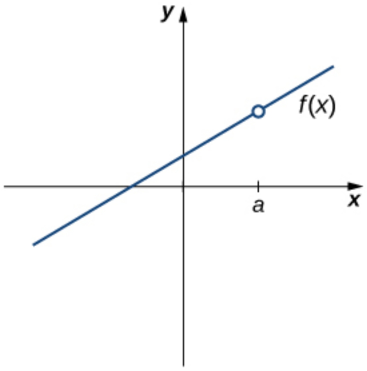
We begin our investigation of continuity by exploring what it means for a function to have *continuity at a point*. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Continuity at a Point

Before we look at a formal definition of what it means for a function to be continuous at a point, let’s consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that prevent such failures.

Our first function of interest is shown in [link]. We see that the graph of $f(x)$ has a hole at *a*. In fact, $f(a)$ is undefined. At the very least, for $f(x)$ to be continuous at *a*, we need the following condition:
Equation:

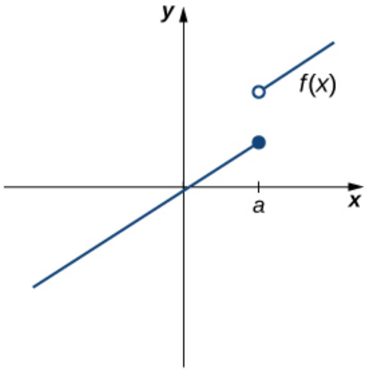
$$\text{i. } f(a) \text{ is defined.}$$



The function $f(x)$ is not continuous at *a* because $f(a)$ is undefined.

However, as we see in [link], this condition alone is insufficient to guarantee continuity at the point *a*. Although $f(a)$ is defined, the function has a gap at *a*. In this example, the gap exists because $\lim\limits_{x \to a} f(x)$ does not exist. We must add another condition for continuity at *a*—namely,
Equation:

$$\text{ii. } \lim_{x \to a} f(x) \text{ exists.}$$

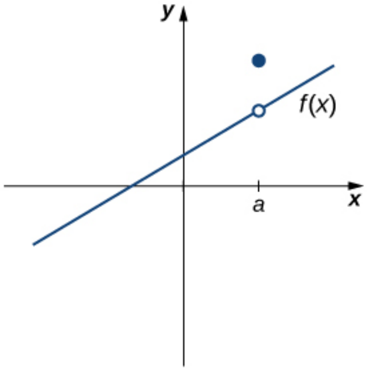


The function $f(x)$ is not continuous at *a* because $\lim_{x \to a} f(x)$ does not exist.

However, as we see in [link], these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but is still not continuous at *a*. We must add a third condition to our list:

Equation:

$$\text{iii. } \lim_{x \to a} f(x) = f(a).$$



The function $f(x)$ is not continuous at *a* because $\lim_{x \to a} f(x) \neq f(a)$.

Now we put our list of conditions together and form a definition of continuity at a point.

Note:
Definition
A function $f\left(x\right)$ is **continuous at a point** a if and only if the following three conditions are satisfied:

i. $f\left(a\right)$ is defined
ii. $\underset{x\to a}{\lim}f\left(x\right)$ exists
iii. $\underset{x\to a}{\lim}f\left(x\right)=f\left(a\right)$

A function is **discontinuous at a point** a if it fails to be continuous at a.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Note:
Problem-Solving Strategy: Determining Continuity at a Point

1. Check to see if $f\left(a\right)$ is defined. If $f\left(a\right)$ is undefined, we need go no further. The function is not continuous at a. If $f\left(a\right)$ is defined, continue to step 2.
2. Compute $\underset{x\to a}{\lim}f\left(x\right)$. In some cases, we may need to do this by first computing $\underset{x\to a^-}{\lim}f\left(x\right)$ and $\underset{x\to a^+}{\lim}f\left(x\right)$. If $\underset{x\to a}{\lim}f\left(x\right)$ does not exist (that is, it is not a real number), then the function is not continuous at a and the problem is solved. If $\underset{x\to a}{\lim}f\left(x\right)$ exists, then continue to step 3.
3. Compare $f\left(a\right)$ and $\underset{x\to a}{\lim}f\left(x\right)$. If $\underset{x\to a}{\lim}f\left(x\right)\ne f\left(a\right)$, then the function is not continuous at a. If $\underset{x\to a}{\lim}f\left(x\right)=f\left(a\right)$, then the function is continuous at a.

The next three examples demonstrate how to apply this definition to determine whether a function is continuous at a given point. These examples illustrate situations in which each of the conditions for continuity in the definition succeed or fail.

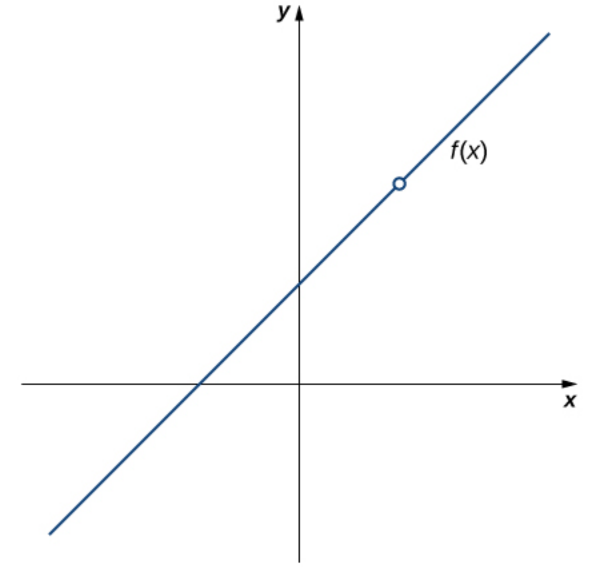
Example:
Exercise:

Problem:
Determining Continuity at a Point, Condition 1

Using the definition, determine whether the function $f\left(x\right)=\left(x^2-4\right)/\left(x-2\right)$ is continuous at $x=2$. Justify the conclusion.

Solution:

Let's begin by trying to calculate $f\left(2\right)$. We can see that $f\left(2\right)=0/0$, which is undefined. Therefore, $f\left(x\right)=\frac{x^2-4}{x-2}$ is discontinuous at 2 because $f\left(2\right)$ is undefined. The graph of $f\left(x\right)$ is shown in [link].



The function $f(x)$ is discontinuous at 2 because $f(2)$ is undefined.

Example:
Exercise:

Problem:
Determining Continuity at a Point, Condition 2

Using the definition, determine whether the function $f(x) = \begin{cases} -x^2 + 4 & \text{if } x \le 3 \\ 4x - 8 & \text{if } x > 3 \end{cases}$ is continuous at $x = 3$. Justify the conclusion.

Solution:

Let's begin by trying to calculate $f(3)$.
Equation:

$$f(3) = -(3^2) + 4 = -5.$$

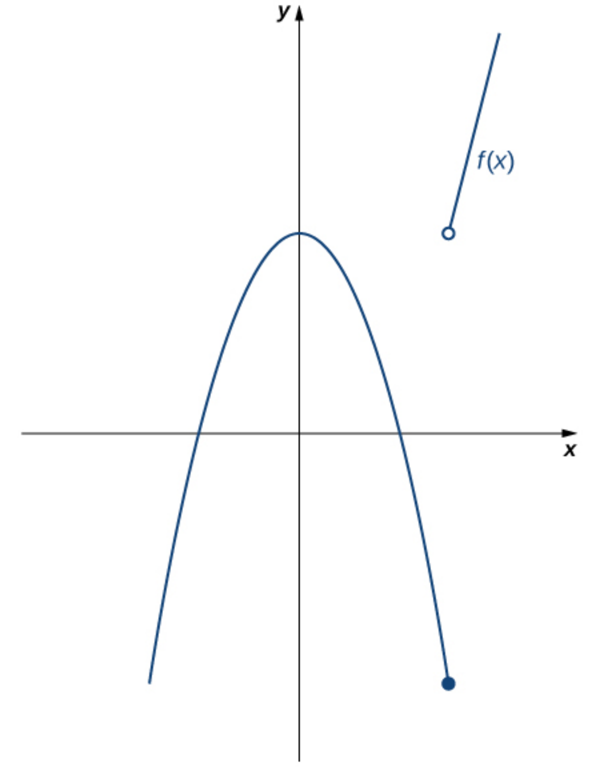
Thus, $f(3)$ is defined. Next, we calculate $\lim\limits_{x \to 3} f(x)$. To do this, we must compute $\lim\limits_{x \to 3^-} f(x)$ and $\lim\limits_{x \to 3^+} f(x)$:
Equation:

$$\lim_{x \to 3^-} f(x) = -(3^2) + 4 = -5$$

and
Equation:

$$\lim_{x \to 3^+} f(x) = 4(3) - 8 = 4.$$

Therefore, $\lim_{x \to 3} f(x)$ does not exist. Thus, $f(x)$ is not continuous at 3. The graph of $f(x)$ is shown in [link].



The function $f(x)$ is not continuous at 3 because $\lim_{x \to 3} f(x)$ does not exist.

Example:
Exercise:

Problem:
Determining Continuity at a Point, Condition 3

Using the definition, determine whether the function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$.

Solution:

First, observe that
Equation:

$$f(0) = 1.$$

Next,
Equation:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Last, compare $f(0)$ and $\lim_{x \to 1} f(x)$. We see that

Equation:

$$f(0) = 1 = \lim_{x \to 0} f(x).$$

Since all three of the conditions in the definition of continuity are satisfied, $f(x)$ is continuous at $x = 0$.

Note:

Exercise:

Problem:

Using the definition, determine whether the function $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x+4 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. If the function is not continuous at 1, indicate the condition for continuity at a point that fails to hold.

Solution:

f is not continuous at 1 because $f(1) = 2 \neq 3 = \lim_{x \to 1} f(x)$.

Hint

Check each condition of the definition.

By applying the definition of continuity and previously established theorems concerning the evaluation of limits, we can state the following theorem.

Note:

Continuity of Polynomials and Rational Functions

Polynomials and rational functions are continuous at every point in their domains.

Proof

Previously, we showed that if $p(x)$ and $q(x)$ are polynomials, $\lim_{x \to a} p(x) = p(a)$ for every polynomial $p(x)$ and $\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$ as long as $q(a) \neq 0$. Therefore, polynomials and rational functions are continuous on their domains.

□

We now apply [link] to determine the points at which a given rational function is continuous.

Example:
Exercise:

Problem:
Continuity of a Rational Function

For what values of x is $f\left(x\right)=\frac{x+1}{x-5}$ continuous?

Solution:

The rational function $f\left(x\right)=\frac{x+1}{x-5}$ is continuous for every value of x except $x=5.$

Note:
Exercise:

Problem: For what values of x is $f\left(x\right)=3x^4-4x^2$ continuous?

Solution:

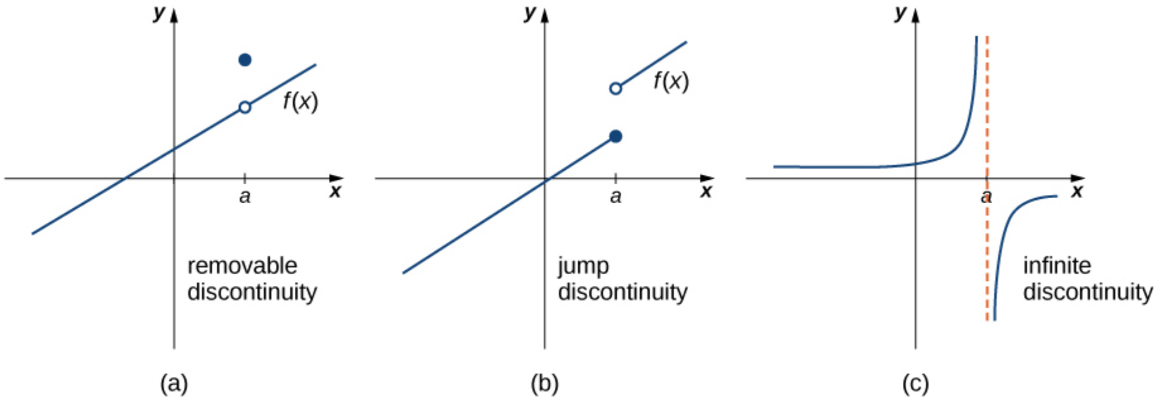
$f\left(x\right)$ is continuous at every real number.

Hint

Use [link]

Types of Discontinuities

As we have seen in [link] and [link], discontinuities take on several different appearances. We classify the types of discontinuities we have seen thus far as removable discontinuities, infinite discontinuities, or jump discontinuities. Intuitively, a **removable discontinuity** is a discontinuity for which there is a hole in the graph, a **jump discontinuity** is a noninfinite discontinuity for which the sections of the function do not meet up, and an **infinite discontinuity** is a discontinuity located at a vertical asymptote. [link] illustrates the differences in these types of discontinuities. Although these terms provide a handy way of describing three common types of discontinuities, keep in mind that not all discontinuities fit neatly into these categories.



Discontinuities are classified as (a) removable, (b) jump, or (c) infinite.

These three discontinuities are formally defined as follows:

Note:
Definition
If $f\left(x\right)$ is discontinuous at a, then

1. f has a **removable discontinuity** at a if $\lim_{x\to a} f\left(x\right)$ exists. (Note: When we state that $\lim_{x\to a} f\left(x\right)$ exists, we mean that $\lim_{x\to a} f\left(x\right) = L$, where L is a real number.)
2. f has a **jump discontinuity** at a if $\lim_{x\to a^-} f\left(x\right)$ and $\lim_{x\to a^+} f\left(x\right)$ both exist, but $\lim_{x\to a^-} f\left(x\right) \ne \lim_{x\to a^+} f\left(x\right)$. (Note: When we state that $\lim_{x\to a^-} f\left(x\right)$ and $\lim_{x\to a^+} f\left(x\right)$ both exist, we mean that both are real-valued and that neither take on the values ±∞.)
3. f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f\left(x\right) = \pm\infty$ or $\lim_{x\to a^+} f\left(x\right) = \pm\infty$.

Example:
Exercise:

Problem:
Classifying a Discontinuity

In [link], we showed that $f\left(x\right) = \frac{x^2-4}{x-2}$ is discontinuous at $x = 2$. Classify this discontinuity as removable, jump, or infinite.

Solution:

To classify the discontinuity at 2 we must evaluate $\lim_{x\to 2} f\left(x\right)$:

Equation:

$$\begin{aligned}\lim_{x\to 2} f\left(x\right) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2} \left(x+2\right) \\ &= 4.\end{aligned}$$

Since f is discontinuous at 2 and $\lim_{x\to 2} f\left(x\right)$ exists, f has a removable discontinuity at $x = 2$.

Example:
Exercise:

Problem:
Classifying a Discontinuity

In [link], we showed that $f(x) = \begin{cases} -x^2+4 & \text{if } x \le 3 \\ 4x-8 & \text{if } x > 3 \end{cases}$ is discontinuous at $x = 3$. Classify this discontinuity as removable, jump, or infinite.

Solution:

Earlier, we showed that *f* is discontinuous at 3 because $\lim\limits_{x \to 3} f(x)$ does not exist. However, since $\lim\limits_{x \to 3^-} f(x) = -5$ and $\lim\limits_{x \to 3^+} f(x) = 4$ both exist, we conclude that the function has a jump discontinuity at 3.

Example:
Exercise:

Problem:
Classifying a Discontinuity

Determine whether $f(x) = \frac{x+2}{x+1}$ is continuous at -1. If the function is discontinuous at -1, classify the discontinuity as removable, jump, or infinite.

Solution:

The function value $f(-1)$ is undefined. Therefore, the function is not continuous at -1. To determine the type of discontinuity, we must determine the limit at -1. We see that $\lim\limits_{x \to -1^-} \frac{x+2}{x+1} = -\infty$ and $\lim\limits_{x \to -1^+} \frac{x+2}{x+1} = +\infty$. Therefore, the function has an infinite discontinuity at -1.

Note:
Exercise:

Problem:

For $f(x) = \begin{cases} x^2 & \text{if } x \ne 1 \\ 3 & \text{if } x = 1 \end{cases}$, decide whether *f* is continuous at 1. If *f* is not continuous at 1, classify the discontinuity as removable, jump, or infinite.

Solution:

Discontinuous at 1; removable

Hint

Follow the steps in [link]. If the function is discontinuous at 1, look at $\lim\limits_{x \to 1} f(x)$ and use the definition to determine the type of discontinuity.

Continuity over an Interval

Now that we have explored the concept of continuity at a point, we extend that idea to **continuity over an interval**. As we develop this idea for different types of intervals, it may be useful to keep in mind the intuitive idea that a function is continuous over an interval if we can use a pencil to trace the function between any two points in the interval without lifting the pencil from the paper. In preparation for defining continuity on an interval, we begin by looking at the definition of what it means for a function to be continuous from the right at a point and continuous from the left at a point.

Note:
Continuity from the Right and from the Left
A function $f(x)$ is said to be **continuous from the right** at a if $\lim_{x \to a^+} f(x) = f(a)$.
A function $f(x)$ is said to be **continuous from the left** at a if $\lim_{x \to a^-} f(x) = f(a)$.

A function is continuous over an open interval if it is continuous at every point in the interval. A function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b) and is continuous from the right at a and is continuous from the left at b. Analogously, a function $f(x)$ is continuous over an interval of the form $(a, b]$ if it is continuous over (a, b) and is continuous from the left at b. Continuity over other types of intervals are defined in a similar fashion.

Requiring that $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$ ensures that we can trace the graph of the function from the point $(a, f(a))$ to the point $(b, f(b))$ without lifting the pencil. If, for example, $\lim_{x \to a^+} f(x) \neq f(a)$, we would need to lift our pencil to jump from $f(a)$ to the graph of the rest of the function over $(a, b]$.

Example:
Exercise:

Problem:
Continuity on an Interval

State the interval(s) over which the function $f(x) = \frac{x-1}{x^2+2x}$ is continuous.

Solution:

Since $f(x) = \frac{x-1}{x^2+2x}$ is a rational function, it is continuous at every point in its domain. The domain of $f(x)$ is the set $(-\infty, -2) \cup (-2, 0) \cup (0, +\infty)$. Thus, $f(x)$ is continuous over each of the intervals $(-\infty, -2)$, $(-2, 0)$, and $(0, +\infty)$.

Example:
Exercise:

Problem:
Continuity over an Interval

State the interval(s) over which the function $f(x) = \sqrt{4 - x^2}$ is continuous.

Solution:

From the limit laws, we know that $\lim_{x \to a} \sqrt{4 - x^2} = \sqrt{4 - a^2}$ for all values of a in $(-2, 2)$. We also know that $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ exists and $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ exists. Therefore, $f(x)$ is continuous over the interval $[-2, 2]$.

Note:
Exercise:

Problem: State the interval(s) over which the function $f\left(x\right)=\sqrt{x+3}$ is continuous.

Solution:

$\left[-3,+\infty\right)$

Hint

Use [link] as a guide for solving.

The [link] allows us to expand our ability to compute limits. In particular, this theorem ultimately allows us to demonstrate that trigonometric functions are continuous over their domains.

Note:
Composite Function Theorem
If $f\left(x\right)$ is continuous at L and $\lim\limits_{x\to a}g\left(x\right)=L$, then
Equation:

$$\lim\limits_{x\to a}f\left(g\left(x\right)\right)=f\left(\lim\limits_{x\to a}g\left(x\right)\right)=f\left(L\right).$$

Before we move on to [link], recall that earlier, in the section on limit laws, we showed $\lim\limits_{x\to 0}\cos x=1=\cos\left(0\right)$. Consequently, we know that $f\left(x\right)=\cos x$ is continuous at 0. In [link] we see how to combine this result with the composite function theorem.

Example:
Exercise:

Problem:
Limit of a Composite Cosine Function

Evaluate $\lim\limits_{x\to\pi/2}\cos\left(x-\frac{\pi}{2}\right)$.

Solution:

The given function is a composite of $\cos x$ and $x-\frac{\pi}{2}$. Since $\lim\limits_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)=0$ and $\cos x$ is continuous at 0, we may apply the composite function theorem. Thus,
Equation:

$$\lim\limits_{x\to\pi/2}\cos\left(x-\frac{\pi}{2}\right)=\cos\left(\lim\limits_{x\to\pi/2}\left(x-\frac{\pi}{2}\right)\right)=\cos\left(0\right)=1.$$

Note:
Exercise:

Problem: Evaluate $\lim\limits_{x \to \pi} \sin\left(x - \pi\right)$.

Solution:

0

Hint

$f\left(x\right) = \sin x$ is continuous at 0. Use [link] as a guide.

The proof of the next theorem uses the composite function theorem as well as the continuity of $f\left(x\right) = \sin x$ and $g\left(x\right) = \cos x$ at the point 0 to show that trigonometric functions are continuous over their entire domains.

Note:
Continuity of Trigonometric Functions
Trigonometric functions are continuous over their entire domains.

Proof

We begin by demonstrating that $\cos x$ is continuous at every real number. To do this, we must show that $\lim\limits_{x \to a} \cos x = \cos a$ for all values of a.

$$\begin{aligned}
\lim_{x \to a} \cos x &= \lim_{x \to a} \cos\left(\left(x - a\right) + a\right) && \text{rewrite } x = x - a + a \\
&= \lim_{x \to a} \left(\cos\left(x - a\right)\cos a - \sin\left(x - a\right)\sin a\right) && \text{apply the identity for the cosine of the} \\
&= \cos\left(\lim_{x \to a}\left(x - a\right)\right)\cos a - \sin\left(\lim_{x \to a}\left(x - a\right)\right)\sin a && \lim_{x \to a}\left(x - a\right) = 0, \text{ and } \sin x \text{ and } \cos x \text{ are} \\
&= \cos\left(0\right)\cos a - \sin\left(0\right)\sin a && \text{evaluate } \cos(0) \text{ and } \sin(0) \text{ and simplify} \\
&= 1 \cdot \cos a - 0 \cdot \sin a = \cos a.
\end{aligned}$$

The proof that $\sin x$ is continuous at every real number is analogous. Because the remaining trigonometric functions may be expressed in terms of $\sin x$ and $\cos x$, their continuity follows from the quotient limit law.

□

As you can see, the composite function theorem is invaluable in demonstrating the continuity of trigonometric functions. As we continue our study of calculus, we revisit this theorem many times.

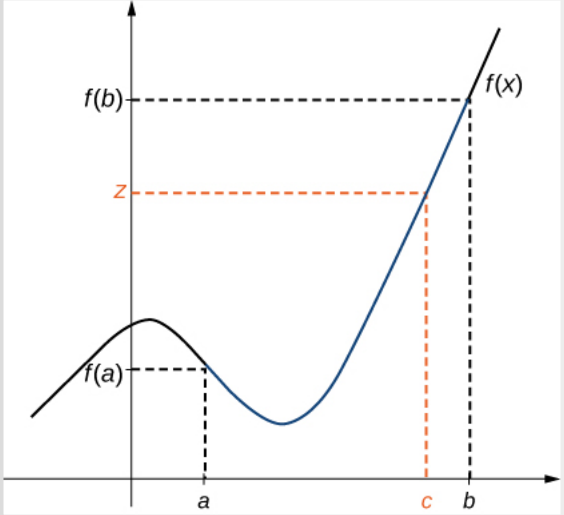
The Intermediate Value Theorem

Functions that are continuous over intervals of the form $[a, b]$, where a and b are real numbers, exhibit many useful properties. Throughout our study of calculus, we will encounter many powerful theorems concerning such functions. The first of these theorems is the **Intermediate Value Theorem**.

Note:

The Intermediate Value Theorem

Let f be continuous over a closed, bounded interval $[a,b]$. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ satisfying $f(c)=z$ in [link].



There is a number $c \in [a,b]$ that satisfies $f(c)=z$.

Example:

Exercise:

Problem:

Application of the Intermediate Value Theorem

Show that $f(x)=x-\cos x$ has at least one zero.

Solution:

Since $f(x)=x-\cos x$ is continuous over $(-\infty,+\infty)$, it is continuous over any closed interval of the form $[a,b]$. If you can find an interval $[a,b]$ such that $f(a)$ and $f(b)$ have opposite signs, you can use the Intermediate Value Theorem to conclude there must be a real number c in (a,b) that satisfies $f(c)=0$. Note that

Equation:

$$f(0)=0-\cos(0)=-1<0$$

and

Equation:

$$f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}-\cos\frac{\pi}{2}=\frac{\pi}{2}>0.$$

Using the Intermediate Value Theorem, we can see that there must be a real number c in $[0,\pi/2]$ that satisfies $f(c)=0$. Therefore, $f(x)=x-\cos x$ has at least one zero.

Example:
Exercise:

Problem:
When Can You Apply the Intermediate Value Theorem?

If $f\left(x\right)$ is continuous over $\left[0,2\right]$, $f\left(0\right)>0$ and $f\left(2\right)>0$, can we use the Intermediate Value Theorem to conclude that $f\left(x\right)$ has no zeros in the interval $\left[0,2\right]$? Explain.

Solution:

No. The Intermediate Value Theorem only allows us to conclude that we can find a value between $f\left(0\right)$ and $f\left(2\right)$; it doesn't allow us to conclude that we can't find other values. To see this more clearly, consider the function $f\left(x\right)={\left(x-1\right)}^{2}$. It satisfies $f\left(0\right)=1>0$, $f\left(2\right)=1>0$, and $f\left(1\right)=0$.

Example:
Exercise:

Problem:
When Can You Apply the Intermediate Value Theorem?

For $f\left(x\right)=1/x$, $f\left(-1\right)=-1<0$ and $f\left(1\right)=1>0$. Can we conclude that $f\left(x\right)$ has a zero in the interval $\left[-1,1\right]$?

Solution:

No. The function is not continuous over $\left[-1,1\right]$. The Intermediate Value Theorem does not apply here.

Note:
Exercise:

Problem: Show that $f\left(x\right)={x}^{3}-{x}^{2}-3x+1$ has a zero over the interval $\left[0,1\right]$.

Solution:

$f\left(0\right)=1>0,f\left(1\right)=-2<0$; $f\left(x\right)$ is continuous over $\left[0,1\right]$. It must have a zero on this interval.

Hint

Find $f\left(0\right)$ and $f\left(1\right)$. Apply the Intermediate Value Theorem.

Key Concepts

- For a function to be continuous at a point, it must be defined at that point, its limit must exist at the point, and the value of the function at that point must equal the value of the limit at that point.
- Discontinuities may be classified as removable, jump, or infinite.
- A function is continuous over an open interval if it is continuous at every point in the interval. It is continuous over a closed interval if it is continuous at every point in its interior and is continuous at its endpoints.
- The composite function theorem states: If $f\left(x\right)$ is continuous at L and $\underset{x\to a}{\lim}g\left(x\right)=L$, then $\underset{x\to a}{\lim}f\left(g\left(x\right)\right)=f\left(\underset{x\to a}{\lim}g\left(x\right)\right)=f\left(L\right)$.

- The Intermediate Value Theorem guarantees that if a function is continuous over a closed interval, then the function takes on every value between the values at its endpoints.

For the following exercises, determine the point(s), if any, at which each function is discontinuous. Classify any discontinuity as jump, removable, infinite, or other.

Exercise:

Problem: $f\left(x\right)=\frac{1}{\sqrt{x}}$

Solution:

The function is defined for all *x* in the interval $\left(0,\infty\right)$.

Exercise:

Problem: $f\left(x\right)=\frac{2}{x^{2}+1}$

Exercise:

Problem: $f\left(x\right)=\frac{x}{x^{2}-x}$

Solution:

Removable discontinuity at $x=0$; infinite discontinuity at $x=1$

Exercise:

Problem: $g\left(t\right)=t^{-1}+1$

Exercise:

Problem: $f\left(x\right)=\frac{5}{e^{x}-2}$

Solution:

Infinite discontinuity at $x=\ln 2$

Exercise:

Problem: $f\left(x\right)=\frac{|x-2|}{x-2}$

Exercise:

Problem: $H\left(x\right)=\tan 2x$

Solution:

Infinite discontinuities at $x=\frac{(2k+1)\pi}{4}$, for $k=0,\pm 1,\pm 2,\pm 3,\ldots$

Exercise:

Problem: $f\left(t\right)=\frac{t+3}{t^{2}+5t+6}$

For the following exercises, decide if the function continuous at the given point. If it is discontinuous, what type of discontinuity is it?

Exercise:

Problem: $f(x)\frac{2x^2-5x+3}{x-1}$ at $x=1$

Solution:

No. It is a removable discontinuity.

Exercise:

Problem: $h\left(\theta\right)=\frac{\sin\theta-\cos\theta}{\tan\theta}$ at $\theta=\pi$

Exercise:

Problem: $g\left(u\right)=\begin{cases}\frac{6u^2+u-2}{2u-1} & \text{if } u\neq\frac{1}{2}\\ \frac{7}{2} & \text{if } u=\frac{1}{2}\end{cases}$, at $u=\frac{1}{2}$

Solution:

Yes. It is continuous.

Exercise:

Problem: $f\left(y\right)=\frac{\sin\left(\pi y\right)}{\tan\left(\pi y\right)}$, at $y=1$

Exercise:

Problem: $f\left(x\right)=\begin{cases}x^2-e^x & \text{if } x<0\\ x-1 & \text{if } x\geq 0\end{cases}$, at $x=0$

Solution:

Yes. It is continuous.

Exercise:

Problem: $f\left(x\right)=\begin{cases}x\sin\left(x\right) \text{ if } x\leq\pi\\ x\tan\left(x\right) \text{ if } x>\pi\end{cases}$, at $x=\pi$

In the following exercises, find the value(s) of *k* that makes each function continuous over the given interval.

Exercise:

Problem: $f\left(x\right)=\begin{cases}3x+2, & x<k\\ 2x-3, & k\leq x\leq 8\end{cases}$

Solution:

$k=-5$

Exercise:

Problem: $f\left(\theta\right)=\begin{cases}\sin\theta, & 0\leq\theta<\frac{\pi}{2}\\ \cos\left(\theta+k\right), & \frac{\pi}{2}\leq\theta\leq\pi\end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} \frac{x^2+3x+2}{x+2}, & x \neq -2 \\ k, & x = -2 \end{cases}$

Solution:

$k = -1$

Exercise:

Problem: $f(x) = \begin{cases} e^{kx}, & 0 \leq x < 4 \\ x+3, & 4 \leq x \leq 8 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} \sqrt{kx}, & 0 \leq x \leq 3 \\ x+1, & 3 < x \leq 10 \end{cases}$

Solution:

$k = \frac{16}{3}$

In the following exercises, use the Intermediate Value Theorem (IVT).

Exercise:

Problem:

Let $h(x) = \begin{cases} 3x^2 - 4, & x \leq 2 \\ 5 + 4x, & x > 2 \end{cases}$ Over the interval $[0, 4]$, there is no value of *x* such that $h(x) = 10$, although $h(0) < 10$ and $h(4) > 10$. Explain why this does not contradict the IVT.

Exercise:

Problem:

A particle moving along a line has at each time *t* a position function $s(t)$, which is continuous. Assume $s(2) = 5$ and $s(5) = 2$. Another particle moves such that its position is given by $h(t) = s(t) - t$. Explain why there must be a value *c* for $2 < c < 5$ such that $h(c) = 0$.

Solution:

Since both *s* and $y = t$ are continuous everywhere, then $h(t) = s(t) - t$ is continuous everywhere and, in particular, it is continuous over the closed interval $[2, 5]$. Also, $h(2) = 3 > 0$ and $h(5) = -3 < 0$. Therefore, by the IVT, there is a value $x = c$ such that $h(c) = 0$.

Exercise:

Problem: **[T]** Use the statement “The cosine of *t* is equal to *t* cubed.”

a. Write a mathematical equation of the statement.
b. Prove that the equation in part a. has at least one real solution.
c. Use a calculator to find an interval of length 0.01 that contains a solution.

Exercise:

Problem:

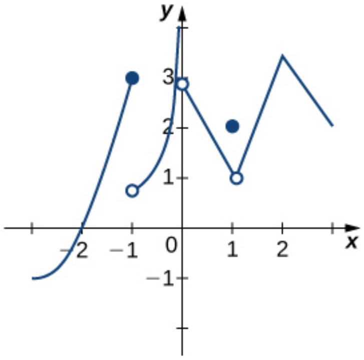
Apply the IVT to determine whether $2^x = x^3$ has a solution in one of the intervals $[1.25, 1.375]$ or $[1.375, 1.5]$. Briefly explain your response for each interval.

Solution:

The function $f(x) = 2^x - x^3$ is continuous over the interval $[1.25, 1.375]$ and has opposite signs at the endpoints.

Exercise:

Problem: Consider the graph of the function $y = f(x)$ shown in the following graph.



a. Find all values for which the function is discontinuous.
b. For each value in part a., state why the formal definition of continuity does not apply.
c. Classify each discontinuity as either jump, removable, or infinite.

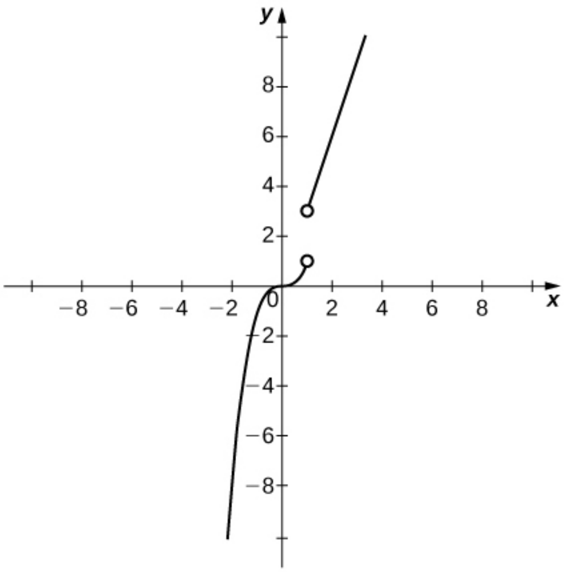
Exercise:

Problem: Let $f(x) = \begin{cases} 3x, x > 1 \\ x^3, x < 1 \end{cases}$.

a. Sketch the graph of *f*.
b. Is it possible to find a value *k* such that $f(1) = k$, which makes $f(x)$ continuous for all real numbers? Briefly explain.

Solution:

a.



b. It is not possible to redefine $f\left(1\right)$ since the discontinuity is a jump discontinuity.

Exercise:

Problem: Let $f\left(x\right)=\frac{x^4-1}{x^2-1}$ for $x\ne -1,1$.

a. Sketch the graph of *f*.
b. Is it possible to find values k_1 and k_2 such that $f\left(-1\right)=k_1$ and $f\left(1\right)=k_2$, and that makes $f\left(x\right)$ continuous for all real numbers? Briefly explain.

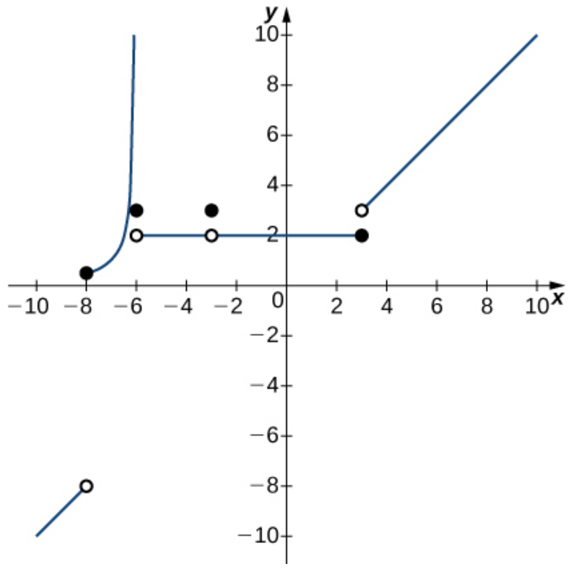
Exercise:

Problem: Sketch the graph of the function $y=f\left(x\right)$ with properties i. through vii.

i. The domain of *f* is $\left(-\infty ,+\infty \right)$.
ii. *f* has an infinite discontinuity at $x=-6$.
iii. $f\left(-6\right)=3$
iv. $\lim_{x\to -3^-}f\left(x\right)=\lim_{x\to -3^+}f\left(x\right)=2$
v. $f\left(-3\right)=3$
vi. *f* is left continuous but not right continuous at $x=3$.
vii. $\lim_{x\to -\infty}f\left(x\right)=-\infty$ and $\lim_{x\to +\infty}f\left(x\right)=+\infty$

Solution:

Answers may vary; see the following example:



Exercise:

Problem: Sketch the graph of the function $y=f\left(x\right)$ with properties i. through iv.

i. The domain of *f* is $\left[0,5\right]$.
ii. $\underset{x\to 1^{+}}{\lim}f\left(x\right)$ and $\underset{x\to 1^{-}}{\lim}f\left(x\right)$ exist and are equal.
iii. $f\left(x\right)$ is left continuous but not continuous at $x=2$, and right continuous but not continuous at $x=3$.
iv. $f\left(x\right)$ has a removable discontinuity at $x=1$, a jump discontinuity at $x=2$, and the following limits hold: $\underset{x\to 3^{-}}{\lim}f\left(x\right)=-\infty$ and $\underset{x\to 3^{+}}{\lim}f\left(x\right)=2$.

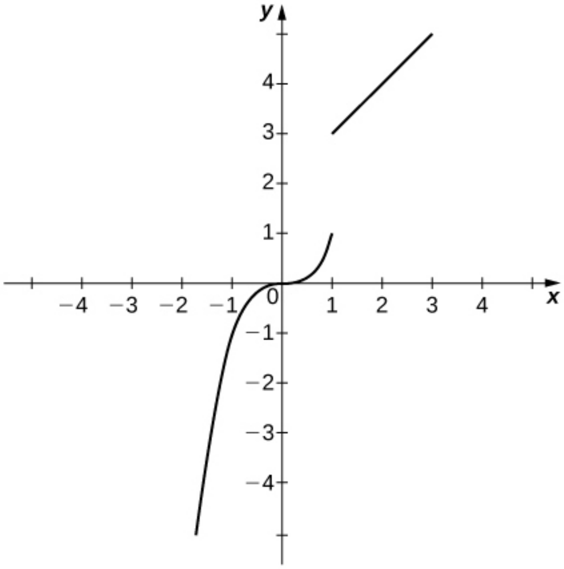
In the following exercises, suppose $y=f\left(x\right)$ is defined for all *x*. For each description, sketch a graph with the indicated property.

Exercise:

Problem: Discontinuous at $x=1$ with $\underset{x\to -1}{\lim}f\left(x\right)=-1$ and $\underset{x\to 2}{\lim}f\left(x\right)=4$

Solution:

Answers may vary; see the following example:



Exercise:

Problem: Discontinuous at $x = 2$ but continuous elsewhere with $\lim\limits_{x \to 0} f\left(x\right) = \frac{1}{2}$

Determine whether each of the given statements is true. Justify your response with an explanation or counterexample.

Exercise:

Problem: $f\left(t\right) = \frac{2}{e^{t} - e^{-t}}$ is continuous everywhere.

Solution:

False. It is continuous over $\left(-\infty, 0\right) \cup \left(0, \infty\right)$.

Exercise:

Problem:

If the left- and right-hand limits of $f\left(x\right)$ as $x \to a$ exist and are equal, then f cannot be discontinuous at $x = a$.

Exercise:

Problem: If a function is not continuous at a point, then it is not defined at that point.

Solution:

False. Consider $f\left(x\right) = \begin{cases} x \text{ if } x \neq 0 \\ 4 \text{ if } x = 0 \end{cases}$.

Exercise:

Problem: According to the IVT, $\cos x - \sin x - x = 2$ has a solution over the interval $\left[-1, 1\right]$.

Exercise:

Problem:

If $f(x)$ is continuous such that $f(a)$ and $f(b)$ have opposite signs, then $f(x) = 0$ has exactly one solution in $[a, b]$.

Solution:

False. IVT only says that there is at least one solution; it does not guarantee that there is exactly one. Consider $f(x) = \cos(x)$ on $[-\pi, 2\pi]$.

Exercise:

Problem: The function $f(x) = \frac{x^2-4x+3}{x^2-1}$ is continuous over the interval $[0, 3]$.

Exercise:

Problem:

If $f(x)$ is continuous everywhere and $f(a), f(b) > 0$, then there is no root of $f(x)$ in the interval $[a, b]$.

Solution:

False. The IVT does *not* work in reverse! Consider $(x-1)^2$ over the interval $[-2, 2]$.

[T] The following problems consider the scalar form of Coulomb's law, which describes the electrostatic force between two point charges, such as electrons. It is given by the equation $F(r) = k_e \frac{|q_1 q_2|}{r^2}$, where k_e is Coulomb's constant, q_i are the magnitudes of the charges of the two particles, and *r* is the distance between the two particles.

Exercise:

Problem:

To simplify the calculation of a model with many interacting particles, after some threshold value $r = R$, we approximate *F* as zero.

a. Explain the physical reasoning behind this assumption.
b. What is the force equation?
c. Evaluate the force *F* using both Coulomb's law and our approximation, assuming two protons with a charge magnitude of 1.6022×10^{-19} coulombs (C), and the Coulomb constant $k_e = 8.988 \times 10^9 \mathrm{Nm}^2/\mathrm{C}^2$ are 1 m apart. Also, assume $R < 1\,\mathrm{m}$. How much inaccuracy does our approximation generate? Is our approximation reasonable?
d. Is there any finite value of *R* for which this system remains continuous at *R*?

Exercise:

Problem:

Instead of making the force 0 at *R*, instead we let the force be 10^{-20} for $r \geq R$. Assume two protons, which have a magnitude of charge $1.6022 \times 10^{-19}\,\mathrm{C}$, and the Coulomb constant $k_e = 8.988 \times 10^9 \mathrm{Nm}^2/\mathrm{C}^2$. Is there a value *R* that can make this system continuous? If so, find it.

Solution:

$R = 0.0001519\,\mathrm{m}$

Recall the discussion on spacecraft from the chapter opener. The following problems consider a rocket launch from Earth's surface. The force of gravity on the rocket is given by $F(d) = -mk/d^2$, where *m* is the mass of the rocket, *d* is the distance of the rocket from the center of Earth, and *k* is a constant.

Exercise:

Problem:

[T] Determine the value and units of *k* given that the mass of the rocket is 3 million kg. (*Hint*: The distance from the center of Earth to its surface is 6378 km.)

Exercise:

Problem:

[T] After a certain distance *D* has passed, the gravitational effect of Earth becomes quite negligible, so we can approximate the force function by $F\left(d\right)=\begin{cases}-\frac{mk}{d^2} & \text{if } d<D\\ 10{,}000 & \text{if } d\ge D\end{cases}$. Using the value of k found in the previous exercise, find the necessary condition *D* such that the force function remains continuous.

Solution:

$D=345{,}826\ \text{km}$

Exercise:

Problem:

As the rocket travels away from Earth's surface, there is a distance *D* where the rocket sheds some of its mass, since it no longer needs the excess fuel storage. We can write this function as $F\left(d\right)=\begin{cases}-\frac{m_1k}{d^2} \text{ if } d<D\\ -\frac{m_2k}{d^2} \text{ if } d\ge D\end{cases}$. Is there a *D* value such that this function is continuous, assuming $m_1\ne m_2$?

Prove the following functions are continuous everywhere

Exercise:

Problem: $f\left(\theta\right)=\sin\theta$

Solution:

For all values of a, $f\left(a\right)$ is defined, $\lim\limits_{\theta\to a}f\left(\theta\right)$ exists, and $\lim\limits_{\theta\to a}f\left(\theta\right)=f\left(a\right)$. Therefore, $f(\theta)$ is continuous everywhere.

Exercise:

Problem: $g\left(x\right)=|x|$

Exercise:

Problem: Where is $f\left(x\right)=\begin{cases}0 \text{ if } x \text{ is irrational}\\ 1 \text{ if } x \text{ is rational}\end{cases}$ continuous?

Solution:

Nowhere

Glossary

continuity at a point

A function $f(x)$ is continuous at a point *a* if and only if the following three conditions are satisfied: (1) $f(a)$ is defined, (2) $\lim\limits_{x \to a} f(x)$ exists, and (3) $\lim\limits_{x \to a} f(x) = f(a)$

continuity from the left

A function is continuous from the left at *b* if $\lim\limits_{x \to b^-} f(x) = f(b)$

continuity from the right

A function is continuous from the right at *a* if $\lim\limits_{x \to a^+} f(x) = f(a)$

continuity over an interval

a function that can be traced with a pencil without lifting the pencil; a function is continuous over an open interval if it is continuous at every point in the interval; a function $f(x)$ is continuous over a closed interval of the form $[a, b]$ if it is continuous at every point in (a, b), and it is continuous from the right at *a* and from the left at *b*

discontinuity at a point

A function is discontinuous at a point or has a discontinuity at a point if it is not continuous at the point

infinite discontinuity

An infinite discontinuity occurs at a point *a* if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$

Intermediate Value Theorem

Let *f* be continuous over a closed bounded interval [a, b]; if *z* is any real number between $f(a)$ and $f(b)$, then there is a number *c* in $[a, b]$ satisfying $f(c) = z$

jump discontinuity

A jump discontinuity occurs at a point *a* if $\lim\limits_{x \to a^-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

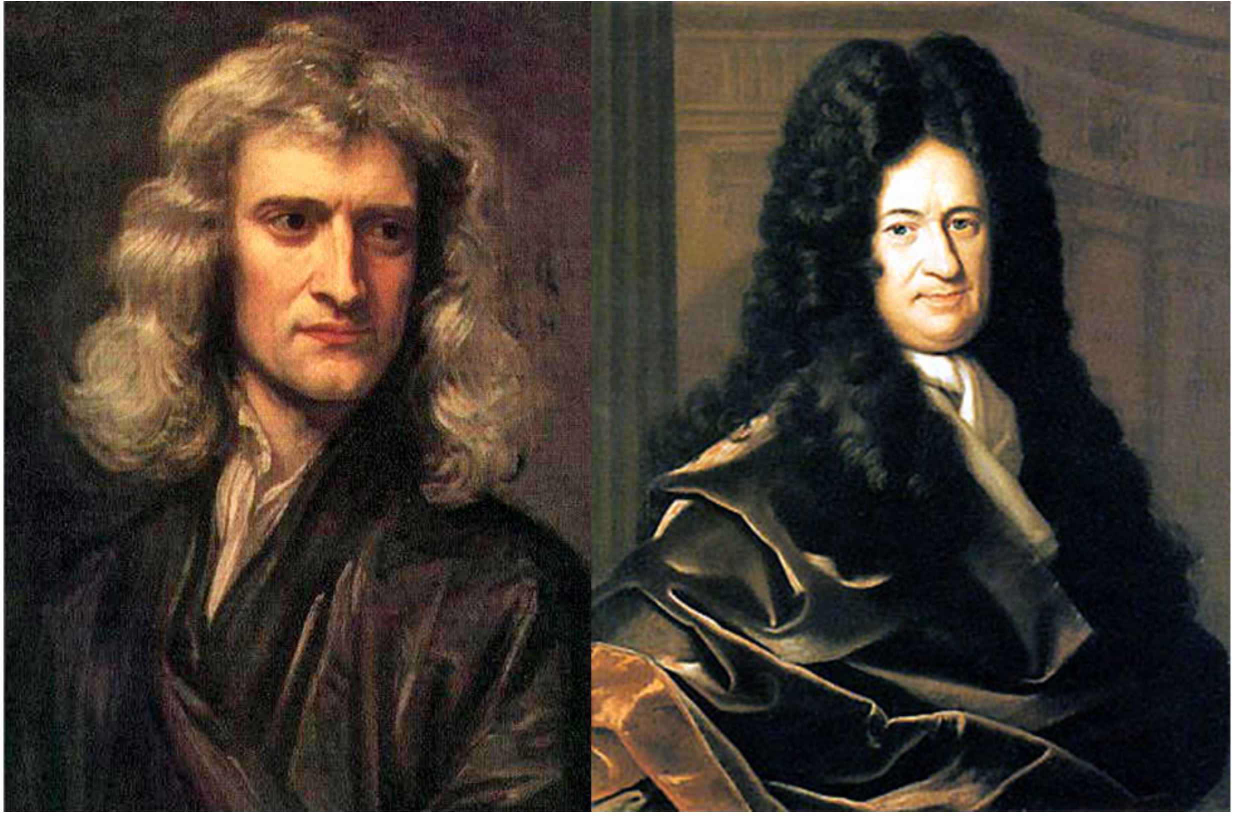
removable discontinuity

A removable discontinuity occurs at a point *a* if $f(x)$ is discontinuous at *a*, but $\lim\limits_{x \to a} f(x)$ exists

Defining the Derivative

- Recognize the meaning of the tangent to a curve at a point.
- Calculate the slope of a tangent line.
- Identify the derivative as the limit of a difference quotient.
- Calculate the derivative of a given function at a point.
- Describe the velocity as a rate of change.
- Explain the difference between average velocity and instantaneous velocity.
- Estimate the derivative from a table of values.

Now that we have both a conceptual understanding of a limit and the practical ability to compute limits, we have established the foundation for our study of calculus, the branch of mathematics in which we compute derivatives and integrals. Most mathematicians and historians agree that calculus was developed independently by the Englishman Isaac Newton $(1643–1727)$ and the German Gottfried Leibniz $(1646–1716)$, whose images appear in [link]. When we credit Newton and Leibniz with developing calculus, we are really referring to the fact that Newton and Leibniz were the first to understand the relationship between the derivative and the integral. Both mathematicians benefited from the work of predecessors, such as Barrow, Fermat, and Cavalieri. The initial relationship between the two mathematicians appears to have been amicable; however, in later years a bitter controversy erupted over whose work took precedence. Although it seems likely that Newton did, indeed, arrive at the ideas behind calculus first, we are indebted to Leibniz for the notation that we commonly use today.



Newton and Leibniz are credited with developing calculus independently.

Tangent Lines

We begin our study of calculus by revisiting the notion of secant lines and tangent lines. Recall that we used the slope of a secant line to a function at a point $\left(a, f\left(a\right)\right)$ to estimate the rate of change, or the rate at which one variable changes in relation to another variable. We can obtain the slope of the secant by choosing a value of x near a and drawing a line through the points $\left(a, f\left(a\right)\right)$ and $\left(x, f\left(x\right)\right)$, as shown in [link]. The slope of this line is given by an equation in the form of a difference quotient:

Equation:

$$m_{\text{sec}} = \frac{f\left(x\right) - f\left(a\right)}{x - a}.$$

We can also calculate the slope of a secant line to a function at a value a by using this equation and replacing x with $a + h$, where h is a value close to 0. We can then calculate the slope of the line through the points $\left(a, f\left(a\right)\right)$ and $\left(a + h, f\left(a + h\right)\right)$. In this case, we find the secant line has a slope given by the following difference quotient with increment h:

Equation:

$$m_{\text{sec}} = \frac{f\left(a + h\right) - f\left(a\right)}{a + h - a} = \frac{f\left(a + h\right) - f\left(a\right)}{h}.$$

Note:

Definition

Let f be a function defined on an interval I containing a. If $x \neq a$ is in I, then

Equation:

$$Q = \frac{f\left(x\right) - f(a)}{x - a}$$

is a **difference quotient**.

Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then

Equation:

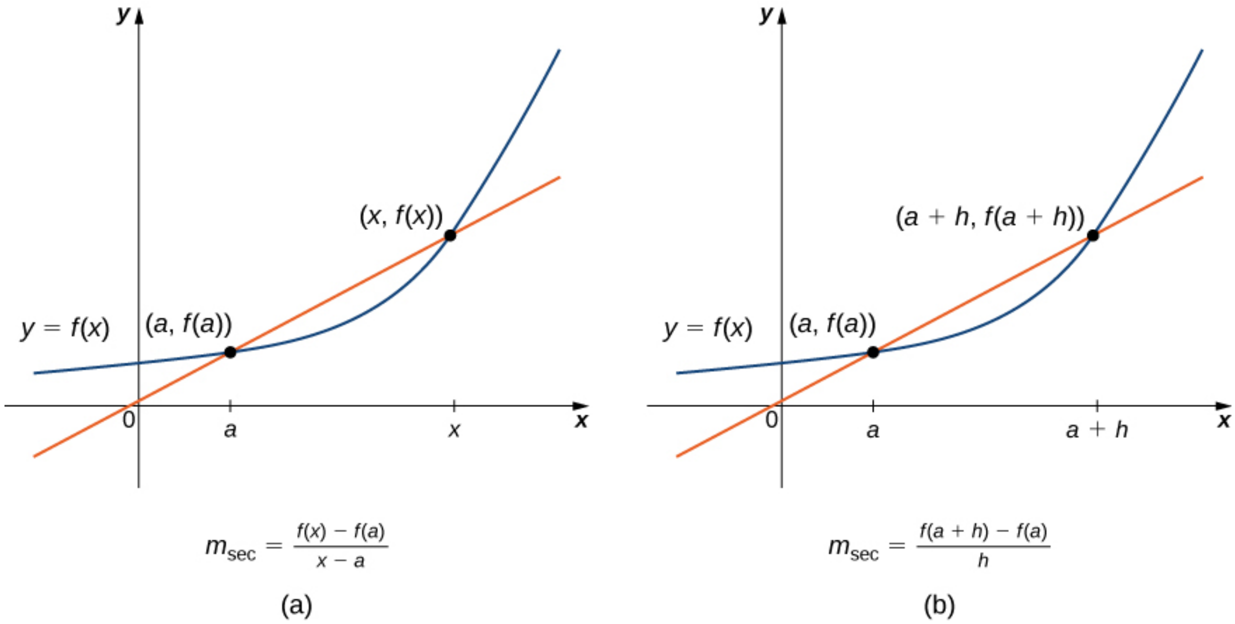
$$Q = \frac{f\left(a + h\right) - f(a)}{h}$$

is a difference quotient with increment h.

Note:

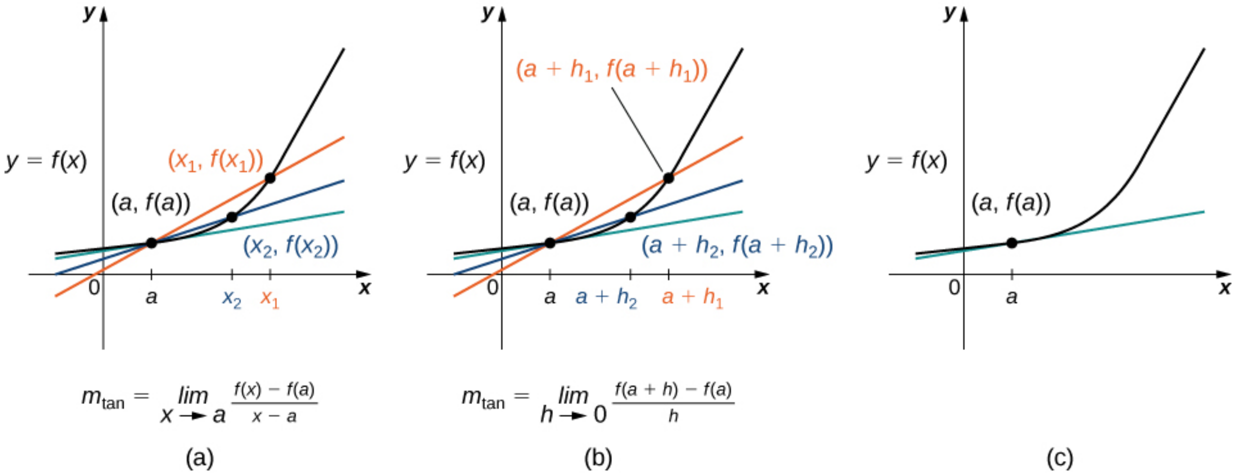
View the development of the derivative with this applet.

These two expressions for calculating the slope of a secant line are illustrated in [link]. We will see that each of these two methods for finding the slope of a secant line is of value. Depending on the setting, we can choose one or the other. The primary consideration in our choice usually depends on ease of calculation.


y
x
y = f(x)
(a, f(a))
(x, f(x))
0
a
x
$m_{\text{sec}} = \frac{f(x) - f(a)}{x - a}$
(a)
y
x
y = f(x)
(a, f(a))
(a + h, f(a + h))
0
a
a + h
$m_{\text{sec}} = \frac{f(a + h) - f(a)}{h}$
(b)

We can calculate the slope of a secant line in either of two ways.

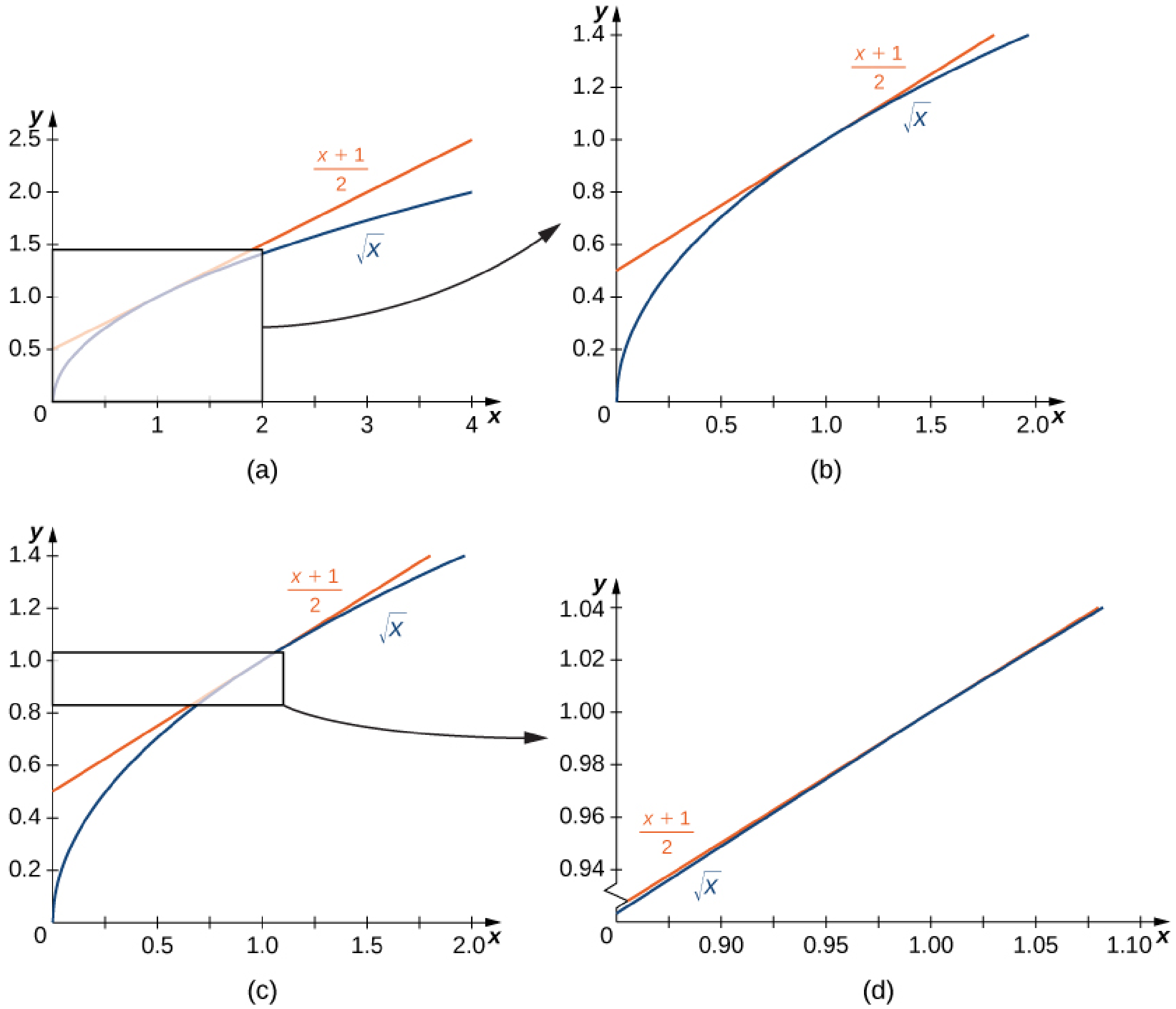
In [link](a) we see that, as the values of x approach a, the slopes of the secant lines provide better estimates of the rate of change of the function at a. Furthermore, the secant lines themselves approach the tangent line to the function at a, which represents the limit of the secant lines. Similarly, [link](b) shows that as the values of h get closer to 0, the secant lines also approach the tangent line. The slope of the tangent line at a is the rate of change of the function at a, as shown in [link](c).


y
y = f(x)
(x₁, f(x₁))
(a, f(a))
(x₂, f(x₂))
0
a
x₂
x₁
x
m_tan = lim x→a (f(x) − f(a))/(x − a)
(a)
y
(a + h₁, f(a + h₁))
y = f(x)
(a, f(a))
(a + h₂, f(a + h₂))
0
a
a + h₂
a + h₁
x
m_tan = lim h→0 (f(a + h) − f(a))/h
(b)
y
y = f(x)
(a, f(a))
0
a
x
(c)

The secant lines approach the tangent line (shown in green) as the second point approaches the first.

Note:
You can use this site to explore graphs to see if they have a tangent line at a point.

In [link] we show the graph of $f(x) = \sqrt{x}$ and its tangent line at $(1, 1)$ in a series of tighter intervals about $x = 1$. As the intervals become narrower, the graph of the function and its tangent line appear to coincide, making the values on the tangent line a good approximation to the values of the function for choices of x close to 1. In fact, the graph of $f(x)$ itself appears to be locally linear in the immediate vicinity of $x = 1$.



For values of x close to 1, the graph of $f\left(x\right)=\sqrt{x}$ and its tangent line appear to coincide.

Formally we may define the tangent line to the graph of a function as follows.

Note:
Definition
Let $f(x)$ be a function defined in an open interval containing a. The *tangent line* to $f(x)$ at a is the line passing through the point $(a, f\left(a\right))$ having slope
Equation:

$$m_{\tan}=\lim_{x\to a}\frac{f\left(x\right)-f(a)}{x-a}$$

provided this limit exists.

Equivalently, we may define the tangent line to $f(x)$ at a to be the line passing through the point $(a, f(a))$ having slope

Equation:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Just as we have used two different expressions to define the slope of a secant line, we use two different forms to define the slope of the tangent line. In this text we use both forms of the definition. As before, the choice of definition will depend on the setting. Now that we have formally defined a tangent line to a function at a point, we can use this definition to find equations of tangent lines.

Example:

Exercise:

Problem:

Finding a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution:

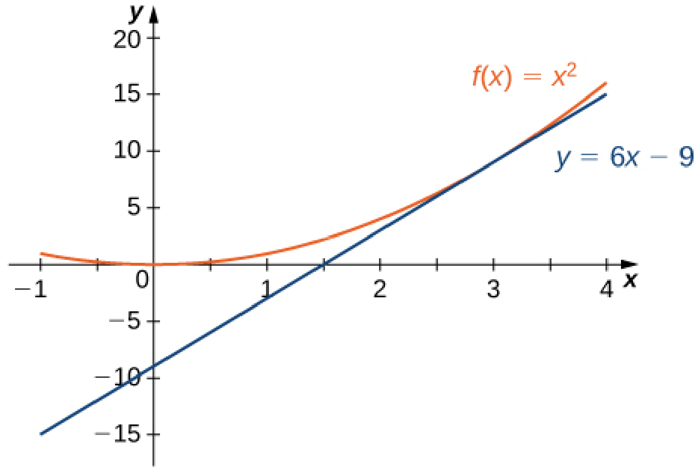
First find the slope of the tangent line. In this example, use [link].

Equation:

$$\begin{aligned} m_{\tan} &= \lim_{x \to 3} \tfrac{f(x) - f(3)}{x - 3} && \text{Apply the definition.} \\ &= \lim_{x \to 3} \tfrac{x^2 - 9}{x - 3} && \text{Substitute } f(x) = x^2 \text{ and } f(3) = 9. \\ &= \lim_{x \to 3} \tfrac{(x-3)(x+3)}{x - 3} = \lim_{x \to 3} (x + 3) = 6 && \text{Factor the numerator to evaluate the limit.} \end{aligned}$$

Next, find a point on the tangent line. Since the line is tangent to the graph of $f(x)$ at $x = 3$, it passes through the point $(3, f(3))$. We have $f(3) = 9$, so the tangent line passes through the point $(3, 9)$.

Using the point-slope equation of the line with the slope $m = 6$ and the point $(3, 9)$, we obtain the line $y - 9 = 6(x - 3)$. Simplifying, we have $y = 6x - 9$. The graph of $f(x) = x^2$ and its tangent line at 3 are shown in [link].



The tangent line to $f(x)$ at $x = 3$.

Example:
Exercise:

Problem:
The Slope of a Tangent Line Revisited

Use [link] to find the slope of the line tangent to the graph of $f(x) = x^2$ at $x = 3$.

Solution:

The steps are very similar to [link]. See [link] for the definition.
Equation:

$$\begin{aligned} m_{\tan} &= \lim_{h\to 0}\frac{f(3+h)-f(3)}{h} && \text{Apply the definition.}\\ &= \lim_{h\to 0}\frac{(3+h)^2-9}{h} && \text{Substitute } f(3+h)=(3+h)^2 \text{ and } f(3)=9.\\ &= \lim_{h\to 0}\frac{9+6h+h^2-9}{h} && \text{Expand and simplify to evaluate the limit.}\\ &= \lim_{h\to 0}\frac{h(6+h)}{h} = \lim_{h\to 0}(6+h) = 6 \end{aligned}$$

We obtained the same value for the slope of the tangent line by using the other definition, demonstrating that the formulas can be interchanged.

Example:

Exercise:

Problem:
Finding the Equation of a Tangent Line

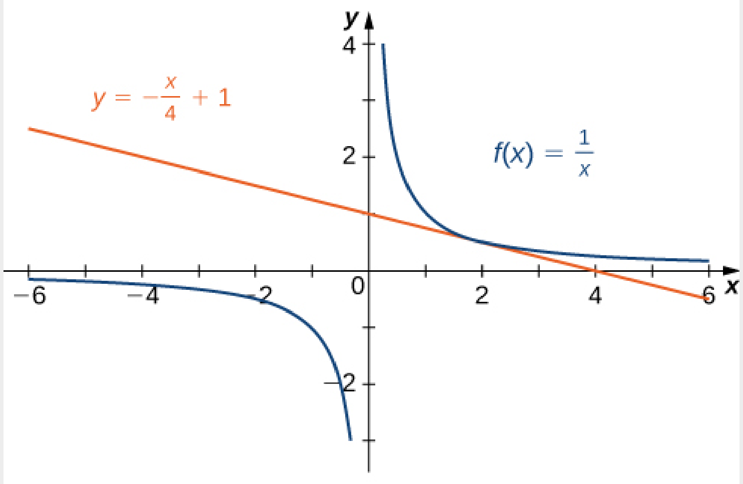
Find the equation of the line tangent to the graph of $f\left(x\right)=1/x$ at $x=2$.

Solution:

We can use [link], but as we have seen, the results are the same if we use [link].
Equation:

$$\begin{aligned}
m_{\text{tan}} &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} && \text{Apply the definition.}\\
&= \lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2} && \text{Substitute } f\left(x\right)=\frac{1}{x} \text{ and } f\left(2\right)=\frac{1}{2}.\\
&= \lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}\cdot\frac{2x}{2x} && \text{Multiply numerator and denominator by } 2x \text{ to simplify fractions.}\\
&= \lim_{x\to 2}\frac{(2-x)}{(x-2)(2x)} && \text{Simplify.}\\
&= \lim_{x\to 2}\frac{-1}{2x} && \text{Simplify using } \frac{2-x}{x-2}=-1, \text{ for } x\ne 2.\\
&= -\frac{1}{4} && \text{Evaluate the limit.}
\end{aligned}$$

We now know that the slope of the tangent line is $-\frac{1}{4}$. To find the equation of the tangent line, we also need a point on the line. We know that $f\left(2\right)=\frac{1}{2}$. Since the tangent line passes through the point $\left(2,\frac{1}{2}\right)$ we can use the point-slope equation of a line to find the equation of the tangent line. Thus the tangent line has the equation $y=-\frac{1}{4}x+1$. The graphs of $f\left(x\right)=\frac{1}{x}$ and $y=-\frac{1}{4}x+1$ are shown in [link].



The line is tangent to $f\left(x\right)$ at $x=2$.

Note:
Exercise:

Problem: Find the slope of the line tangent to the graph of $f\left(x\right)=\sqrt{x}$ at $x=4$.

Solution:

$\frac{1}{4}$

Hint

Use either [link] or [link]. Multiply the numerator and the denominator by a conjugate.

The Derivative of a Function at a Point

The type of limit we compute in order to find the slope of the line tangent to a function at a point occurs in many applications across many disciplines. These applications include velocity and acceleration in physics, marginal profit functions in business, and growth rates in biology. This limit occurs so frequently that we give this value a special name: the **derivative**. The process of finding a derivative is called **differentiation**.

Note:
Definition
Let $f(x)$ be a function defined in an open interval containing a. The derivative of the function $f(x)$ at a, denoted by $f'\left(a\right)$, is defined by
Equation:

$$f'\left(a\right)=\lim_{x\to a}\frac{f\left(x\right)-f(a)}{x-a}$$

provided this limit exists.
Alternatively, we may also define the derivative of $f(x)$ at a as
Equation:

$$f'\left(a\right)=\lim_{h\to 0}\frac{f\left(a+h\right)-f\left(a\right)}{h}.$$

Example:
Exercise:

Problem:
Estimating a Derivative

For $f\left(x\right)=x^2$, use a table to estimate $f'(3)$ using [link].

Solution:

Create a table using values of x just below 3 and just above 3.

x	$\frac{x^2-9}{x-3}$
2.9	5.9
2.99	5.99
2.999	5.999
3.001	6.001
3.01	6.01
3.1	6.1

After examining the table, we see that a good estimate is $f'(3) = 6$.

Note:
Exercise:

Problem: For $f(x) = x^2$, use a table to estimate $f'(3)$ using [link].

Solution:

6

Hint

Evaluate $\frac{(x+h)-x^2}{h}$ at $h = -0.1, -0.01, -0.001, 0.001, 0.01, 0.1$

Example:
Exercise:

Problem:
Finding a Derivative

For $f(x) = 3x^2 - 4x + 1$, find $f'(2)$ by using [link].

Solution:

Substitute the given function and value directly into the equation.

Equation:

$$\begin{aligned}
f'(x) &= \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} && \text{Apply the definition.}\\
&= \lim_{x\to 2}\frac{(3x^2-4x+1)-5}{x-2} && \text{Substitute } f(x)=3x^2-4x+1 \text{ and } f(2)=5.\\
&= \lim_{x\to 2}\frac{(x-2)(3x+2)}{x-2} && \text{Simplify and factor the numerator.}\\
&= \lim_{x\to 2}(3x+2) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}$$

Example:

Exercise:

Problem:

Revisiting the Derivative

For $f(x)=3x^2-4x+1$, find $f'(2)$ by using [link].

Solution:

Using this equation, we can substitute two values of the function into the equation, and we should get the same value as in [link].

Equation:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} && \text{Apply the definition.}\\
&= \lim_{h\to 0}\frac{(3(2+h)^2-4(2+h)+1)-5}{h} && \text{Substitute } f(2)=5 \text{ and}\\
& && f(2+h)=3(2+h)^2-4(2+h)+1.\\
&= \lim_{h\to 0}\frac{3h^2+8h}{h} && \text{Simplify the numerator.}\\
&= \lim_{h\to 0}\frac{h(3h+8)}{h} && \text{Factor the numerator.}\\
&= \lim_{h\to 0}(3h+8) && \text{Cancel the common factor.}\\
&= 8 && \text{Evaluate the limit.}
\end{aligned}$$

The results are the same whether we use [link] or [link].

Note:
Exercise:

Problem: For $f(x) = x^2 + 3x + 2$, find $f'(1)$.

Solution:

$f'(1) = 5$

Hint

Use either [link], [link], or try both. Use either [link] or [link] as a guide.

Velocities and Rates of Change

Now that we can evaluate a derivative, we can use it in velocity applications. Recall that if $s(t)$ is the position of an object moving along a coordinate axis, the average velocity of the object over a time interval $[a, t]$ if $t > a$ or $[t, a]$ if $t < a$ is given by the difference quotient

Equation:

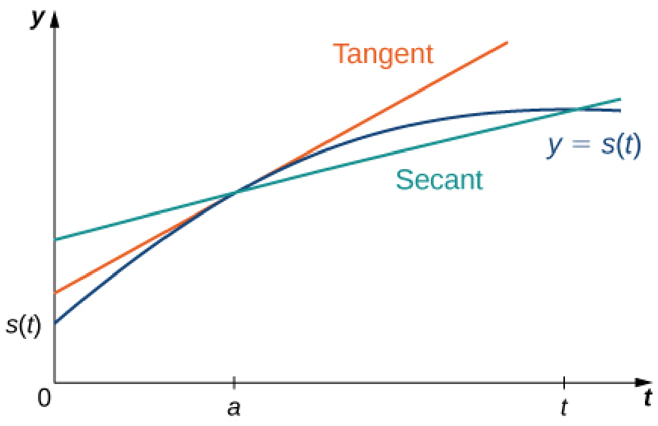
$$v_{\text{ave}} = \frac{s(t) - s(a)}{t - a}.$$

As the values of t approach a, the values of v_{ave} approach the value we call the instantaneous velocity at a. That is, instantaneous velocity at a, denoted $v(a)$, is given by

Equation:

$$v(a) = s'(a) = \lim_{t \to a} \frac{s(t) - s(a)}{t - a}.$$

To better understand the relationship between average velocity and instantaneous velocity, see [link]. In this figure, the slope of the tangent line (shown in red) is the instantaneous velocity of the object at time $t = a$ whose position at time t is given by the function $s(t)$. The slope of the secant line (shown in green) is the average velocity of the object over the time interval $[a, t]$.



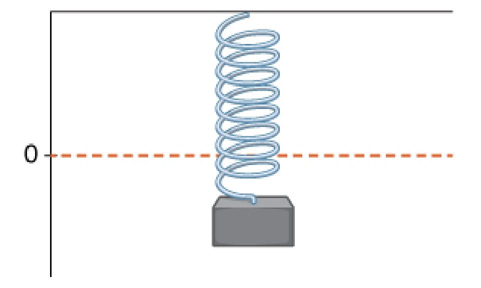
The slope of the secant line is the average velocity over the interval $[a, t]$. The slope of the tangent line is the instantaneous velocity.

We can use [link] to calculate the instantaneous velocity, or we can estimate the velocity of a moving object by using a table of values. We can then confirm the estimate by using [link].

Example:
Exercise:

Problem:
Estimating Velocity

A lead weight on a spring is oscillating up and down. Its position at time t with respect to a fixed horizontal line is given by $s(t) = \sin t$ ([link]). Use a table of values to estimate $v(0)$. Check the estimate by using [link].



A lead weight suspended from a spring in vertical oscillatory motion.

Solution:

We can estimate the instantaneous velocity at $t = 0$ by computing a table of average velocities using values of t approaching 0, as shown in [link].

t	$\frac{\sin t - \sin 0}{t - 0} = \frac{\sin t}{t}$
-0.1	0.998334166
-0.01	0.9999833333
-0.001	0.999999833
0.001	0.999999833
0.01	0.9999833333
0.1	0.998334166

Average velocities using values of *t* approaching 0

From the table we see that the average velocity over the time interval $[-0.1, 0]$ is 0.998334166, the average velocity over the time interval $[-0.01, 0]$ is 0.9999833333, and so forth. Using this table of values, it appears that a good estimate is $v(0) = 1$.

By using [link], we can see that
Equation:

$$v(0) = s'(0) = \lim_{t \to 0} \frac{\sin t - \sin 0}{t - 0} = \lim_{t \to 0} \frac{\sin t}{t} = 1.$$

Thus, in fact, $v(0) = 1$.

Note:
Exercise:

Problem:

A rock is dropped from a height of 64 feet. Its height above ground at time t seconds later is given by $s(t) = -16t^2 + 64, 0 \leq t \leq 2$. Find its instantaneous velocity 1 second after it is dropped, using [link].

Solution:

-32 ft/s

Hint

$v(t) = s'(t)$. Follow the earlier examples of the derivative using [link].

As we have seen throughout this section, the slope of a tangent line to a function and instantaneous velocity are related concepts. Each is calculated by computing a derivative and each measures the instantaneous rate of change of a function, or the rate of change of a function at any point along the function.

Note:
Definition
The **instantaneous rate of change** of a function $f(x)$ at a value a is its derivative $f'(a)$.

Example:
Exercise:

Problem:
Chapter Opener: Estimating Rate of Change of Velocity

(credit: modification of work by Codex41, Flickr)

Reaching a top speed of 270.49 mph, the Hennessey Venom GT is one of the fastest cars in the world. In tests it went from 0 to 60 mph in 3.05 seconds, from 0 to 100 mph in 5.88 seconds, from 0 to 200 mph in 14.51 seconds, and from 0 to 229.9 mph in 19.96 seconds. Use this data to draw a conclusion about the rate of change of velocity (that is, its acceleration) as it approaches 229.9 mph. Does the rate at which the car is accelerating appear to be increasing, decreasing, or constant?

Solution:

First observe that 60 mph = 88 ft/s, 100 mph ≈ 146.67 ft/s, 200 mph ≈ 293.33 ft/s, and 229.9 mph ≈ 337.19 ft/s. We can summarize the information in a table.

t	$v(t)$
0	0
3.05	88
5.88	147.67
14.51	293.33
19.96	337.19

$v(t)$ at different values of t

Now compute the average acceleration of the car in feet per second per second on intervals of the form $[t, 19.96]$ as t approaches 19.96, as shown in the following table.

t	$\frac{v(t)-v(19.96)}{t-19.96} = \frac{v(t)-337.19}{t-19.96}$
0.0	16.89
3.05	14.74
5.88	13.46
14.51	8.05

Average acceleration

The rate at which the car is accelerating is decreasing as its velocity approaches 229.9 mph (337.19 ft/s).

Example:
Exercise:

Problem:
Rate of Change of Temperature

A homeowner sets the thermostat so that the temperature in the house begins to drop from $70\,^\circ\mathrm{F}$ at 9 p.m., reaches a low of 60° during the night, and rises back to 70° by 7 a.m. the next morning. Suppose that the temperature in the house is given by $T(t) = 0.4t^2 - 4t + 70$ for $0 \le t \le 10$, where t is the number of hours past 9 p.m. Find the instantaneous rate of change of the temperature at midnight.

Solution:

Since midnight is 3 hours past 9 p.m., we want to compute $T'(3)$. Refer to [link].
Equation:

$$\begin{aligned}
T'(3) &= \lim_{t\to 3}\frac{T(t)-T(3)}{t-3} && \text{Apply the definition.}\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+70-61.6}{t-3} && \text{Substitute } T(t) = 0.4t^2 - 4t + 70 \text{ and}\\
& && T(3) = 61.6.\\
&= \lim_{t\to 3}\frac{0.4t^2-4t+8.4}{t-3} && \text{Simplify.}\\
&= \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3} && = \lim_{t\to 3}\frac{0.4(t-3)(t-7)}{t-3}\\
&= \lim_{t\to 3}0.4(t-7) && \text{Cancel.}\\
&= -1.6 && \text{Evaluate the limit.}
\end{aligned}$$

The instantaneous rate of change of the temperature at midnight is $-1.6\,^\circ\mathrm{F}$ per hour.

Example:
Exercise:

Problem:
Rate of Change of Profit

A toy company can sell x electronic gaming systems at a price of $p = -0.01x + 400$ dollars per gaming system. The cost of manufacturing x systems is given by $C(x) = 100x + 10{,}000$ dollars. Find the rate of change of profit when 10,000 games are produced. Should the toy company increase or decrease production?

Solution:

The profit $P(x)$ earned by producing x gaming systems is $R(x) - C(x)$, where $R(x)$ is the revenue obtained from the sale of x games. Since the company can sell x games at $p = -0.01x + 400$ per game,
Equation:

$$R(x) = xp = x(-0.01x + 400) = -0.01x^2 + 400x.$$

Consequently,
Equation:

$$P\left(x\right)=-0.01x^{2}+300x-10{,}000.$$

Therefore, evaluating the rate of change of profit gives
Equation:

$$\begin{aligned}P'\left(10000\right)&=\lim_{x\to 10000}\frac{P(x)-P(10000)}{x-10000}\\&=\lim_{x\to 10000}\frac{-0.01x^{2}+300x-10000-1990000}{x-10000}\\&=\lim_{x\to 10000}\frac{-0.01x^{2}+300x-2000000}{x-10000}\\&=100.\end{aligned}$$

Since the rate of change of profit $P'\left(10{,}000\right)>0$ and $P\left(10{,}000\right)>0$, the company should increase production.

Note:
Exercise:

Problem:

A coffee shop determines that the daily profit on scones obtained by charging s dollars per scone is $P\left(s\right)=-20s^{2}+150s-10$. The coffee shop currently charges $\$3.25$ per scone. Find $P'(3.25)$, the rate of change of profit when the price is $\$3.25$ and decide whether or not the coffee shop should consider raising or lowering its prices on scones.

Solution:

$P'\left(3.25\right)=20>0$; raise prices

Hint

Use [link] for a guide.

Key Concepts

- The slope of the tangent line to a curve measures the instantaneous rate of change of a curve. We can calculate it by finding the limit of the difference quotient or the difference quotient with increment h.
- The derivative of a function $f\left(x\right)$ at a value a is found using either of the definitions for the slope of the tangent line.

- Velocity is the rate of change of position. As such, the velocity $v\left(t\right)$ at time t is the derivative of the position $s\left(t\right)$ at time t. Average velocity is given by
  Equation:
  $$v_{\text{ave}}=\frac{s\left(t\right)-s(a)}{t-a}.$$
  Instantaneous velocity is given by
  Equation:
  $$v\left(a\right)=s'\left(a\right)=\lim_{t\to a}\frac{s\left(t\right)-s(a)}{t-a}.$$
- We may estimate a derivative by using a table of values.

Key Equations

- **Difference quotient**
  $Q=\frac{f(x)-f(a)}{x-a}$
- **Difference quotient with increment h**
  $Q=\frac{f(a+h)-f(a)}{a+h-a}=\frac{f(a+h)-f(a)}{h}$
- **Slope of tangent line**
  $m_{\text{tan}}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$
  $m_{\text{tan}}=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$
- **Derivative of $f(x)$ at a**
  $f'\left(a\right)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$
  $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$
- **Average velocity**
  $v_{\text{ave}}=\frac{s(t)-s(a)}{t-a}$
- **Instantaneous velocity**
  $v\left(a\right)=s'\left(a\right)=\lim_{t\to a}\frac{s(t)-s(a)}{t-a}$

For the following exercises, use [link] to find the slope of the secant line between the values x_1 and x_2 for each function $y=f\left(x\right)$.

Exercise:

Problem: $f\left(x\right)=4x+7;x_1=2,x_2=5$

Solution:

4

Exercise:

Problem: $f(x) = 8x - 3; x_1 = -1, x_2 = 3$

Exercise:

Problem: $f(x) = x^2 + 2x + 1; x_1 = 3, x_2 = 3.5$

Solution:

8.5

Exercise:

Problem: $f(x) = -x^2 + x + 2; x_1 = 0.5, x_2 = 1.5$

Exercise:

Problem: $f(x) = \frac{4}{3x-1}; x_1 = 1, x_2 = 3$

Solution:

$-\frac{3}{4}$

Exercise:

Problem: $f(x) = \frac{x-7}{2x+1}; x_1 = 0, x_2 = 2$

Exercise:

Problem: $f(x) = \sqrt{x}; x_1 = 1, x_2 = 16$

Solution:

0.2

Exercise:

Problem: $f(x) = \sqrt{x-9}; x_1 = 10, x_2 = 13$

Exercise:

Problem: $f(x) = x^{1/3} + 1; x_1 = 0, x_2 = 8$

Solution:

0.25

Exercise:

Problem: $f(x) = 6x^{2/3} + 2x^{1/3}; x_1 = 1, x_2 = 27$

For the following functions,

a. use [link] to find the slope of the tangent line $m_{\text{tan}} = f'(a)$, and
b. find the equation of the tangent line to f at $x = a$.

Exercise:

Problem: $f(x) = 3 - 4x, a = 2$

Solution:

a. -4 b. $y = 3 - 4x$

Exercise:

Problem: $f(x) = \frac{x}{5} + 6, a = -1$

Exercise:

Problem: $f(x) = x^2 + x, a = 1$

Solution:

a. 3 b. $y = 3x - 1$

Exercise:

Problem: $f(x) = 1 - x - x^2, a = 0$

Exercise:

Problem: $f(x) = \frac{7}{x}, a = 3$

Solution:

a. $\frac{-7}{9}$ b. $y = \frac{-7}{9}x + \frac{14}{3}$

Exercise:

Problem: $f(x) = \sqrt{x + 8}, a = 1$

Exercise:

Problem: $f(x) = 2 - 3x^2, a = -2$

Solution:

a. 12 b. $y = 12x + 14$

Exercise:

Problem: $f\left(x\right)=\frac{-3}{x-1},a=4$

Exercise:

Problem: $f\left(x\right)=\frac{2}{x+3},a=-4$

Solution:

a. -2 b. $y=-2x-10$

Exercise:

Problem: $f\left(x\right)=\frac{3}{x^{2}},a=3$

For the following functions $y=f\left(x\right)$, find $f'\left(a\right)$ using [link].

Exercise:

Problem: $f\left(x\right)=5x+4,a=-1$

Solution:

5

Exercise:

Problem: $f\left(x\right)=-7x+1,a=3$

Exercise:

Problem: $f\left(x\right)=x^{2}+9x,a=2$

Solution:

13

Exercise:

Problem: $f\left(x\right)=3x^{2}-x+2,a=1$

Exercise:

Problem: $f\left(x\right)=\sqrt{x},a=4$

Solution:

$\frac{1}{4}$

Exercise:

Problem: $f\left(x\right)=\sqrt{x-2},a=6$

Exercise:

Problem: $f\left(x\right)=\frac{1}{x},a=2$

Solution:

$-\frac{1}{4}$

Exercise:

Problem: $f\left(x\right)=\frac{1}{x-3},a=-1$

Exercise:

Problem: $f\left(x\right)=\frac{1}{x^3},a=1$

Solution:

-3

Exercise:

Problem: $f\left(x\right)=\frac{1}{\sqrt{x}},a=4$

For the following exercises, given the function $y=f\left(x\right)$,

a. find the slope of the secant line PQ for each point $Q\left(x,f\left(x\right)\right)$ with x value given in the table.
b. Use the answers from a. to estimate the value of the slope of the tangent line at P.
c. Use the answer from b. to find the equation of the tangent line to f at point P.

Exercise:

Problem: **[T]** $f\left(x\right)=x^2+3x+4,P\left(1,8\right)$ (Round to 6 decimal places.)

x	**Slope** m_{PQ}	***x***	**Slope** m_{PQ}
1.1	(i)	0.9	(vii)
1.01	(ii)	0.99	(viii)

x	**Slope** m_{PQ}	*x*	**Slope** m_{PQ}
1.001	(iii)	0.999	(ix)
1.0001	(iv)	0.9999	(x)
1.00001	(v)	0.99999	(xi)
1.000001	(vi)	0.999999	(xii)

Solution:

a. (i) 5.100000, (ii) 5.010000, (iii) 5.001000, (iv) 5.000100, (v) 5.000010, (vi) 5.000001, (vii) 4.900000, (viii) 4.990000, (ix) 4.999000, (x) 4.999900, (xi) 4.999990, (x) 4.999999 b. $m_{\tan} = 5$ c. $y = 5x + 3$

Exercise:

Problem: **[T]** $f(x) = \frac{x+1}{x^2-1}, P(0, -1)$

x	**Slope** m_{PQ}	*x*	**Slope** m_{PQ}
0.1	(i)	-0.1	(vii)
0.01	(ii)	-0.01	(viii)
0.001	(iii)	-0.001	(ix)
0.0001	(iv)	-0.0001	(x)
0.00001	(v)	-0.00001	(xi)
0.000001	(vi)	-0.000001	(xii)

Exercise:

Problem: **[T]** $f(x) = 10e^{0.5x}$, $P(0, 10)$ (Round to 4 decimal places.)

x	**Slope** m_{PQ}
-0.1	(i)
-0.01	(ii)
-0.001	(iii)
-0.0001	(iv)
-0.00001	(v)
−0.000001	(vi)

Solution:

a. (i) 4.8771, (ii) 4.9875 (iii) 4.9988, (iv) 4.9999, (v) 4.9999, (vi) 4.9999 b. $m_{\tan} = 5$ c. $y = 5x + 10$

Exercise:

Problem: [T] $f(x) = \tan(x), P(\pi, 0)$

x	**Slope** m_{PQ}
3.1	(i)
3.14	(ii)
3.141	(iii)
3.1415	(iv)
3.14159	(v)
3.141592	(vi)

[T] For the following position functions $y = s(t)$, an object is moving along a straight line, where t is in seconds and s is in meters. Find

a. the simplified expression for the average velocity from $t = 2$ to $t = 2 + h$;

b. the average velocity between $t = 2$ and $t = 2 + h$, where (i) $h = 0.1$, (ii) $h = 0.01$, (iii) $h = 0.001$, and (iv) $h = 0.0001$; and

c. use the answer from a. to estimate the instantaneous velocity at $t = 2$ second.

Exercise:

Problem: $s(t) = \frac{1}{3}t + 5$

Solution:

a. $\frac{1}{3}$; b. (i) $0.\bar{3}$ m/s, (ii) $0.\bar{3}$ m/s, (iii) $0.\bar{3}$ m/s, (iv) $0.\bar{3}$ m/s; c. $0.\bar{3} = \frac{1}{3}$ m/s

Exercise:

Problem: $s(t) = t^2 - 2t$

Exercise:

Problem: $s(t) = 2t^3 + 3$

Solution:

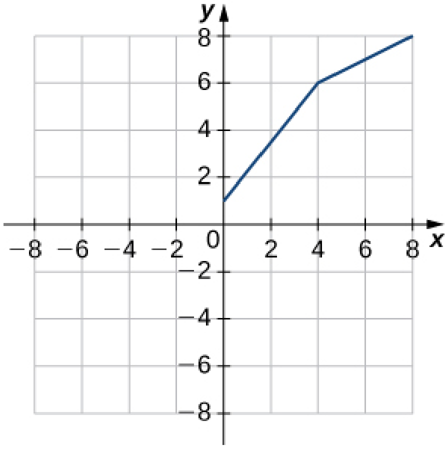
a. $2\left(h^2 + 6h + 12\right)$; b. (i) 25.22 m/s, (ii) 24.12 m/s, (iii) 24.01 m/s, (iv) 24 m/s; c. 24 m/s

Exercise:

Problem: $s(t) = \frac{16}{t^2} - \frac{4}{t}$

Exercise:

Problem: Use the following graph to evaluate a. $f'(1)$ and b. $f'(6)$.

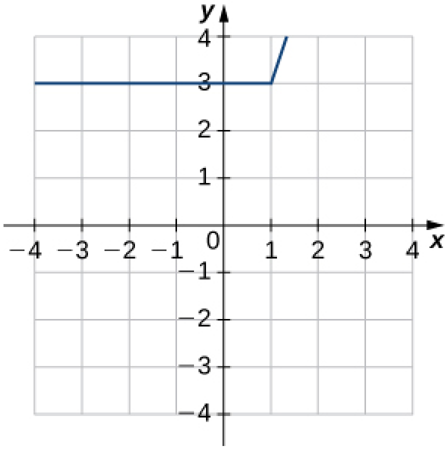


Solution:

a. 1.25; b. 0.5

Exercise:

Problem: Use the following graph to evaluate a. $f'(-3)$ and b. $f'(1.5)$.



For the following exercises, use the limit definition of derivative to show that the derivative does not exist at $x = a$ for each of the given functions.

Exercise:

Problem: $f(x) = x^{1/3}, x = 0$

Solution:

$$\lim_{x \to 0^-} \frac{x^{1/3} - 0}{x - 0} = \lim_{x \to 0^-} \frac{1}{x^{2/3}} = \infty$$

Exercise:

Problem: $f(x) = x^{2/3}, x = 0$

Exercise:

Problem: $f(x) = \begin{cases} 1, x < 1 \\ x, x \geq 1 \end{cases}, x = 1$

Solution:

$$\lim_{x \to 1^-} \frac{1-1}{x-1} = 0 \neq 1 = \lim_{x \to 1^+} \frac{x-1}{x-1}$$

Exercise:

Problem: $f(x) = \frac{|x|}{x}, x = 0$

Exercise:

Problem:

[T] The position in feet of a race car along a straight track after t seconds is modeled by the function $s\left(t\right)=8{t}^{2}-\frac{1}{16}{t}^{3}$.

a. Find the average velocity of the vehicle over the following time intervals to four decimal places:
   i. [4, 4.1]
   ii. [4, 4.01]
   iii. [4, 4.001]
   iv. [4, 4.0001]
b. Use a. to draw a conclusion about the instantaneous velocity of the vehicle at $t=4$ seconds.

Solution:

a. $\left(\text{i}\right)$ 61.7244 ft/s, $\left(\text{ii}\right)$ 61.0725 ft/s $\left(\text{iii}\right)$ 61.0072 ft/s $\left(\text{iv}\right)$ 61.0007 ft/s b. At 4 seconds the race car is traveling at a rate/velocity of 61 ft/s.

Exercise:

Problem:

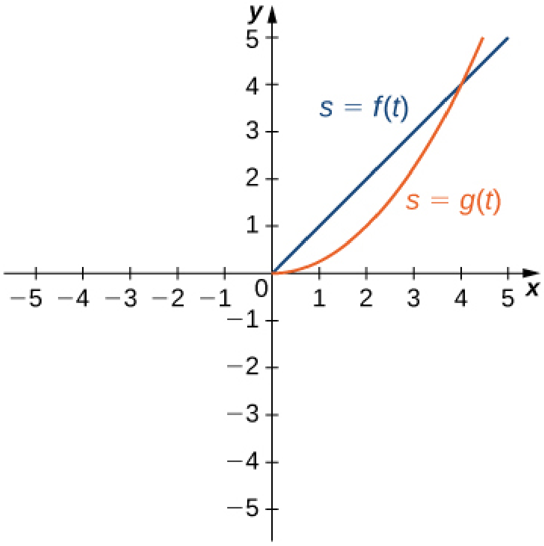
[T] The distance in feet that a ball rolls down an incline is modeled by the function $s\left(t\right)=14{t}^{2}$, where t is seconds after the ball begins rolling.

a. Find the average velocity of the ball over the following time intervals:
   i. [5, 5.1]
   ii. [5, 5.01]
   iii. [5, 5.001]
   iv. [5, 5.0001]
b. Use the answers from a. to draw a conclusion about the instantaneous velocity of the ball at $t=5$ seconds.

Exercise:

Problem:

Two vehicles start out traveling side by side along a straight road. Their position functions, shown in the following graph, are given by $s=f(t)$ and $s=g\left(t\right)$, where s is measured in feet and t is measured in seconds.



a. Which vehicle has traveled farther at $t = 2$ seconds?
b. What is the approximate velocity of each vehicle at $t = 3$ seconds?
c. Which vehicle is traveling faster at $t = 4$ seconds?
d. What is true about the positions of the vehicles at $t = 4$ seconds?

Solution:

a. The vehicle represented by $f\left(t\right)$, because it has traveled 2 feet, whereas $g\left(t\right)$ has traveled 1 foot. b. The velocity of $f\left(t\right)$ is constant at 1 ft/s, while the velocity of $g\left(t\right)$ is approximately 2 ft/s. c. The vehicle represented by $g\left(t\right)$, with a velocity of approximately 4 ft/s. d. Both have traveled 4 feet in 4 seconds.

Exercise:

Problem:

[T] The total cost $C(x)$, in hundreds of dollars, to produce x jars of mayonnaise is given by $C\left(x\right) = 0.000003x^3 + 4x + 300.$

a. Calculate the average cost per jar over the following intervals:

- i. [100, 100.1]
- ii. [100, 100.01]
- iii. [100, 100.001]
- iv. [100, 100.0001]

b. Use the answers from a. to estimate the average cost to produce 100 jars of mayonnaise.

Exercise:

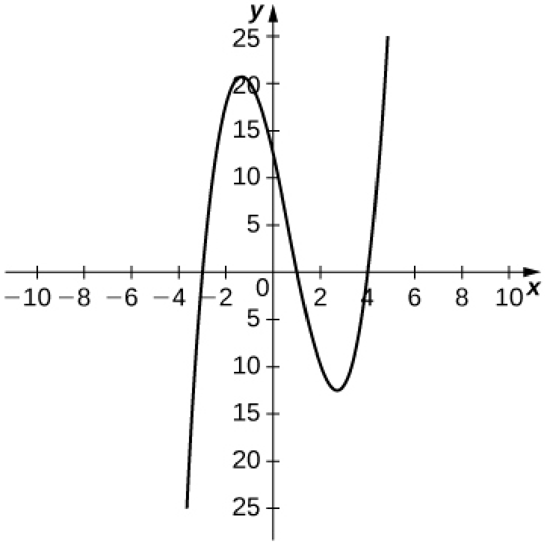
Problem: **[T]** For the function $f\left(x\right) = x^3 - 2x^2 - 11x + 12$, do the following.

a. Use a graphing calculator to graph *f* in an appropriate viewing window.

b. Use the ZOOM feature on the calculator to approximate the two values of $x=a$ for which $m_{\tan}=f'(a)=0$.

Solution:

a.



b. $a\approx -1.361, 2.694$

Exercise:

Problem: **[T]** For the function $f(x)=\frac{x}{1+x^2}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the ZOOM feature on the calculator to approximate the values of $x=a$ for which $m_{\tan}=f'(a)=0$.

Exercise:

Problem:

Suppose that $N(x)$ computes the number of gallons of gas used by a vehicle traveling x miles. Suppose the vehicle gets 30 mpg.

a. Find a mathematical expression for $N(x)$.
b. What is $N(100)$? Explain the physical meaning.
c. What is $N'(100)$? Explain the physical meaning.

Solution:

a. $N(x) = \frac{x}{30}$ b. ~ 3.3 gallons. When the vehicle travels 100 miles, it has used 3.3 gallons of gas. c. $\frac{1}{30}$. The rate of gas consumption in gallons per mile that the vehicle is achieving after having traveled 100 miles.

Exercise:

Problem: **[T]** For the function $f(x) = x^4 - 5x^2 + 4$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function, which numerically finds the derivative, on a graphing calculator to estimate $f'(-2)$, $f'(-0.5)$, $f'(1.7)$, and $f'(2.718)$.

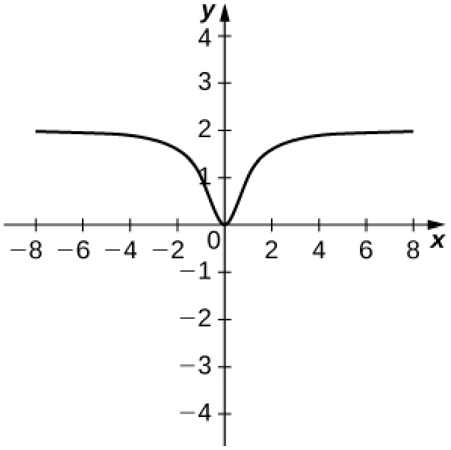
Exercise:

Problem: **[T]** For the function $f(x) = \frac{x^2}{x^2+1}$, do the following.

a. Use a graphing calculator to graph f in an appropriate viewing window.
b. Use the nDeriv function on a graphing calculator to find $f'(-4)$, $f'(-2)$, $f'(2)$, and $f'(4)$.

Solution:

a.



b. $-0.028, -0.16, 0.16, 0.028$

Glossary

derivative
: the slope of the tangent line to a function at a point, calculated by taking the limit of the difference quotient, is the derivative

difference quotient

of a function $f\left(x\right)$ at a is given by

Equation:

$$\frac{f\left(a+h\right)-f(a)}{h}\text{ or }\frac{f\left(x\right)-f(a)}{x-a}$$

differentiation

the process of taking a derivative

instantaneous rate of change

the rate of change of a function at any point along the function a, also called $f'(a)$, or the derivative of the function at a

The Derivative as a Function

- Define the derivative function of a given function.
- Graph a derivative function from the graph of a given function.
- State the connection between derivatives and continuity.
- Describe three conditions for when a function does not have a derivative.
- Explain the meaning of a higher-order derivative.

As we have seen, the derivative of a function at a given point gives us the rate of change or slope of the tangent line to the function at that point. If we differentiate a position function at a given time, we obtain the velocity at that time. It seems reasonable to conclude that knowing the derivative of the function at every point would produce valuable information about the behavior of the function. However, the process of finding the derivative at even a handful of values using the techniques of the preceding section would quickly become quite tedious. In this section we define the derivative function and learn a process for finding it.

Derivative Functions

The derivative function gives the derivative of a function at each point in the domain of the original function for which the derivative is defined. We can formally define a derivative function as follows.

Note:
Definition
Let f be a function. The **derivative function**, denoted by f', is the function whose domain consists of those values of x such that the following limit exists:
Equation:

$$f'\left(x\right)=\lim_{h\to 0}\frac{f\left(x+h\right)-f(x)}{h}.$$

A function $f(x)$ is said to be **differentiable at** a if $f'(a)$ exists. More generally, a function is said to be **differentiable on** S if it is differentiable at every point in an open set S, and a **differentiable function** is one in which $f'(x)$ exists on its domain.

In the next few examples we use [link] to find the derivative of a function.

Example:
Exercise:

Problem:
Finding the Derivative of a Square-Root Function

Find the derivative of $f\left(x\right)=\sqrt{x}$.

Solution:

Start directly with the definition of the derivative function. Use [link].
Equation:

$f'(x) = \lim\limits_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$	Substitute $f(x+h)=\sqrt{x+h}$ and $f(x)=\sqrt{x}$ into $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
$= \lim\limits_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$	Multiply numerator and denominator by $\sqrt{x+h}+\sqrt{x}$ without distributing in the denominator.
$= \lim\limits_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$	Multiply the numerators and simplify.
$= \lim\limits_{h\to 0} \frac{1}{(\sqrt{x+h}+\sqrt{x})}$	Cancel the h.
$= \frac{1}{2\sqrt{x}}$	Evaluate the limit.

Example:
Exercise:

Problem:
Finding the Derivative of a Quadratic Function

Find the derivative of the function $f(x) = x^2 - 2x$.

Solution:

Follow the same procedure here, but without having to multiply by the conjugate.
Equation:

$f'(x) = \lim\limits_{h\to 0} \frac{((x+h)^2-2(x+h))-(x^2-2x)}{h}$	Substitute $f(x+h)=(x+h)^2-2(x+h)$ and $f(x)=x^2-2x$ into $f'(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$.
$= \lim\limits_{h\to 0} \frac{x^2+2xh+h^2-2x-2h-x^2+2x}{h}$	Expand $(x+h)^2-2(x+h)$.
$= \lim\limits_{h\to 0} \frac{2xh-2h+h^2}{h}$	Simplify.
$= \lim\limits_{h\to 0} \frac{h(2x-2+h)}{h}$	Factor out h from the numerator.
$= \lim\limits_{h\to 0} (2x-2+h)$	Cancel the common factor of h.
$= 2x-2$	Evaluate the limit.

Note:
Exercise:

Problem: Find the derivative of $f(x) = x^2$.

Solution:

$f'(x) = 2x$

Hint

Use [link] and follow the example.

We use a variety of different notations to express the derivative of a function. In [link] we showed that if $f\left(x\right)={x}^{2}-2x$, then $f'\left(x\right)=2x-2$. If we had expressed this function in the form $y={x}^{2}-2x$, we could have expressed the derivative as $y'=2x-2$ or $\frac{dy}{dx}=2x-2$. We could have conveyed the same information by writing $\frac{d}{dx}\left({x}^{2}-2x\right)=2x-2$. Thus, for the function $y=f\left(x\right)$, each of the following notations represents the derivative of $f\left(x\right)$:

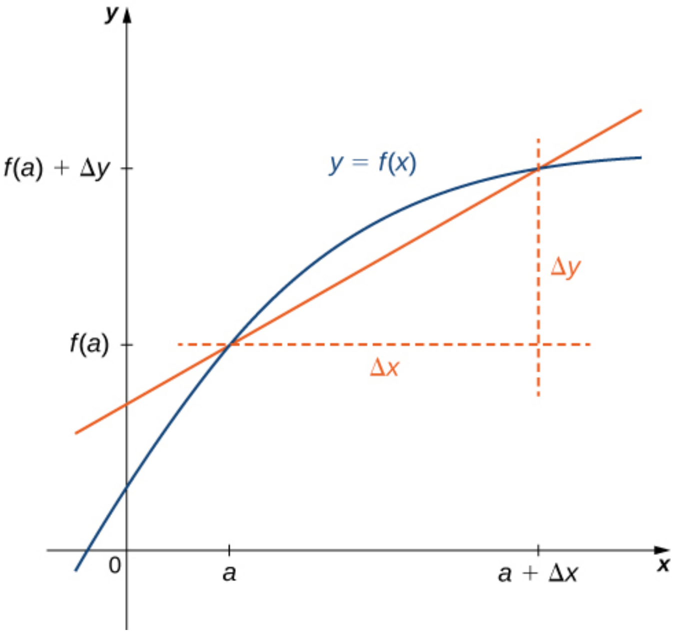
Equation:

$$f'\left(x\right),\ \frac{dy}{dx},\ y',\ \frac{d}{dx}\left(f\left(x\right)\right).$$

In place of $f'\left(a\right)$ we may also use $\frac{dy}{dx}{\Big|}_{x=a}$ Use of the $\frac{dy}{dx}$ notation (called Leibniz notation) is quite common in engineering and physics. To understand this notation better, recall that the derivative of a function at a point is the limit of the slopes of secant lines as the secant lines approach the tangent line. The slopes of these secant lines are often expressed in the form $\frac{\Delta y}{\Delta x}$ where Δy is the difference in the y values corresponding to the difference in the x values, which are expressed as Δx ([link]). Thus the derivative, which can be thought of as the instantaneous rate of change of y with respect to x, is expressed as

Equation:

$$\frac{dy}{dx}=\underset{\Delta x\to 0}{\lim}\frac{\Delta y}{\Delta x}.$$

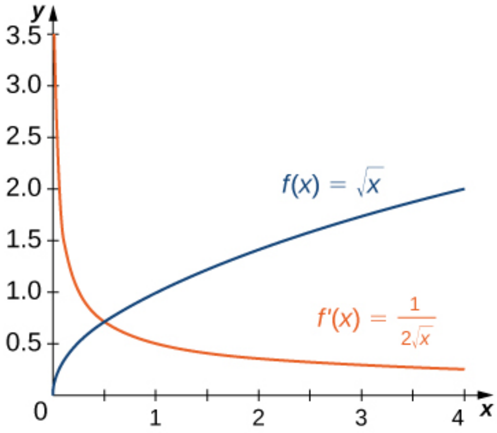


The derivative is expressed as $\frac{dy}{dx}=\underset{\Delta x\to 0}{\lim}\frac{\Delta y}{\Delta x}$.

Graphing a Derivative

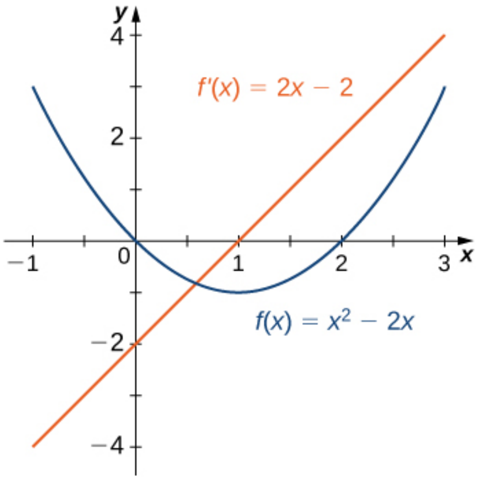
We have already discussed how to graph a function, so given the equation of a function or the equation of a derivative function, we could graph it. Given both, we would expect to see a correspondence between the graphs of these two functions, since $f'(x)$ gives the rate of change of a function $f(x)$ (or slope of the tangent line to $f(x)$).

In [link] we found that for $f(x)=\sqrt{x}, f'(x)=1/2\sqrt{x}$. If we graph these functions on the same axes, as in [link], we can use the graphs to understand the relationship between these two functions. First, we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x)>0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim\limits_{x\to 0^+} f'(x)=+\infty$, corresponding to a vertical tangent to $f(x)$ at 0.



The derivative $f'(x)$ is positive everywhere because the function $f(x)$ is increasing.

In [link] we found that for $f(x)=x^2-2x, f'(x)=2x-2$. The graphs of these functions are shown in [link]. Observe that $f(x)$ is decreasing for $x<1$. For these same values of x, $f'(x)<0$. For values of $x>1$, $f(x)$ is increasing and $f'(x)>0$. Also, $f(x)$ has a horizontal tangent at $x=1$ and $f'(1)=0$.

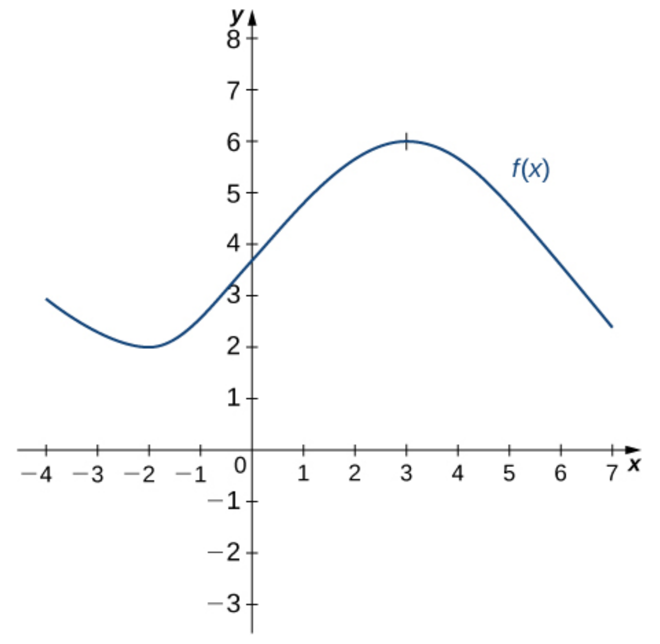


The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

Example:
Exercise:

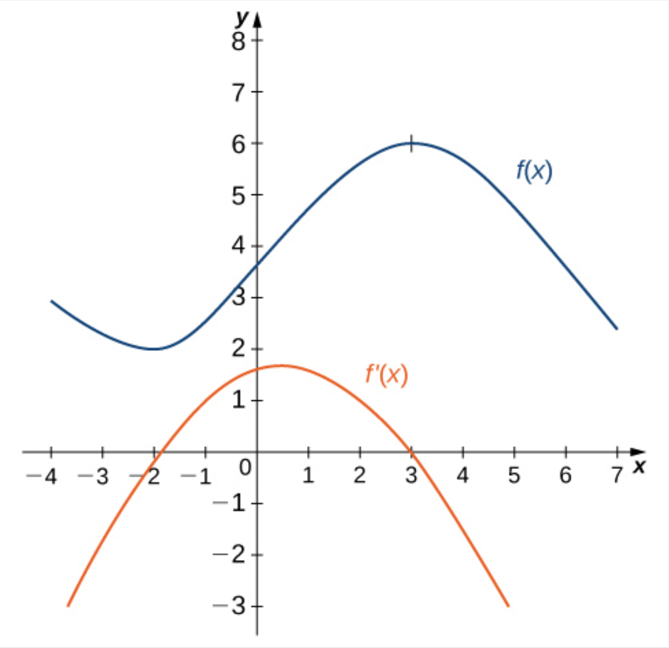
Problem:
Sketching a Derivative Using a Function

Use the following graph of $f(x)$ to sketch a graph of $f'(x)$.



Solution:

The solution is shown in the following graph. Observe that $f(x)$ is increasing and $f'(x) > 0$ on $(-2, 3)$. Also, $f(x)$ is decreasing and $f'(x) < 0$ on $(-\infty, -2)$ and on $(3, +\infty)$. Also note that $f(x)$ has horizontal tangents at -2 and 3, and $f'(-2) = 0$ and $f'(3) = 0$.



Note:
Exercise:

Problem: Sketch the graph of $f(x) = x^2 - 4$. On what interval is the graph of $f'(x)$ above the x-axis?

Solution:

$(0, +\infty)$

Hint

The graph of $f'(x)$ is positive where $f(x)$ is increasing.

Derivatives and Continuity

Now that we can graph a derivative, let’s examine the behavior of the graphs. First, we consider the relationship between differentiability and continuity. We will see that if a function is differentiable at a point, it must be continuous there; however, a function that is continuous at a point need not be differentiable at that point. In fact, a function may be continuous at a point and fail to be differentiable at the point for one of several reasons.

Note:
Differentiability Implies Continuity
Let $f(x)$ be a function and a be in its domain. If $f(x)$ is differentiable at a, then f is continuous at a.

Proof

If $f(x)$ is differentiable at a, then $f'(a)$ exists and
Equation:

$$f'\left(a\right)=\lim_{x\to a}\frac{f\left(x\right)-f(a)}{x-a}.$$

We want to show that $f(x)$ is continuous at a by showing that $\lim\limits_{x\to a}f(x)=f(a)$. Thus,
Equation:

$$\begin{aligned}\lim_{x\to a}f\left(x\right) &= \lim_{x\to a}\left(f\left(x\right)-f\left(a\right)+f(a)\right) \\ &= \lim_{x\to a}\left(\tfrac{f(x)-f(a)}{x-a}\cdot\left(x-a\right)+f(a)\right) && \text{Multiply and divide } f\left(x\right)-f(a) \text{ by } x-a. \\ &= \left(\lim_{x\to a}\tfrac{f(x)-f(a)}{x-a}\right)\cdot\left(\lim_{x\to a}\left(x-a\right)\right)+\lim_{x\to a}f(a) \\ &= f\prime(a)\cdot 0+f\left(a\right) \\ &= f(a\ ).\end{aligned}$$

Therefore, since $f\left(a\right)$ is defined and $\lim\limits_{x\to a}f\left(x\right)=f(a)$, we conclude that f is continuous at a.

□

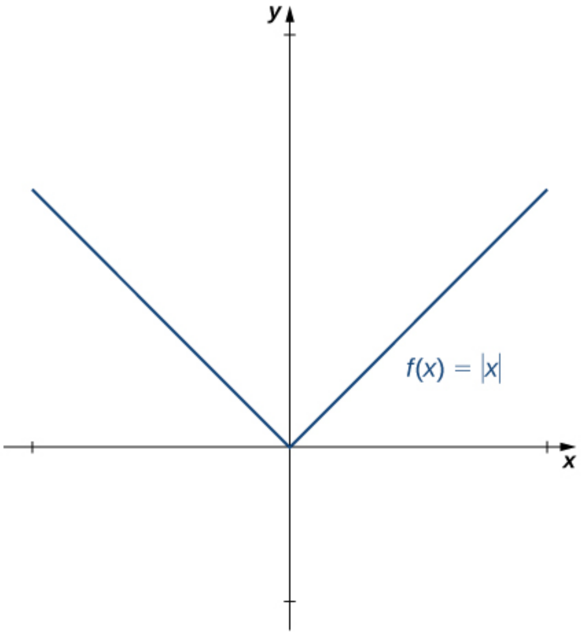
We have just proven that differentiability implies continuity, but now we consider whether continuity implies differentiability. To determine an answer to this question, we examine the function $f\left(x\right)=|x|$. This function is continuous everywhere; however, $f'(0)$ is undefined. This observation leads us to believe that continuity does not imply differentiability. Let's explore further. For $f\left(x\right)=|x|$,
Equation:

$$f'\left(0\right)=\lim_{x\to 0}\frac{f\left(x\right)-f\left(0\right)}{x-0}=\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}.$$

This limit does not exist because
Equation:

$$\lim_{x\to 0^-}\frac{|x|}{x}=-1\text{ and }\lim_{x\to 0^+}\frac{|x|}{x}=1.$$

See [link].



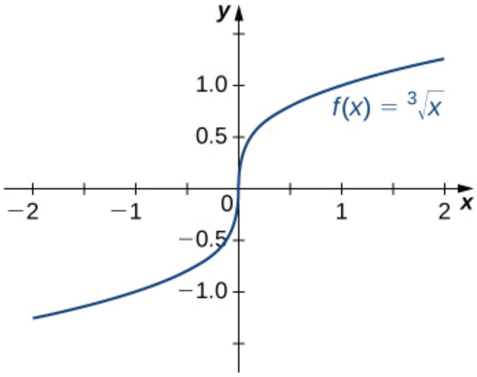
The function $f\left(x\right)=|x|$ is continuous at 0 but is not differentiable at 0.

Let’s consider some additional situations in which a continuous function fails to be differentiable. Consider the function $f\left(x\right)=\sqrt[3]{x}$:

Equation:

$$f'\left(0\right)=\underset{x\to 0}{\lim}\frac{\sqrt[3]{x}-0}{x-0}=\underset{x\to 0}{\lim}\frac{1}{\sqrt[3]{x^2}}=+\infty .$$

Thus $f'\left(0\right)$ does not exist. A quick look at the graph of $f\left(x\right)=\sqrt[3]{x}$ clarifies the situation. The function has a vertical tangent line at 0 ([link]).



The function $f\left(x\right)=\sqrt[3]{x}$ has a vertical tangent at $x=0$. It is continuous at 0 but is not differentiable at

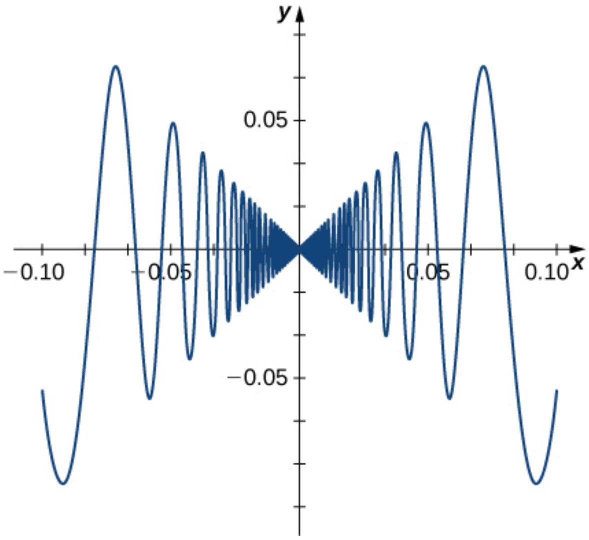
$0.$

The function $f\left(x\right)=\begin{cases}x\sin\left(\frac{1}{x}\right) & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ also has a derivative that exhibits interesting behavior at 0. We see that

Equation:

$$f'\left(0\right)=\lim_{x\to 0}\frac{x\sin\left(1/x\right)-0}{x-0}=\lim_{x\to 0}\sin\left(\frac{1}{x}\right).$$

This limit does not exist, essentially because the slopes of the secant lines continuously change direction as they approach zero ([link]).



The function $f\left(x\right)=\begin{cases}x\sin\left(\frac{1}{x}\right) & \text{if } x\ne 0\\ 0 & \text{if } x=0\end{cases}$ is not differentiable at 0.

In summary:

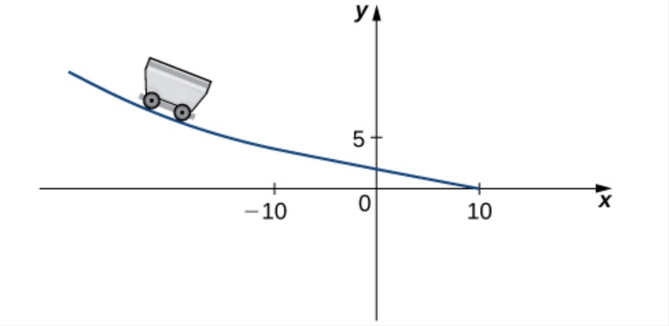
1. We observe that if a function is not continuous, it cannot be differentiable, since every differentiable function must be continuous. However, if a function is continuous, it may still fail to be differentiable.
2. We saw that $f\left(x\right)=\left|x\right|$ failed to be differentiable at 0 because the limit of the slopes of the tangent lines on the left and right were not the same. Visually, this resulted in a sharp corner on the graph of the function at 0. From this we conclude that in order to be differentiable at a point, a function must be “smooth” at that point.
3. As we saw in the example of $f\left(x\right)=\sqrt[3]{x}$, a function fails to be differentiable at a point where there is a vertical tangent line.

4. As we saw with $f\left(x\right)=\begin{cases}x\sin\left(\frac{1}{x}\right)\text{ if }x\ne 0\\ 0\text{ if }x=0\end{cases}$ a function may fail to be differentiable at a point in more complicated ways as well.

Example:
Exercise:

Problem:
A Piecewise Function that is Continuous and Differentiable

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line ([link]). The function that describes the track is to have the form $f\left(x\right)=\begin{cases}\frac{1}{10}x^2+bx+c\text{if}\,x<-10\\ -\frac{1}{4}x+\frac{5}{2}\text{if}\,x\ge -10\end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.



For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution:

For the function to be continuous at $x=-10$, $\lim\limits_{x\to -10^-}f\left(x\right)=f\left(-10\right)$. Thus, since

Equation:

$$\lim_{x\to -10^-}f\left(x\right)=\frac{1}{10}(-10)^2-10b+c=10-10b+c$$

and $f\left(-10\right)=5$, we must have $10-10b+c=5$. Equivalently, we have $c=10b-5$.

For the function to be differentiable at -10,
Equation:

$$f'\left(10\right)=\lim_{x\to -10}\frac{f\left(x\right)-f(-10)}{x+10}$$

must exist. Since $f\left(x\right)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

Equation:

$$\begin{aligned}
\lim_{x\to -10^-}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+c-5}{x+10} \\
&= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+(10b-5)-5}{x+10} && \text{Substitute } c=10b-5. \\
&= \lim_{x\to -10^-}\frac{x^2-100+10bx+100b}{10(x+10)} \\
&= \lim_{x\to -10^-}\frac{(x+10)(x-10+10b)}{10(x+10)} && \text{Factor by grouping.} \\
&= b-2.
\end{aligned}$$

We also have

Equation:

$$\begin{aligned}
\lim_{x\to -10^+}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^+}\frac{-\frac{1}{4}x+\frac{5}{2}-5}{x+10} \\
&= \lim_{x\to -10^+}\frac{-(x+10)}{4(x+10)} \\
&= -\frac{1}{4}.
\end{aligned}$$

This gives us $b-2=-\frac{1}{4}$. Thus $b=\frac{7}{4}$ and $c=10\left(\frac{7}{4}\right)-5=\frac{25}{2}$.

Note:

Exercise:

Problem:

Find values of a and b that make $f(x)=\begin{cases} ax+b \text{ if } x<3 \\ x^2 \text{ if } x\geq 3\end{cases}$ both continuous and differentiable at 3.

Solution:

$a=6$ and $b=-9$

Hint

Use [link] as a guide.

Higher-Order Derivatives

The derivative of a function is itself a function, so we can find the derivative of a derivative. For example, the derivative of a position function is the rate of change of position, or velocity. The derivative of velocity is the rate of change of velocity, which is acceleration. The new function obtained by differentiating the derivative is called the second derivative. Furthermore, we can continue to take derivatives to obtain the third derivative, fourth derivative, and so on. Collectively, these are referred to as **higher-order derivatives**. The notation for the higher-order derivatives of $y=f(x)$ can be expressed in any of the following forms:

Equation:

$$f''(x),\ f'''(x), f^{(4)}(x), \ldots, f^{(n)}(x)$$

Equation:

$$y''(x), y'''(x), y^{(4)}(x), \ldots, y^{(n)}(x)$$

Equation:

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}.$$

It is interesting to note that the notation for $\frac{d^2y}{dx^2}$ may be viewed as an attempt to express $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ more compactly. Analogously, $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$.

Example:
Exercise:

Problem:
Finding a Second Derivative

For $f(x) = 2x^2 - 3x + 1$, find $f''(x)$.

Solution:

First find $f'(x)$.
Equation:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\left(2(x+h)^2 - 3(x+h) + 1\right) - (2x^2 - 3x + 1)}{h} && \text{Substitute } f(x) = 2x^2 - 3x + 1 \text{ and } f(x+h) = 2(x+h)^2 - 3(x+h) + 1 \text{ into } f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}. \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 - 3h}{h} && \text{Simplify the numerator.} \\ &= \lim_{h\to 0} (4x + 2h - 3) && \text{Factor out the } h \text{ in the numerator and cancel with the } h \text{ in the denominator.} \\ &= 4x - 3 && \text{Take the limit.} \end{aligned}$$

Next, find $f''(x)$ by taking the derivative of $f'(x) = 4x - 3$.
Equation:

$$\begin{aligned}
f''(x) &= \lim_{h\to 0}\frac{f'(x+h)-f'(x)}{h} && \text{Use } f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \text{ with } f'(x) \text{ in place of } f(x). \\
&= \lim_{h\to 0}\frac{(4(x+h)-3)-(4x-3)}{h} && \text{Substitute } f'(x+h)=4(x+h)-3 \text{ and } f'(x)=4x-3. \\
&= \lim_{h\to 0}4 && \text{Simplify.} \\
&= 4 && \text{Take the limit.}
\end{aligned}$$

Note:
Exercise:

Problem: Find $f''(x)$ for $f(x)=x^2$.

Solution:

$f''(x)=2$

Hint

We found $f'(x)=2x$ in a previous checkpoint. Use [link] to find the derivative of $f'(x)$

Example:
Exercise:

Problem:
Finding Acceleration

The position of a particle along a coordinate axis at time t (in seconds) is given by $s(t)=3t^2-4t+1$ (in meters). Find the function that describes its acceleration at time t.

Solution:

Since $v(t)=s'(t)$ and $a(t)=v'(t)=s''(t)$, we begin by finding the derivative of $s(t)$:
Equation:

$$\begin{aligned}
s'(t) &= \lim_{h\to 0}\frac{s(t+h)-s(t)}{h} \\
&= \lim_{h\to 0}\frac{3(t+h)^2-4(t+h)+1-\left(3t^2-4t+1\right)}{h} \\
&= 6t-4.
\end{aligned}$$

Next,
Equation:

$$\begin{aligned} s''(t) &= \lim_{h\to 0} \frac{s'(t+h)-s'(t)}{h} \\ &= \lim_{h\to 0} \frac{6(t+h)-4-(6t-4)}{h} \\ &= 6. \end{aligned}$$

Thus, $a = 6\,\text{m/s}^2$.

Note:
Exercise:

Problem: For $s(t) = t^3$, find $a(t)$.

Solution:

$a(t) = 6t$

Hint

Use [link] as a guide.

Key Concepts

- The derivative of a function $f(x)$ is the function whose value at x is $f'(x)$.
- The graph of a derivative of a function $f(x)$ is related to the graph of $f(x)$. Where $f(x)$ has a tangent line with positive slope, $f'(x) > 0$. Where $f(x)$ has a tangent line with negative slope, $f'(x) < 0$. Where $f(x)$ has a horizontal tangent line, $f'(x) = 0$.
- If a function is differentiable at a point, then it is continuous at that point. A function is not differentiable at a point if it is not continuous at the point, if it has a vertical tangent line at the point, or if the graph has a sharp corner or cusp.
- Higher-order derivatives are derivatives of derivatives, from the second derivative to the $n\text{th}$ derivative.

Key Equations

- **The derivative function**
  $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

For the following exercises, use the definition of a derivative to find $f'(x)$.

Exercise:

Problem: $f(x) = 6$

Exercise:

Problem: $f(x) = 2 - 3x$

Solution:

-3

Exercise:

Problem: $f(x) = \frac{2x}{7} + 1$

Exercise:

Problem: $f(x) = 4x^2$

Solution:

$8x$

Exercise:

Problem: $f(x) = 5x - x^2$

Exercise:

Problem: $f(x) = \sqrt{2x}$

Solution:

$\frac{1}{\sqrt{2x}}$

Exercise:

Problem: $f(x) = \sqrt{x - 6}$

Exercise:

Problem: $f(x) = \frac{9}{x}$

Solution:

$\frac{-9}{x^2}$

Exercise:

Problem: $f(x) = x + \frac{1}{x}$

Exercise:

Problem: $f(x) = \frac{1}{\sqrt{x}}$

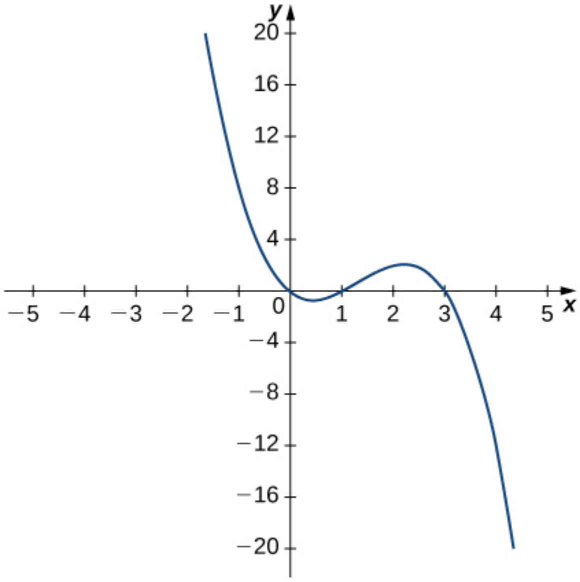
Solution:

$\frac{-1}{2x^{3/2}}$

For the following exercises, use the graph of $y = f(x)$ to sketch the graph of its derivative $f'(x)$.

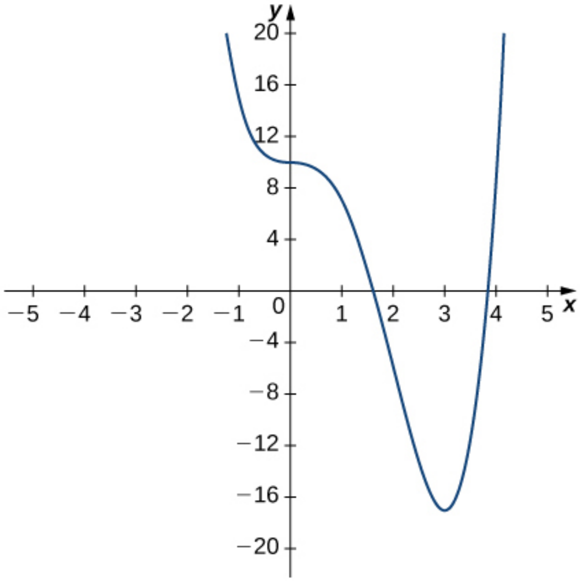
Exercise:

Problem:

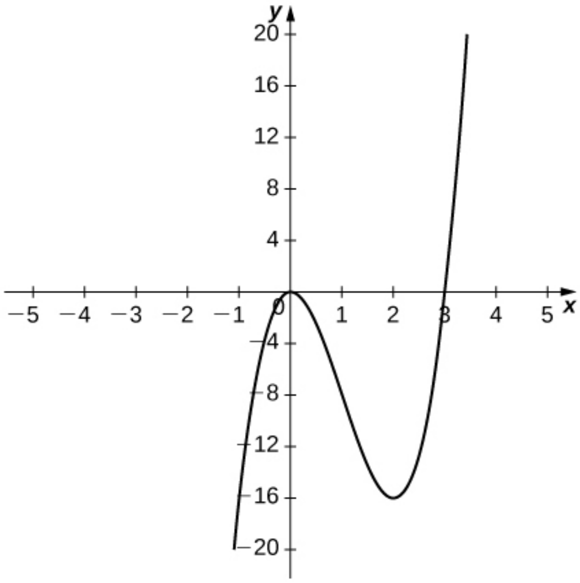


Exercise:

Problem:

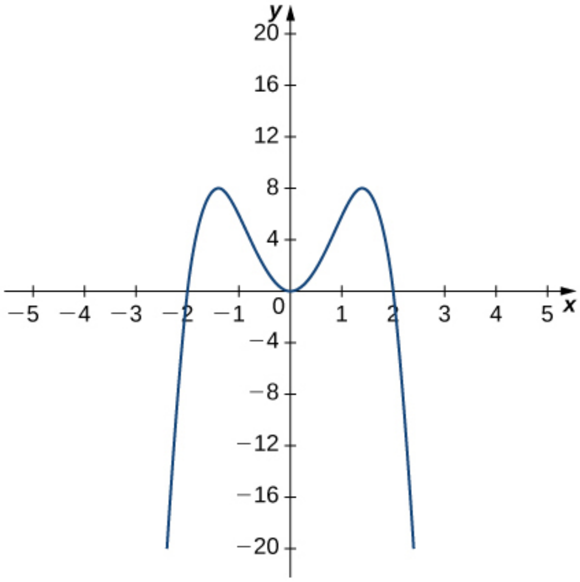


Solution:



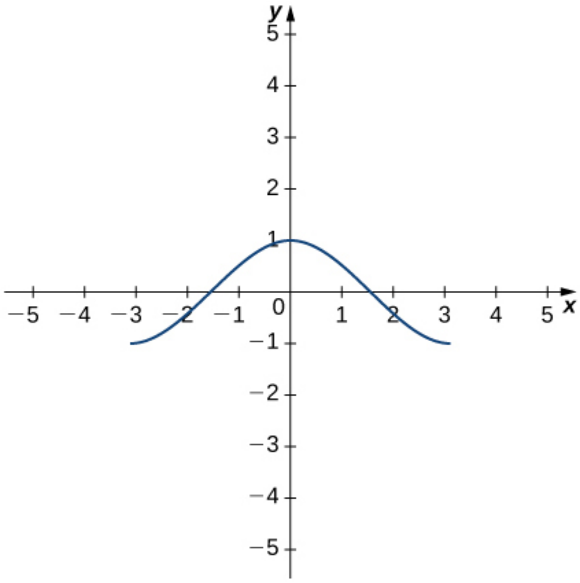
Exercise:

Problem:

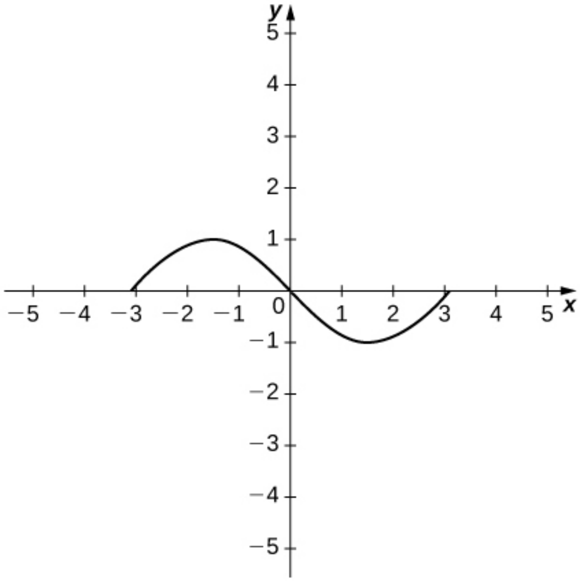


Exercise:

Problem:



Solution:



For the following exercises, the given limit represents the derivative of a function $y = f\left(x\right)$ at $x = a$. Find $f\left(x\right)$ and a.

Exercise:

Problem: $\lim\limits_{h \to 0} \frac{(1+h)^{2/3}-1}{h}$

Exercise:

Problem: $\lim\limits_{h\to 0}\frac{\left[3(2+h)^2+2\right]-14}{h}$

Solution:

$f(x) = 3x^2 + 2, a = 2$

Exercise:

Problem: $\lim\limits_{h\to 0}\frac{\cos(\pi+h)+1}{h}$

Exercise:

Problem: $\lim\limits_{h\to 0}\frac{(2+h)^4-16}{h}$

Solution:

$f(x) = x^4, a = 2$

Exercise:

Problem: $\lim\limits_{h\to 0}\frac{[2(3+h)^2-(3+h)]-15}{h}$

Exercise:

Problem: $\lim\limits_{h\to 0}\frac{e^h-1}{h}$

Solution:

$f(x) = e^x, a = 0$

For the following functions,

a. sketch the graph and
b. use the definition of a derivative to show that the function is not differentiable at $x = 1$.

Exercise:

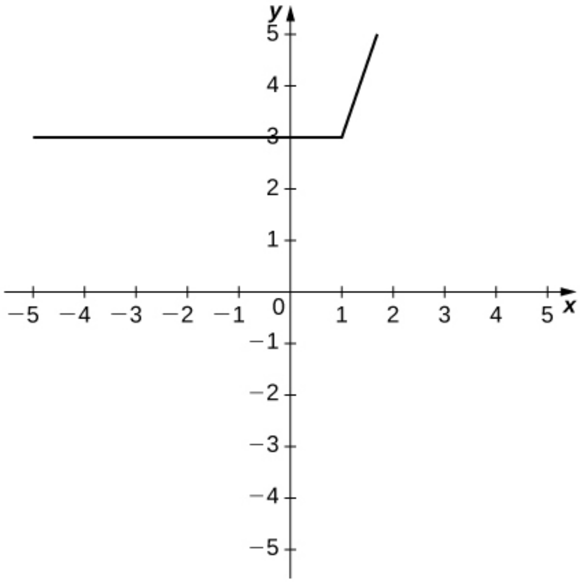
Problem: $f(x) = \begin{cases} 2\sqrt{x}, 0 \le x \le 1 \\ 3x-1, x > 1 \end{cases}$

Exercise:

Problem: $f(x) = \begin{cases} 3, x < 1 \\ 3x, x \ge 1 \end{cases}$

Solution:

a.



b. $\lim\limits_{h\to1^-}\frac{3-3}{h}\neq\lim\limits_{h\to1^+}\frac{3h}{h}$

Exercise:

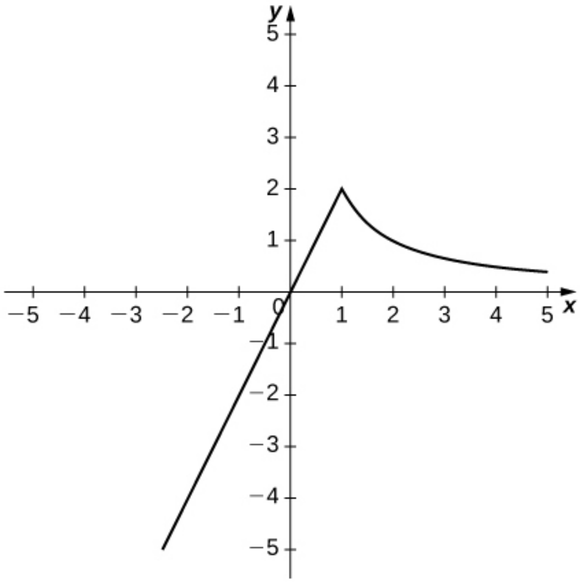
Problem: $f(x)=\begin{cases}-x^2+2, x\leq1\\ x, x>1\end{cases}$

Exercise:

Problem: $f(x)=\begin{cases}2x, x\leq1\\ \frac{2}{x}, x>1\end{cases}$

Solution:

a.



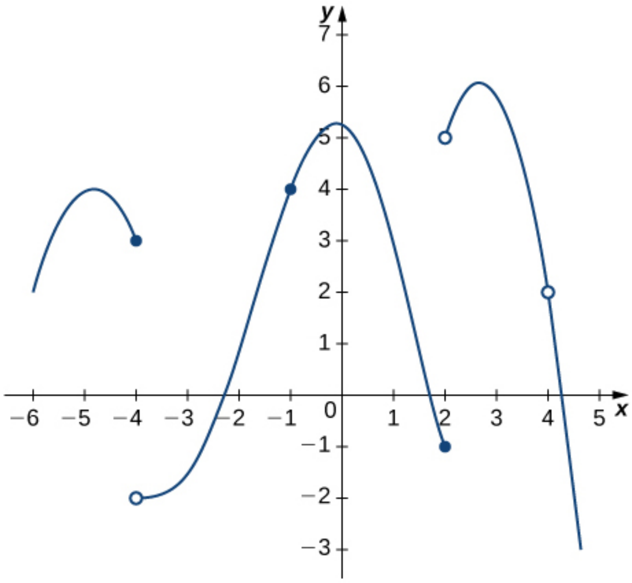
b. $\lim\limits_{h\to1^-}\frac{2h}{h} \neq \lim\limits_{h\to1^+}\frac{\frac{2}{x+h}-\frac{2}{x}}{h}$.

For the following graphs,

a. determine for which values of $x=a$ the $\lim\limits_{x\to a}f(x)$ exists but f is not continuous at $x=a$, and
b. determine for which values of $x=a$ the function is continuous but not differentiable at $x=a$.

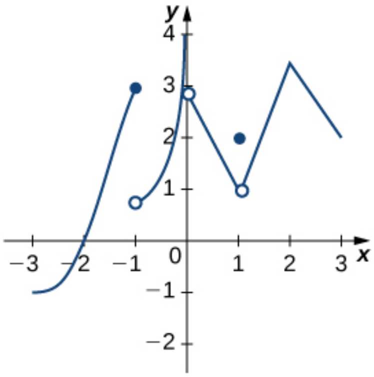
Exercise:

Problem:



Exercise:

Problem:

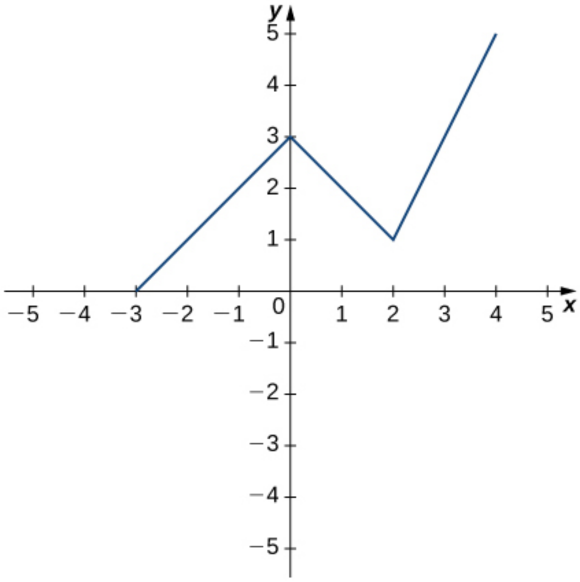


Solution:

a. $x = 1$, b. $x = 2$

Exercise:

Problem: Use the graph to evaluate a. $f'(-0.5)$, b. $f'(0)$, c. $f'(1)$, d. $f'(2)$, and e. $f'(3)$, if it exists.



For the following functions, use $f\prime\prime(x) = \lim\limits_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$ to find $f\prime\prime(x)$.

Exercise:

Problem: $f(x) = 2 - 3x$

Solution:

0

Exercise:

Problem: $f(x) = 4x^2$

Exercise:

Problem: $f(x) = x + \frac{1}{x}$

Solution:

$\frac{2}{x^3}$

For the following exercises, use a calculator to graph $f\left(x\right)$. Determine the function $f'\left(x\right)$, then use a calculator to graph $f'\left(x\right)$.

Exercise:

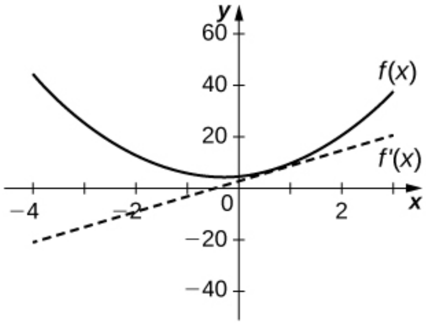
Problem: **[T]** $f\left(x\right) = -\frac{5}{x}$

Exercise:

Problem: **[T]** $f\left(x\right) = 3x^2 + 2x + 4.$

Solution:

$f'\left(x\right) = 6x + 2$



Exercise:

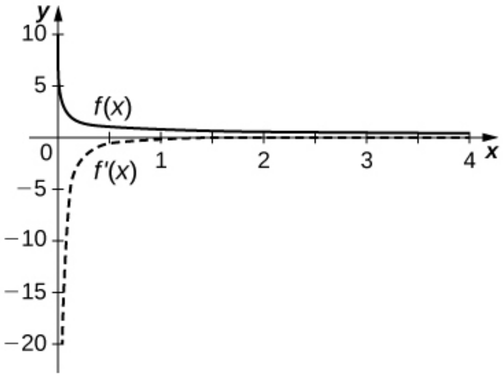
Problem: **[T]** $f\left(x\right) = \sqrt{x} + 3x$

Exercise:

Problem: **[T]** $f\left(x\right) = \frac{1}{\sqrt{2x}}$

Solution:

$f'\left(x\right) = -\frac{1}{\left(2x\right)^{3/2}}$



Exercise:

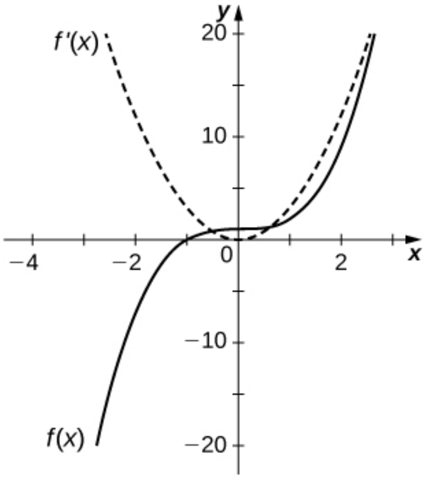
Problem: **[T]** $f\left(x\right)=1+x+\frac{1}{x}$

Exercise:

Problem: **[T]** $f\left(x\right)={x}^{3}+1$

Solution:

$f^{\prime}\left(x\right)=3{x}^{2}$



For the following exercises, describe what the two expressions represent in terms of each of the given situations. Be sure to include units.

a. $\frac{f\left(x+h\right)-f\left(x\right)}{h}$
b. $f^{\prime}\left(x\right)=\underset{h\to 0}{\lim}\frac{f\left(x+h\right)-f\left(x\right)}{h}$

Exercise:

Problem: $P(x)$ denotes the population of a city at time x in years.

Exercise:

Problem:

$C(x)$ denotes the total amount of money (in thousands of dollars) spent on concessions by x customers at an amusement park.

Solution:

a. Average rate at which customers spent on concessions in thousands per customer. b. Rate (in thousands per customer) at which x customers spent money on concessions in thousands per customer.

Exercise:

Problem: $R(x)$ denotes the total cost (in thousands of dollars) of manufacturing x clock radios.

Exercise:

Problem: $g(x)$ denotes the grade (in percentage points) received on a test, given x hours of studying.

Solution:

a. Average grade received on the test with an average study time between two values. b. Rate (in percentage points per hour) at which the grade on the test increased or decreased for a given average study time of x hours.

Exercise:

Problem:

$B(x)$ denotes the cost (in dollars) of a sociology textbook at university bookstores in the United States in x years since 1990.

Exercise:

Problem: $p\left(x\right)$ denotes atmospheric pressure at an altitude of x feet.

Solution:

a. Average change of atmospheric pressure between two different altitudes. b. Rate (torr per foot) at which atmospheric pressure is increasing or decreasing at x feet.

Exercise:

Problem: Sketch the graph of a function $y=f\left(x\right)$ with all of the following properties:

a. $f'\left(x\right)>0$ for $-2\le x<1$
b. $f'\left(2\right)=0$
c. $f'\left(x\right)>0$ for $x>2$
d. $f\left(2\right)=2$ and $f\left(0\right)=1$
e. $\lim_{x\to -\infty}f\left(x\right)=0$ and $\lim_{x\to \infty}f\left(x\right)=\infty$
f. $f'\left(1\right)$ does not exist.

Exercise:

Problem:

Suppose temperature T in degrees Fahrenheit at a height x in feet above the ground is given by $y=T(x)$.

a. Give a physical interpretation, with units, of $T'(x)$.
b. If we know that $T'(1000) = -0.1$, explain the physical meaning.

Solution:

a. The rate (in degrees per foot) at which temperature is increasing or decreasing for a given height x. b. The rate of change of temperature as altitude changes at 1000 feet is -0.1 degrees per foot.

Exercise:

Problem:

Suppose the total profit of a company is $y = P(x)$ thousand dollars when x units of an item are sold.

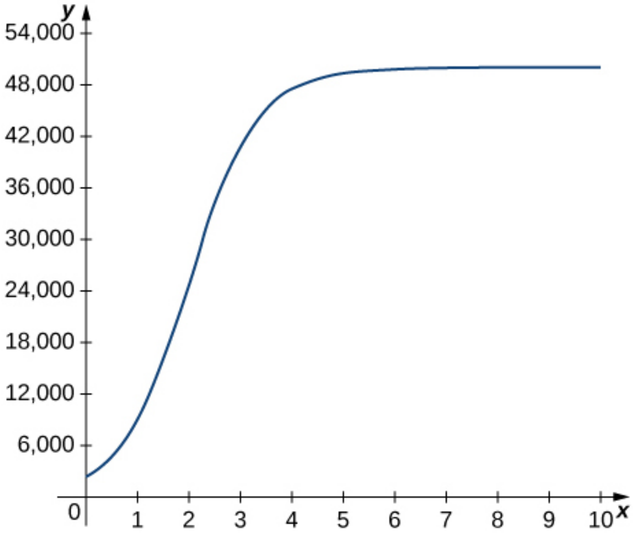
a. What does $\frac{P(b)-P(a)}{b-a}$ for $0 < a < b$ measure, and what are the units?
b. What does $P'(x)$ measure, and what are the units?
c. Suppose that $P'(30) = 5$, what is the approximate change in profit if the number of items sold increases from 30 to 31?

Exercise:

Problem:

The graph in the following figure models the number of people $N(t)$ who have come down with the flu t weeks after its initial outbreak in a town with a population of 50,000 citizens.

a. Describe what $N'(t)$ represents and how it behaves as t increases.
b. What does the derivative tell us about how this town is affected by the flu outbreak?



Solution:

a. The rate at which the number of people who have come down with the flu is changing t weeks after the initial outbreak. b. The rate is increasing sharply up to the third week, at which point it slows down and then becomes constant.

For the following exercises, use the following table, which shows the height h of the Saturn V rocket for the Apollo 11 mission t seconds after launch.

Time (seconds)	**Height (meters)**
0	0
1	2
2	4
3	13
4	25
5	32

Exercise:

Problem: What is the physical meaning of $h'(t)$? What are the units?

Exercise:

Problem:

[T] Construct a table of values for $h'(t)$ and graph both $h(t)$ and $h'(t)$ on the same graph. (*Hint:* for **interior points**, estimate both the left limit and right limit and average them. An interior point of an interval I is an element of I which is not an endpoint of I.)

Solution:

Time (seconds)	$h'(t)\ (\text{m/s})$
0	2
1	2
2	5.5
3	10.5
4	9.5
5	7

Exercise:

Problem:

[T] The best linear fit to the data is given by $H(t) = 7.229t - 4.905$, where H is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $H'(t)$. Graph $H(t)$ with the given data and, on a separate coordinate plane, graph $H'(t)$.

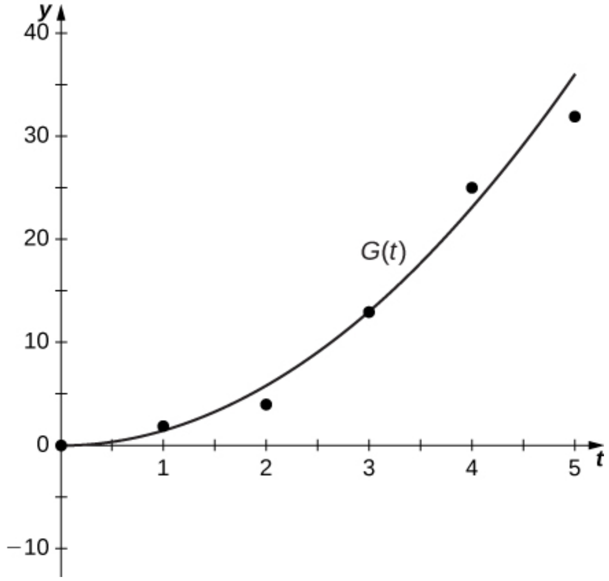
Exercise:

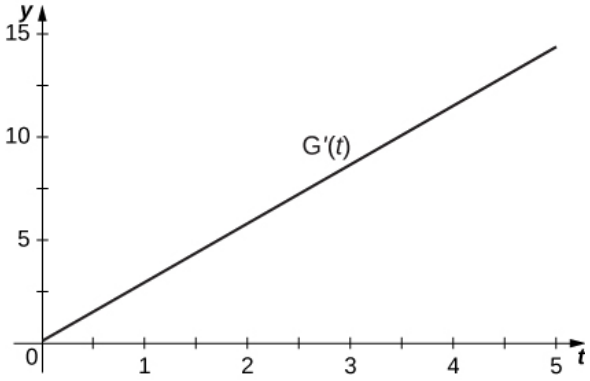
Problem:

[T] The best quadratic fit to the data is given by $G(t) = 1.429t^2 + 0.0857t - 0.1429$, where G is the height of the rocket (in meters) and t is the time elapsed since takeoff. From this equation, determine $G'(t)$. Graph $G(t)$ with the given data and, on a separate coordinate plane, graph $G'(t)$.

Solution:

$G'(t) = 2.858t + 0.0857$





Exercise:

Problem:

[T] The best cubic fit to the data is given by $F(t) = 0.2037t^3 + 2.956t^2 - 2.705t + 0.4683$, where F is the height of the rocket (in m) and t is the time elapsed since take off. From this equation, determine $F'(t)$. Graph $F(t)$ with the given data and, on a separate coordinate plane, graph $F'(t)$. Does the linear, quadratic, or cubic function fit the data best?

Exercise:

Problem:

Using the best linear, quadratic, and cubic fits to the data, determine what $H''(t), G''(t) \text{ and } F''(t)$ are. What are the physical meanings of $H''(t), G''(t) \text{ and } F''(t)$, and what are their units?

Solution:

$H''(t) = 0, G''(t) = 2.858 \text{ and } f''(t) = 1.222t + 5.912$ represent the acceleration of the rocket, with units of meters per second squared $\left(\text{m/s}^2\right)$.

Glossary

derivative function

gives the derivative of a function at each point in the domain of the original function for which the derivative is defined

differentiable at ***a***

a function for which $f'(a)$ exists is differentiable at a

differentiable on ***S***

a function for which $f'(x)$ exists for each x in the open set S is differentiable on S

differentiable function

a function for which $f'(x)$ exists is a differentiable function

higher-order derivative

a derivative of a derivative, from the second derivative to the *n*th derivative, is called a higher-order derivative

Differentiation Rules

- State the constant, constant multiple, and power rules.
- Apply the sum and difference rules to combine derivatives.
- Use the product rule for finding the derivative of a product of functions.
- Use the quotient rule for finding the derivative of a quotient of functions.
- Extend the power rule to functions with negative exponents.
- Combine the differentiation rules to find the derivative of a polynomial or rational function.

Finding derivatives of functions by using the definition of the derivative can be a lengthy and, for certain functions, a rather challenging process. For example, previously we found that $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$ by using a process that involved multiplying an expression by a conjugate prior to evaluating a limit. The process that we could use to evaluate $\frac{d}{dx}\left(\sqrt[3]{x}\right)$ using the definition, while similar, is more complicated. In this section, we develop rules for finding derivatives that allow us to bypass this process. We begin with the basics.

The Basic Rules

The functions $f\left(x\right) = c$ and $g\left(x\right) = x^n$ where n is a positive integer are the building blocks from which all polynomials and rational functions are constructed. To find derivatives of polynomials and rational functions efficiently without resorting to the limit definition of the derivative, we must first develop formulas for differentiating these basic functions.

The Constant Rule

We first apply the limit definition of the derivative to find the derivative of the constant function, $f\left(x\right) = c$. For this function, both $f\left(x\right) = c$ and $f\left(x+h\right) = c$, so we obtain the following result:
Equation:

$$\begin{aligned} f'\left(x\right) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{c-c}{h} \\ &= \lim_{h\to 0}\frac{0}{h} \\ &= \lim_{h\to 0}0 = 0. \end{aligned}$$

The rule for differentiating constant functions is called the **constant rule**. It states that the derivative of a constant function is zero; that is, since a constant function is a horizontal line, the slope, or the rate of change, of a constant function is 0. We restate this rule in the following theorem.

Note:
The Constant Rule
Let c be a constant.
If $f\left(x\right) = c$, then $f'(c) = 0$.
Alternatively, we may express this rule as
Equation:

$$\frac{d}{dx}\left(c\right) = 0.$$

Example:
Exercise:

Problem:
Applying the Constant Rule

Find the derivative of $f\left(x\right)=8$.

Solution:

This is just a one-step application of the rule:
Equation:

$$f'(x)=0.$$

Note:
Exercise:

Problem: Find the derivative of $g\left(x\right)=-3$.

Solution:

0

Hint

Use the preceding example as a guide.

The Power Rule

We have shown that
Equation:

$$\frac{d}{dx}\left(x^2\right)=2x \text{ and } \frac{d}{dx}\left(x^{1/2}\right)=\frac{1}{2}x^{-1/2}.$$

At this point, you might see a pattern beginning to develop for derivatives of the form $\frac{d}{dx}\left(x^n\right)$. We continue our examination of derivative formulas by differentiating power functions of the form $f\left(x\right)=x^n$ where n is a positive integer. We develop formulas for derivatives of this type of function in stages, beginning with positive integer powers. Before stating and proving the general rule for derivatives of functions of this form, we take a look at a specific case, $\frac{d}{dx}\left(x^3\right)$. As we go through this derivation, note that the technique used in this case is essentially the same as the technique used to prove the general case.

Example:
Exercise:

Problem:
Differentiating x^3

Find $\frac{d}{dx}\left(x^3\right)$.

Solution:
Equation:

$\frac{d}{dx}\left(x^3\right) = \lim\limits_{h \to 0} \frac{(x+h)^3 - x^3}{h}$	
$= \lim\limits_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$	Notice that the first term in the expansion of $(x+h)^3$ is x^3 and the second term is $3x^2h.$ All other terms contain powers of h that are two or greater.
$= \lim\limits_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$	In this step the x^3 terms have been cancelled, leaving only terms containing h.
$= \lim\limits_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$	Factor out the common factor of h.
$= \lim\limits_{h \to 0} (3x^2 + 3xh + h^2)$	After cancelling the common factor of h, the only term not containing h is $3x^2$.
$= 3x^2$	Let h go to 0.

Note:
Exercise:

Problem: Find $\frac{d}{dx}\left(x^4\right)$.

Solution:

$4x^3$

Hint

Use $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$ and follow the procedure outlined in the preceding example.

As we shall see, the procedure for finding the derivative of the general form $f(x) = x^n$ is very similar. Although it is often unwise to draw general conclusions from specific examples, we note that when we differentiate $f(x) = x^3$, the power on x becomes the coefficient of x^2 in the derivative and the power on x in the derivative decreases by 1. The following theorem states that the **power rule** holds for all positive integer powers of x. We will eventually extend this result to negative integer powers. Later, we will see that this rule may also be extended first to rational powers of x and then to arbitrary powers of x. Be aware, however, that this rule does not apply to functions in which a constant is raised to a variable power, such as $f(x) = 3^x$.

Note:
The Power Rule
Let n be a positive integer. If $f(x) = x^n$, then
Equation:

$$f'(x) = nx^{n-1}.$$

Alternatively, we may express this rule as

Equation:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof

For $f\left(x\right) = x^n$ where n is a positive integer, we have
Equation:

$$f'\left(x\right) = \lim_{h \to 0} \frac{\left(x+h\right)^n - x^n}{h}.$$

Equation:

$$\text{Since } \left(x+h\right)^n = x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \binom{n}{3} x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n,$$

we see that
Equation:

$$\left(x+h\right)^n - x^n = nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \binom{n}{3} x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n.$$

Next, divide both sides by h:
Equation:

$$\frac{\left(x+h\right)^n - x^n}{h} = \frac{nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \binom{n}{3} x^{n-3}h^3 + \ldots + nxh^{n-1} + h^n}{h}.$$

Thus,
Equation:

$$\frac{\left(x+h\right)^n - x^n}{h} = nx^{n-1} + \binom{n}{2} x^{n-2}h + \binom{n}{3} x^{n-3}h^2 + \ldots + nxh^{n-2} + h^{n-1}.$$

Finally,
Equation:

$$\begin{aligned} f'\left(x\right) &= \lim_{h \to 0} \left(nx^{n-1} + \binom{n}{2} x^{n-2}h + \binom{n}{3} x^{n-3}h^2 + \ldots + nxh^{n-1} + h^n \right) \\ &= nx^{n-1}. \end{aligned}$$

□

Example:

Exercise:

Problem:
Applying the Power Rule

Find the derivative of the function $f\left(x\right)=x^{10}$ by applying the power rule.

Solution:

Using the power rule with $n=10$, we obtain
Equation:

$$f\prime(x)=10x^{10-1}=10x^{9}.$$

Note:
Exercise:

Problem: Find the derivative of $f\left(x\right)=x^{7}$.

Solution:

$f'\left(x\right)=7x^{6}$

Hint

Use the power rule with $n=7$.

The Sum, Difference, and Constant Multiple Rules

We find our next differentiation rules by looking at derivatives of sums, differences, and constant multiples of functions. Just as when we work with functions, there are rules that make it easier to find derivatives of functions that we add, subtract, or multiply by a constant. These rules are summarized in the following theorem.

Note:
Sum, Difference, and Constant Multiple Rules
Let $f(x)$ and $g(x)$ be differentiable functions and k be a constant. Then each of the following equations holds.
Sum Rule. The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
Equation:

$$\frac{d}{dx}\left(f\left(x\right)+g\left(x\right)\right)=\frac{d}{dx}\left(f\left(x\right)\right)+\frac{d}{dx}\left(g\left(x\right)\right);$$

that is,
Equation:

$$\text{for } j\left(x\right)=f\left(x\right)+g\left(x\right), j'\left(x\right)=f'\left(x\right)+g'(x).$$

Difference Rule. The derivative of the difference of a function *f* and a function g is the same as the difference of the derivative of *f* and the derivative of g:
Equation:

$$\frac{d}{dx}\left(f\left(x\right)-g\left(x\right)\right)=\frac{d}{dx}\left(f\left(x\right)\right)-\frac{d}{dx}\left(g\left(x\right)\right);$$

that is,
Equation:

$$\text{for } j\left(x\right)=f\left(x\right)-g\left(x\right), j'\left(x\right)=f'\left(x\right)-g'(x).$$

Constant Multiple Rule. The derivative of a constant *k* multiplied by a function *f* is the same as the constant multiplied by the derivative:
Equation:

$$\frac{d}{dx}\left(kf\left(x\right)\right)=k\frac{d}{dx}\left(f\left(x\right)\right);$$

that is,
Equation:

$$\text{for } j\left(x\right)=kf(x), j'\left(x\right)=kf'(x).$$

Proof

We provide only the proof of the sum rule here. The rest follow in a similar manner.

For differentiable functions $f\left(x\right)$ and $g\left(x\right)$, we set $j\left(x\right)=f\left(x\right)+g\left(x\right)$. Using the limit definition of the derivative we have
Equation:

$$j'\left(x\right)=\lim_{h\to 0}\frac{j\left(x+h\right)-j\left(x\right)}{h}.$$

By substituting $j\left(x+h\right)=f\left(x+h\right)+g\left(x+h\right)$ and $j\left(x\right)=f\left(x\right)+g\left(x\right)$, we obtain
Equation:

$$j'(x)=\lim_{h\to 0}\frac{\left(f\left(x+h\right)+g\left(x+h\right)\right)-\left(f\left(x\right)+g\left(x\right)\right)}{h}.$$

Rearranging and regrouping the terms, we have
Equation:

$$j'(x)=\lim_{h\to 0}\left(\frac{f\left(x+h\right)-f(x)}{h}+\frac{g\left(x+h\right)-g(x)}{h}\right).$$

We now apply the sum law for limits and the definition of the derivative to obtain
Equation:

$$j'(x)=\lim_{h\to 0}\left(\frac{f\left(x+h\right)-f(x)}{h}\right)+\lim_{h\to 0}\left(\frac{g\left(x+h\right)-g(x)}{h}\right)=f'\left(x\right)+g'\left(x\right).$$

□

Example:
Exercise:

Problem:
Applying the Constant Multiple Rule

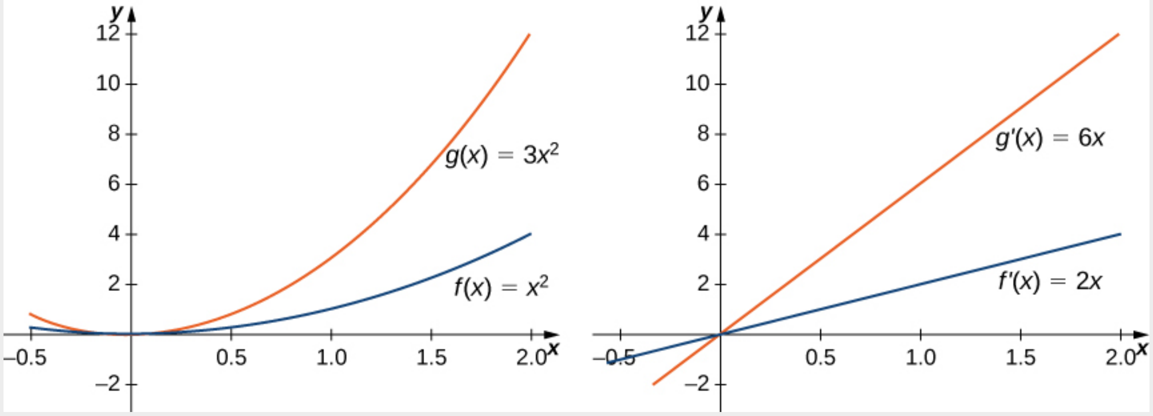
Find the derivative of $g\left(x\right)=3{x}^{2}$ and compare it to the derivative of $f\left(x\right)={x}^{2}$.

Solution:

We use the power rule directly:
Equation:

$$g\prime \left(x\right)=\frac{d}{dx}\left(3{x}^{2}\right)=3\frac{d}{dx}\left({x}^{2}\right)=3\left(2x\right)=6x.$$

Since $f\left(x\right)={x}^{2}$ has derivative $f\prime \left(x\right)=2x$, we see that the derivative of $g\left(x\right)$ is 3 times the derivative of $f\left(x\right)$. This relationship is illustrated in [link].



The derivative of $g\left(x\right)$ is 3 times the derivative of $f\left(x\right)$.

Example:
Exercise:

Problem:
Applying Basic Derivative Rules

Find the derivative of $f\left(x\right)=2{x}^{5}+7$.

Solution:

We begin by applying the rule for differentiating the sum of two functions, followed by the rules for differentiating constant multiples of functions and the rule for differentiating powers. To better understand the sequence in which the differentiation rules are applied, we use Leibniz notation throughout the solution:
Equation:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x^5+7\right) & \\ &= \frac{d}{dx}\left(2x^5\right)+\frac{d}{dx}(7) && \text{Apply the sum rule.} \\ &= 2\frac{d}{dx}\left(x^5\right)+\frac{d}{dx}(7) && \text{Apply the constant multiple rule.} \\ &= 2\left(5x^4\right)+0 && \text{Apply the power rule and the constant rule.} \\ &= 10x^4. && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Find the derivative of $f(x) = 2x^3 - 6x^2 + 3$.

Solution:

$f'(x) = 6x^2 - 12x.$

Hint

Use the preceding example as a guide.

Example:
Exercise:

Problem:
Finding the Equation of a Tangent Line

Find the equation of the line tangent to the graph of $f(x) = x^2 - 4x + 6$ at $x = 1$.

Solution:

To find the equation of the tangent line, we need a point and a slope. To find the point, compute
Equation:

$$f(1) = 1^2 - 4(1) + 6 = 3.$$

This gives us the point $(1, 3)$. Since the slope of the tangent line at 1 is $f'(1)$, we must first find $f'(x)$. Using the definition of a derivative, we have
Equation:

$$f'(x) = 2x - 4$$

so the slope of the tangent line is $f'(1) = -2$. Using the point-slope formula, we see that the equation of the tangent line is
Equation:

$$y - 3 = -2(x - 1).$$

Putting the equation of the line in slope-intercept form, we obtain
Equation:

$$y = -2x + 5.$$

Note:
Exercise:

Problem:

Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 11$ at $x = 2$. Use the point-slope form.

Solution:

$y = 12x - 23$

Hint

Use the preceding example as a guide.

The Product Rule

Now that we have examined the basic rules, we can begin looking at some of the more advanced rules. The first one examines the derivative of the product of two functions. Although it might be tempting to assume that the derivative of the product is the product of the derivatives, similar to the sum and difference rules, the **product rule** does not follow this pattern. To see why we cannot use this pattern, consider the function $f(x) = x^2$, whose derivative is $f'(x) = 2x$ and not $\frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1$.

Note:
Product Rule
Let $f(x)$ and $g(x)$ be differentiable functions. Then
Equation:

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + \frac{d}{dx}(g(x)) \cdot f(x).$$

That is,
Equation:

$$\text{if } j(x) = f(x)g(x), \text{then } j'(x) = f'(x)g(x) + g'(x)f(x).$$

This means that the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.

Proof

We begin by assuming that $f(x)$ and $g(x)$ are differentiable functions. At a key point in this proof we need to use the fact that, since $g(x)$ is differentiable, it is also continuous. In particular, we use the fact that since $g(x)$ is continuous, $\lim\limits_{h \to 0} g(x+h) = g(x)$.

By applying the limit definition of the derivative to $j(x) = f(x)g(x)$, we obtain
Equation:

$$j'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}.$$

By adding and subtracting $f(x)g(x+h)$ in the numerator, we have

Equation:

$$j'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}.$$

After breaking apart this quotient and applying the sum law for limits, the derivative becomes

Equation:

$$j'(x) = \lim_{h\to 0} \left(\frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}\right) + \lim_{h\to 0} \left(\frac{f(x)g(x+h) - f(x)g(x)}{h}\right).$$

Rearranging, we obtain

Equation:

$$j'(x) = \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h)\right) + \lim_{h\to 0} \left(\frac{g(x+h) - g(x)}{h} \cdot f(x)\right).$$

By using the continuity of $g(x)$, the definition of the derivatives of $f(x)$ and $g(x)$, and applying the limit laws, we arrive at the product rule,

Equation:

$$j'(x) = f'(x)g(x) + g'(x)f(x).$$

□

Example:
Exercise:

Problem:
Applying the Product Rule to Functions at a Point

For $j(x) = f(x)g(x)$, use the product rule to find $j'(2)$ if $f(2) = 3$, $f'(2) = -4$, $g(2) = 1$, and $g'(2) = 6$.

Solution:

Since $j(x) = f(x)g(x)$, $j'(x) = f'(x)g(x) + g'(x)f(x)$, and hence

Equation:

$$j'(2) = f'(2)g(2) + g'(2)f(2) = (-4)(1) + (6)(3) = 14.$$

Example:
Exercise:

Problem:
Applying the Product Rule to Binomials

For $j\left(x\right)=\left(x^2+2\right)\left(3x^3-5x\right)$, find $j'\left(x\right)$ by applying the product rule. Check the result by first finding the product and then differentiating.

Solution:

If we set $f\left(x\right)=x^2+2$ and $g\left(x\right)=3x^3-5x$, then $f'\left(x\right)=2x$ and $g'\left(x\right)=9x^2-5$. Thus,
Equation:

$$j'\left(x\right)=f'\left(x\right)g\left(x\right)+g'\left(x\right)f\left(x\right)=\left(2x\right)\left(3x^3-5x\right)+\left(9x^2-5\right)\left(x^2+2\right).$$

Simplifying, we have
Equation:

$$j'\left(x\right)=15x^4+3x^2-10.$$

To check, we see that $j\left(x\right)=3x^5+x^3-10x$ and, consequently, $j'\left(x\right)=15x^4+3x^2-10$.

Note:
Exercise:

Problem: Use the product rule to obtain the derivative of $j\left(x\right)=2x^5\left(4x^2+x\right)$.

Solution:

$j'\left(x\right)=10x^4\left(4x^2+x\right)+\left(8x+1\right)\left(2x^5\right)=56x^6+12x^5.$

Hint

Set $f\left(x\right)=2x^5$ and $g\left(x\right)=4x^2+x$ and use the preceding example as a guide.

The Quotient Rule

Having developed and practiced the product rule, we now consider differentiating quotients of functions. As we see in the following theorem, the derivative of the quotient is not the quotient of the derivatives; rather, it is the derivative of the function in the numerator times the function in the denominator minus the derivative of the function in the denominator times the function in the numerator, all divided by the square of the function in the denominator. In order to better grasp why we cannot simply take the quotient of the derivatives, keep in mind that
Equation:

$$\frac{d}{dx}\left(x^2\right)=2x,\text{not}\ \frac{\frac{d}{dx}\left(x^3\right)}{\frac{d}{dx}\left(x\right)}=\frac{3x^2}{1}=3x^2.$$

Note:
The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then

Equation:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{\frac{d}{dx}\left(f\left(x\right)\right)\cdot g\left(x\right)-\frac{d}{dx}\left(g\left(x\right)\right)\cdot f(x)}{\left(g\left(x\right)\right)^2}.$$

That is,

Equation:

$$\text{if } j\left(x\right)=\frac{f(x)}{g(x)}, \text{then } j'\left(x\right)=\frac{f'\left(x\right)g\left(x\right)-g'\left(x\right)f(x)}{\left(g\left(x\right)\right)^2}.$$

The proof of the **quotient rule** is very similar to the proof of the product rule, so it is omitted here. Instead, we apply this new rule for finding derivatives in the next example.

Example:

Exercise:

Problem:

Applying the Quotient Rule

Use the quotient rule to find the derivative of $k\left(x\right)=\frac{5x^2}{4x+3}$.

Solution:

Let $f\left(x\right)=5x^2$ and $g\left(x\right)=4x+3$. Thus, $f'\left(x\right)=10x$ and $g'\left(x\right)=4$. Substituting into the quotient rule, we have

Equation:

$$k'\left(x\right)=\frac{f'\left(x\right)g\left(x\right)-g'\left(x\right)f(x)}{\left(g\left(x\right)\right)^2}=\frac{10x\left(4x+3\right)-4(5x^2)}{\left(4x+3\right)^2}.$$

Simplifying, we obtain

Equation:

$$k'\left(x\right)=\frac{20x^2+30x}{\left(4x+3\right)^2}.$$

Note:

Exercise:

Problem: Find the derivative of $h\left(x\right)=\frac{3x+1}{4x-3}$.

Solution:

$k'\left(x\right)=-\frac{13}{\left(4x-3\right)^2}$.

Hint

Apply the quotient rule with $f(x) = 3x + 1$ and $g(x) = 4x - 3$.

It is now possible to use the quotient rule to extend the power rule to find derivatives of functions of the form x^k where k is a negative integer.

Note:
Extended Power Rule
If k is a negative integer, then
Equation:

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

Proof

If k is a negative integer, we may set $n = -k$, so that n is a positive integer with $k = -n$. Since for each positive integer $n, x^{-n} = \frac{1}{x^n}$, we may now apply the quotient rule by setting $f(x) = 1$ and $g(x) = x^n$. In this case, $f'(x) = 0$ and $g'(x) = nx^{n-1}$. Thus,
Equation:

$$\frac{d}{\mathrm{dx}}\left(x^{-n}\right) = \frac{0\left(x^n\right) - 1\left(nx^{n-1}\right)}{\left(x^n\right)^2}.$$

Simplifying, we see that
Equation:

$$\frac{d}{\mathrm{dx}}\left(x^{-n}\right) = \frac{-nx^{n-1}}{x^{2n}} = -nx^{(n-1)-2n} = -nx^{-n-1}.$$

Finally, observe that since $k = -n$, by substituting we have
Equation:

$$\frac{d}{dx}\left(x^k\right) = kx^{k-1}.$$

□

Example:
Exercise:

Problem:
Using the Extended Power Rule

Find $\frac{d}{dx}\left(x^{-4}\right)$.

Solution:

By applying the extended power rule with $k = -4$, we obtain

Equation:

$$\frac{d}{dx}\left(x^{-4}\right) = -4x^{-4-1} = -4x^{-5}.$$

Example:

Exercise:

Problem:

Using the Extended Power Rule and the Constant Multiple Rule

Use the extended power rule and the constant multiple rule to find the derivative of $f\left(x\right) = \frac{6}{x^2}$.

Solution:

It may seem tempting to use the quotient rule to find this derivative, and it would certainly not be incorrect to do so. However, it is far easier to differentiate this function by first rewriting it as $f\left(x\right) = 6x^{-2}$.

Equation:

$$\begin{aligned} f'\left(x\right) &= \tfrac{d}{dx}\left(\tfrac{6}{x^2}\right) = \tfrac{d}{dx}\left(6x^{-2}\right) && \text{Rewrite } \tfrac{6}{x^2} \text{ as } 6x^{-2}. \\ &= 6\tfrac{d}{dx}\left(x^{-2}\right) && \text{Apply the constant multiple rule.} \\ &= 6(-2x^{-3}) && \text{Use the extended power rule to differentiate } x^{-2}. \\ &= -12x^{-3} && \text{Simplify.} \end{aligned}$$

Note:

Exercise:

Problem: Find the derivative of $g\left(x\right) = \frac{1}{x^7}$ using the extended power rule.

Solution:

$g'\left(x\right) = -7x^{-8}$.

Hint

Rewrite $g\left(x\right) = \frac{1}{x^7} = x^{-7}$. Use the extended power rule with $k = -7$.

Combining Differentiation Rules

As we have seen throughout the examples in this section, it seldom happens that we are called on to apply just one differentiation rule to find the derivative of a given function. At this point, by combining the differentiation rules, we may find the derivatives of any polynomial or rational function. Later on we will encounter more complex combinations of differentiation rules. A good rule of thumb to use when applying several rules is to apply the rules in reverse of the order in which we would evaluate the function.

Example:
Exercise:

Problem:
Combining Differentiation Rules

For $k(x) = 3h(x) + x^2 g(x)$, find $k'(x)$.

Solution:

Finding this derivative requires the sum rule, the constant multiple rule, and the product rule.
Equation:

$$\begin{aligned} k'(x) &= \tfrac{d}{dx}\left(3h(x) + x^2 g(x)\right) = \tfrac{d}{dx}(3h(x)) + \tfrac{d}{dx}\left(x^2 g(x)\right) \\ &= 3\tfrac{d}{dx}(h(x)) + \left(\tfrac{d}{dx}\left(x^2\right) g(x) + \tfrac{d}{dx}(g(x)) x^2\right) \\ &= 3h'(x) + 2xg(x) + g'(x)x^2 \end{aligned}$$

Apply the sum rule.
Apply the constant multiple rule to differentiate $3h(x)$ and the product rule to differentiate $x^2 g(x)$.

Example:
Exercise:

Problem:
Extending the Product Rule

For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x)$, $g(x)$, $h(x)$, and their derivatives.

Solution:

We can think of the function $k(x)$ as the product of the function $f(x)g(x)$ and the function $h(x)$. That is, $k(x) = (f(x)g(x)) \cdot h(x)$. Thus,
Equation:

$$\begin{aligned} k'(x) &= \tfrac{d}{dx}(f(x)g(x)) \cdot h(x) + \tfrac{d}{dx}(h(x)) \cdot (f(x)g(x)) \\ &= (f'(x)g(x) + g'(x)f(x))h(x) + h'(x)f(x)g(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x). \end{aligned}$$

Apply the product rule to the produ
of $f(x)g(x)$ and $h(x)$.
Apply the product rule to $f(x)g(x)$
Simplify.

Example:
Exercise:

Problem:
Combining the Quotient Rule and the Product Rule

For $h(x) = \frac{2x^3 k(x)}{3x+2}$, find $h'(x)$.

Solution:

This procedure is typical for finding the derivative of a rational function.

Equation:

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}\left(2x^3k(x)\right)\cdot(3x+2)-\frac{d}{dx}(3x+2)\cdot\left(2x^3k(x)\right)}{(3x+2)^2} && \text{Apply the quotient rule.} \\ &= \frac{\left(6x^2k(x)+k'(x)\cdot 2x^3\right)(3x+2)-3\left(2x^3k(x)\right)}{(3x+2)^2} && \text{Apply the product rule to find } \frac{d}{dx}\left(2x^3k\left(x\right)\right)\text{. Use } \frac{d}{dx}\left(3x+2\right)=3. \\ &= \frac{-6x^3k(x)+18x^3k(x)+12x^2k(x)+6x^4k'(x)+4x^3k'(x)}{(3x+2)^2} && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Find $\frac{d}{dx}\left(3f\left(x\right)-2g\left(x\right)\right)$.

Solution:

$3f'\left(x\right)-2g'\left(x\right)$.

Hint

Apply the difference rule and the constant multiple rule.

Example:
Exercise:

Problem:
Determining Where a Function Has a Horizontal Tangent

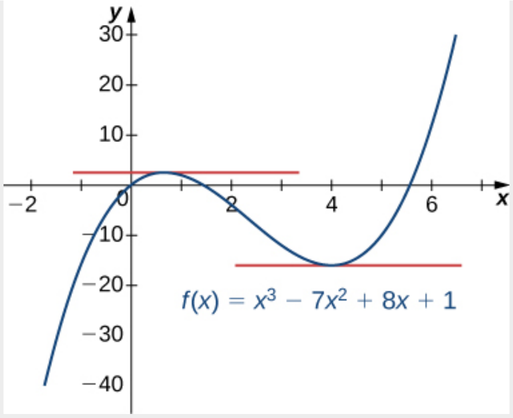
Determine the values of x for which $f\left(x\right)=x^3-7x^2+8x+1$ has a horizontal tangent line.

Solution:

To find the values of x for which $f(x)$ has a horizontal tangent line, we must solve $f'\left(x\right)=0$. Since
Equation:

$$f'\left(x\right)=3x^2-14x+8=\left(3x-2\right)\left(x-4\right),$$

we must solve $\left(3x-2\right)\left(x-4\right)=0$. Thus we see that the function has horizontal tangent lines at $x=\frac{2}{3}$ and $x=4$ as shown in the following graph.



This function has horizontal tangent lines at *x* = 2/3 and *x* = 4.

Example:
Exercise:

Problem:
Finding a Velocity

The position of an object on a coordinate axis at time t is given by $s\left(t\right)=\frac{t}{t^2+1}$. What is the initial velocity of the object?

Solution:

Since the initial velocity is $v\left(0\right)=s'\left(0\right)$, begin by finding $s'(t)$ by applying the quotient rule:
Equation:

$$s'\left(t\right)=\frac{1\left(t^2+1\right)-2t(t)}{\left(t^2+1\right)^2}=\frac{1-t^2}{\left(t^2+1\right)^2}.$$

After evaluating, we see that $v\left(0\right)=1$.

Note:
Exercise:

Problem:

Find the values of x for which the graph of $f\left(x\right)=4x^2-3x+2$ has a tangent line parallel to the line $y=2x+3$.

Solution:

$\frac{5}{8}$

Hint

Solve $f'(x) = 2$.

Note:

Formula One Grandstands

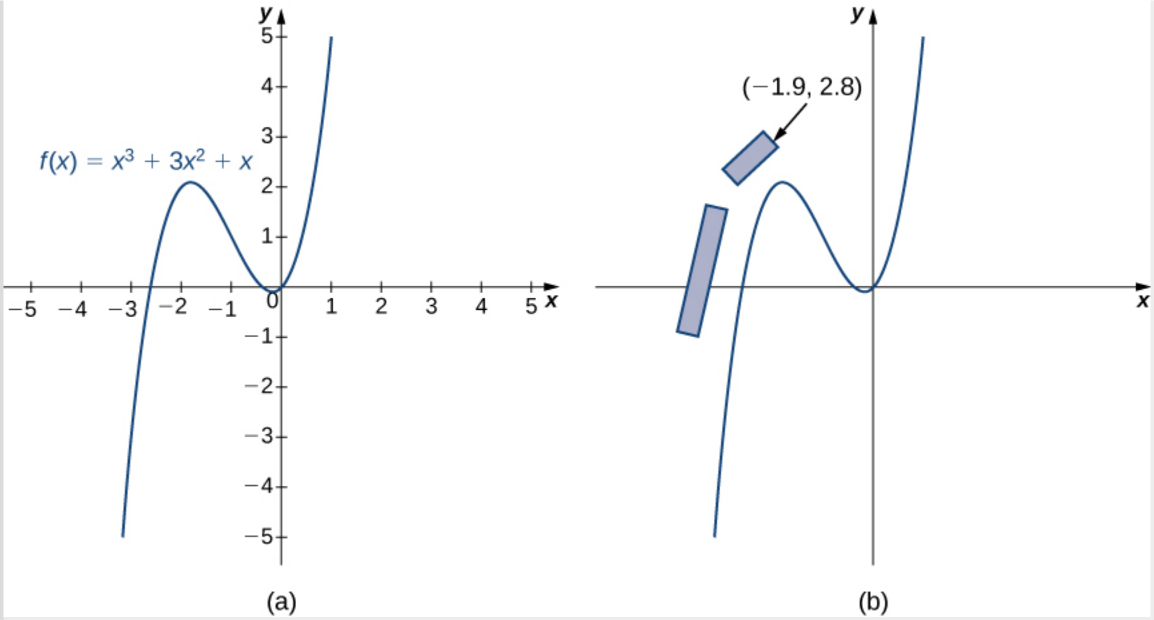
Formula One car races can be very exciting to watch and attract a lot of spectators. Formula One track designers have to ensure sufficient grandstand space is available around the track to accommodate these viewers. However, car racing can be dangerous, and safety considerations are paramount. The grandstands must be placed where spectators will not be in danger should a driver lose control of a car ([link]).

The grandstand next to a straightaway of the Circuit de Barcelona-Catalunya race track, located where the spectators are not in danger.

****

Safety is especially a concern on turns. If a driver does not slow down enough before entering the turn, the car may slide off the racetrack. Normally, this just results in a wider turn, which slows the driver down. But if the driver loses control completely, the car may fly off the track entirely, on a path tangent to the curve of the racetrack.

Suppose you are designing a new Formula One track. One section of the track can be modeled by the function $f(x) = x^3 + 3x^2 + x$ ([link]). The current plan calls for grandstands to be built along the first straightaway and around a portion of the first curve. The plans call for the front corner of the grandstand to be located at the point $(-1.9, 2.8)$. We want to determine whether this location puts the spectators in danger if a driver loses control of the car.



(a) One section of the racetrack can be modeled by the function $f(x) = x^3 + 3x^2 + x$. (b) The front corner of the grandstand is located at $(-1.9, 2.8)$.

1. Physicists have determined that drivers are most likely to lose control of their cars as they are coming into a turn, at the point where the slope of the tangent line is 1. Find the (x, y) coordinates of this point near the turn.
2. Find the equation of the tangent line to the curve at this point.
3. To determine whether the spectators are in danger in this scenario, find the *x*-coordinate of the point where the tangent line crosses the line $y = 2.8$. Is this point safely to the right of the grandstand? Or are the spectators in danger?
4. What if a driver loses control earlier than the physicists project? Suppose a driver loses control at the point $(-2.5, 0.625)$. What is the slope of the tangent line at this point?
5. If a driver loses control as described in part 4, are the spectators safe?
6. Should you proceed with the current design for the grandstand, or should the grandstands be moved?

Key Concepts

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant *c* multiplied by a function *f* is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function *f* and a function *g* is the same as the sum of the derivative of *f* and the derivative of *g*.
- The derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of *g*.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second

function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

For the following exercises, find $f'(x)$ for each function.

Exercise:

Problem: $f\left(x\right)=x^7+10$

Exercise:

Problem: $f\left(x\right)=5x^3-x+1$

Solution:

$f'\left(x\right)=15x^2-1$

Exercise:

Problem: $f\left(x\right)=4x^2-7x$

Exercise:

Problem: $f\left(x\right)=8x^4+9x^2-1$

Solution:

$f'\left(x\right)=32x^3+18x$

Exercise:

Problem: $f\left(x\right)=x^4+\frac{2}{x}$

Exercise:

Problem: $f\left(x\right)=3x\left(18x^4+\frac{13}{x+1}\right)$

Solution:

$f'\left(x\right)=270x^4+\frac{39}{\left(x+1\right)^2}$

Exercise:

Problem: $f\left(x\right)=\left(x+2\right)\left(2x^2-3\right)$

Exercise:

Problem: $f\left(x\right)=x^2\left(\frac{2}{x^2}+\frac{5}{x^3}\right)$

Solution:

$f'\left(x\right)=\frac{-5}{x^2}$

Exercise:

Problem: $f\left(x\right)=\frac{x^3+2x^2-4}{3}$

Exercise:

Problem: $f\left(x\right)=\frac{4x^3-2x+1}{x^2}$

Solution:

$f'\left(x\right)=\frac{4x^4+2x^2-2x}{x^4}$

Exercise:

Problem: $f\left(x\right)=\frac{x^2+4}{x^2-4}$

Exercise:

Problem: $f\left(x\right)=\frac{x+9}{x^2-7x+1}$

Solution:

$f'\left(x\right)=\frac{-x^2-18x+64}{\left(x^2-7x+1\right)^2}$

For the following exercises, find the equation of the tangent line $T\left(x\right)$ to the graph of the given function at the indicated point. Use a graphing calculator to graph the function and the tangent line.

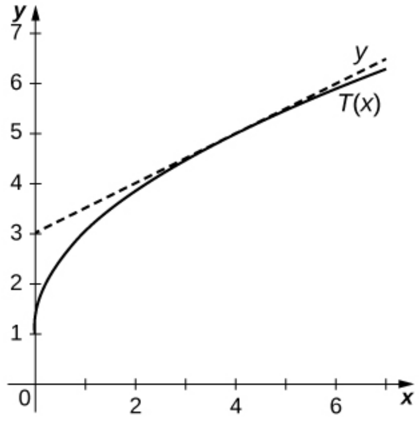
Exercise:

Problem: **[T]** $y=3x^2+4x+1$ at $\left(0,1\right)$

Exercise:

Problem: **[T]** $y=2\sqrt{x}+1$ at $\left(4,5\right)$

Solution:



$T(x)=\frac{1}{2}x+3$

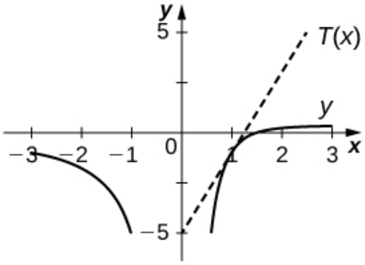
Exercise:

Problem: **[T]** $y = \frac{2x}{x-1}$ at $(-1, 1)$

Exercise:

Problem: **[T]** $y = \frac{2}{x} - \frac{3}{x^2}$ at $(1, -1)$

Solution:



$T\left(x\right) = 4x - 5$

For the following exercises, assume that $f(x)$ and $g\left(x\right)$ are both differentiable functions for all x. Find the derivative of each of the functions $h\left(x\right)$.

Exercise:

Problem: $h\left(x\right) = 4f\left(x\right) + \frac{g(x)}{7}$

Exercise:

Problem: $h\left(x\right) = x^3 f(x)$

Solution:

$h'\left(x\right) = 3x^2 f\left(x\right) + x^3 f'\left(x\right)$

Exercise:

Problem: $h\left(x\right) = \frac{f(x)g(x)}{2}$

Exercise:

Problem: $h\left(x\right) = \frac{3f(x)}{g(x)+2}$

Solution:

$h'\left(x\right) = \frac{3f'(x)(g(x)+2) - 3f(x)g'(x)}{\left(g(x)+2\right)^2}$

For the following exercises, assume that $f(x)$ and $g\left(x\right)$ are both differentiable functions with values as given in the following table. Use the following table to calculate the following derivatives.

x	1	2	3	4
$f(x)$	3	5	-2	0
$g(x)$	2	3	-4	6
$f'(x)$	-1	7	8	-3
$g'(x)$	4	1	2	9

Exercise:

Problem: Find $h'(1)$ if $h(x) = xf(x) + 4g(x)$.

Exercise:

Problem: Find $h'(2)$ if $h(x) = \frac{f(x)}{g(x)}$.

Solution:

$\frac{16}{9}$

Exercise:

Problem: Find $h'(3)$ if $h(x) = 2x + f(x)g(x)$.

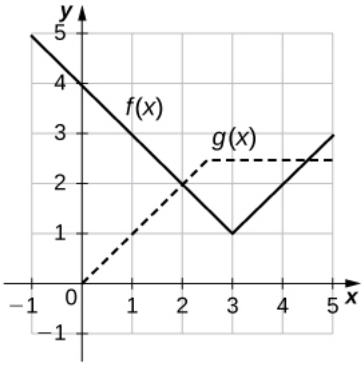
Exercise:

Problem: Find $h'(4)$ if $h(x) = \frac{1}{x} + \frac{g(x)}{f(x)}$.

Solution:

Undefined

For the following exercises, use the following figure to find the indicated derivatives, if they exist.



Exercise:

Problem: Let $h(x) = f(x) + g(x)$. Find

a. $h'(1)$,

b. $h'(3)$, and
c. $h'(4)$.

Exercise:

Problem: Let $h(x) = f(x)g(x)$. Find

a. $h'(1)$,
b. $h'(3)$, and
c. $h'(4)$.

Solution:

a. 2, b. does not exist, c. 2.5

Exercise:

Problem: Let $h(x) = \frac{f(x)}{g(x)}$. Find

a. $h'(1)$,
b. $h'(3)$, and
c. $h'(4)$.

For the following exercises,

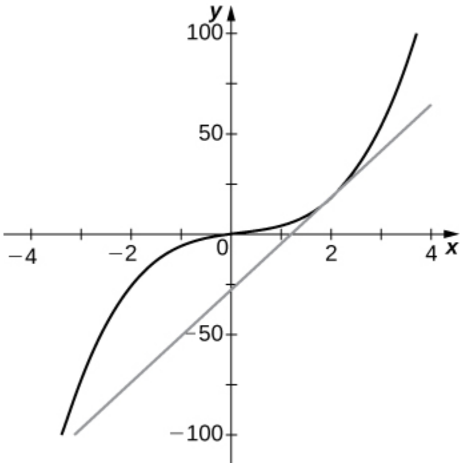
a. evaluate $f'(a)$, and
b. graph the function $f(x)$ and the tangent line at $x = a$.

Exercise:

Problem: **[T]** $f(x) = 2x^3 + 3x - x^2, a = 2$

Solution:

a. 23, b. $y = 23x - 28$



Exercise:

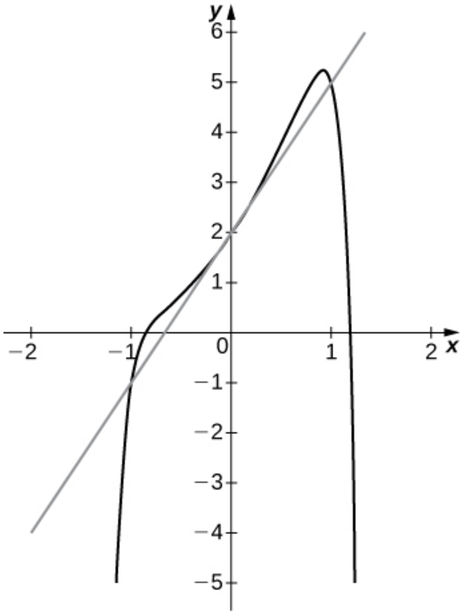
Problem: **[T]** $f\left(x\right)=\frac{1}{x}-x^{2},a=1$

Exercise:

Problem: **[T]** $f\left(x\right)=x^{2}-x^{12}+3x+2,a=0$

Solution:

a. 3, b. $y=3x+2$



Exercise:

Problem: **[T]** $f\left(x\right)=\frac{1}{x}-x^{2/3},a=-1$

Exercise:

Problem: Find the equation of the tangent line to the graph of $f\left(x\right)=2x^{3}+4x^{2}-5x-3$ at $x=-1$.

Solution:

$y=-7x-3$

Exercise:

Problem: Find the equation of the tangent line to the graph of $f\left(x\right)=x^{2}+\frac{4}{x}-10$ at $x=8$.

Exercise:

Problem: Find the equation of the tangent line to the graph of $f\left(x\right)=(3x-x^{2})(3-x-x^{2})$ at $x=1$.

Solution:

$y=-5x+7$

Exercise:

Problem:

Find the point on the graph of $f\left(x\right)=x^3$ such that the tangent line at that point has an x intercept of 6.

Exercise:

Problem:

Find the equation of the line passing through the point $P(3,3)$ and tangent to the graph of $f\left(x\right)=\frac{6}{x-1}$.

Solution:

$y=-\frac{3}{2}x+\frac{15}{2}$

Exercise:

Problem: Determine all points on the graph of $f\left(x\right)=x^3+x^2-x-1$ for which

a. the tangent line is horizontal
b. the tangent line has a slope of -1.

Exercise:

Problem: Find a quadratic polynomial such that $f\left(1\right)=5$, $f'\left(1\right)=3$ and $f''\left(1\right)=-6$.

Solution:

$y=-3x^2+9x-1$

Exercise:

Problem: A car driving along a freeway with traffic has traveled $s\left(t\right)=t^3-6t^2+9t$ meters in t seconds.

a. Determine the time in seconds when the velocity of the car is 0.
b. Determine the acceleration of the car when the velocity is 0.

Exercise:

Problem: **[T]** A herring swimming along a straight line has traveled $s\left(t\right)=\frac{t^2}{t^2+2}$ feet in t seconds.

Determine the velocity of the herring when it has traveled 3 seconds.

Solution:

$\frac{12}{121}$ or 0.0992 ft/s

Exercise:

Problem:

The population in millions of arctic flounder in the Atlantic Ocean is modeled by the function $P\left(t\right)=\frac{8t+3}{0.2t^2+1}$, where t is measured in years.

a. Determine the initial flounder population.
b. Determine $P'\left(10\right)$ and briefly interpret the result.

Exercise:

Problem:

[T] The concentration of antibiotic in the bloodstream t hours after being injected is given by the function $C\left(t\right)=\frac{2t^2+t}{t^3+50}$, where C is measured in milligrams per liter of blood.

a. Find the rate of change of $C\left(t\right)$.
b. Determine the rate of change for $t=8, 12, 24$, and 36.
c. Briefly describe what seems to be occurring as the number of hours increases.

Solution:

a. $\frac{-2t^4-2t^3+200t+50}{\left(t^3+50\right)^2}$ b. -0.02395 mg/L-hr, −0.01344 mg/L-hr, −0.003566 mg/L-hr, −0.001579 mg/L-hr c. The rate at which the concentration of drug in the bloodstream decreases is slowing to 0 as time increases.

Exercise:

Problem:

A book publisher has a cost function given by $C\left(x\right)=\frac{x^3+2x+3}{x^2}$, where *x* is the number of copies of a book in thousands and *C* is the cost, per book, measured in dollars. Evaluate $C'\left(2\right)$ and explain its meaning.

Exercise:

Problem:

[T] According to Newton's law of universal gravitation, the force F between two bodies of constant mass m_1 and m_2 is given by the formula $F=\frac{Gm_1m_2}{d^2}$, where G is the gravitational constant and d is the distance between the bodies.

a. Suppose that $G, m_1, \text{and } m_2$ are constants. Find the rate of change of force F with respect to distance d.
b. Find the rate of change of force F with gravitational constant $G=6.67\ \times\ 10^{-11}\ \text{Nm}^2/\text{kg}^2$, on two bodies 10 meters apart, each with a mass of 1000 kilograms.

Solution:

a. $F\prime(d)=\frac{-2Gm_1m_2}{d^3}$ b. $-1.33\ \times\ 10^{-7}$ N/m

Glossary

constant multiple rule
: the derivative of a constant *c* multiplied by a function *f* is the same as the constant multiplied by the derivative: $\frac{d}{dx}\left(cf\left(x\right)\right)=cf'\left(x\right)$

constant rule
: the derivative of a constant function is zero: $\frac{d}{dx}\left(c\right)=0$, where *c* is a constant

difference rule
: the derivative of the difference of a function *f* and a function *g* is the same as the difference of the derivative of *f* and the derivative of *g*: $\frac{d}{dx}\left(f\left(x\right)-g\left(x\right)\right)=f'\left(x\right)-g'\left(x\right)$

power rule

the derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1: If n is an integer, then $\frac{d}{dx}x^n = nx^{n-1}$

product rule

the derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function: $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + g'(x)f(x)$

quotient rule

the derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$

sum rule

the derivative of the sum of a function *f* and a function g is the same as the sum of the derivative of *f* and the derivative of g: $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

Derivatives as Rates of Change

- Determine a new value of a quantity from the old value and the amount of change.
- Calculate the average rate of change and explain how it differs from the instantaneous rate of change.
- Apply rates of change to displacement, velocity, and acceleration of an object moving along a straight line.
- Predict the future population from the present value and the population growth rate.
- Use derivatives to calculate marginal cost and revenue in a business situation.

In this section we look at some applications of the derivative by focusing on the interpretation of the derivative as the rate of change of a function. These applications include **acceleration** and velocity in physics, **population growth rates** in biology, and marginal functions in economics.

Amount of Change Formula

One application for derivatives is to estimate an unknown value of a function at a point by using a known value of a function at some given point together with its rate of change at the given point. If $f(x)$ is a function defined on an interval $[a, a+h]$, then the **amount of change** of $f(x)$ over the interval is the change in the y values of the function over that interval and is given by
Equation:

$$f\left(a+h\right)-f(a).$$

The **average rate of change** of the function f over that same interval is the ratio of the amount of change over that interval to the corresponding change in the x values. It is given by
Equation:

$$\frac{f\left(a+h\right)-f(a)}{h}.$$

As we already know, the instantaneous rate of change of $f(x)$ at a is its derivative

Equation:

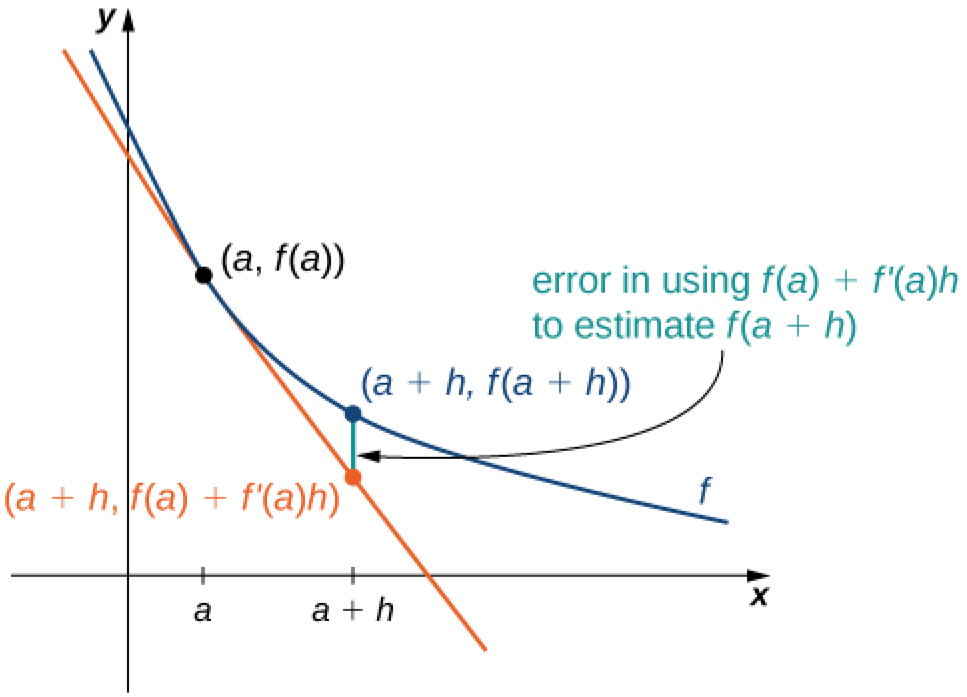
$$f'\left(a\right)=\lim_{h\to 0}\frac{f\left(a+h\right)-f(a)}{h}.$$

For small enough values of h, $f'\left(a\right)\approx\frac{f(a+h)-f(a)}{h}$. We can then solve for $f\left(a+h\right)$ to get the amount of change formula:

Equation:

$$f(a+h)\approx f(a)+f'(a)h.$$

We can use this formula if we know only $f(a)$ and $f'(a)$ and wish to estimate the value of $f\left(a+h\right)$. For example, we may use the current population of a city and the rate at which it is growing to estimate its population in the near future. As we can see in [link], we are approximating $f(a+h)$ by the y coordinate at $a+h$ on the line tangent to $f(x)$ at $x=a$. Observe that the accuracy of this estimate depends on the value of h as well as the value of $f'\left(a\right)$.


y
(a, f(a))
error in using f(a) + f′(a)h
to estimate f(a + h)
(a + h, f(a + h))
(a + h, f(a) + f′(a)h)
f
a
a + h
x

The new value of a changed quantity equals the original value plus the rate of change times the interval of change: $f\left(a+h\right)\approx f\left(a\right)+f'\left(a\right)$h.

Note:
Here is an interesting demonstration of rate of change.

Example:
Exercise:

Problem:
Estimating the Value of a Function

If $f\left(3\right)=2$ and $f'\left(3\right)=5$, estimate $f(3.2)$.

Solution:

Begin by finding h. We have $h = 3.2 - 3 = 0.2$. Thus,

Equation:

$$f(3.2) = f(3 + 0.2) \approx f(3) + (0.2)f'(3) = 2 + 0.2(5) = 3.$$

Note:
Exercise:

Problem: Given $f(10) = -5$ and $f'(10) = 6$, estimate $f(10.1)$.

Solution:

-4.4

Hint

Use the same process as in the preceding example.

Motion along a Line

Another use for the derivative is to analyze motion along a line. We have described velocity as the rate of change of position. If we take the derivative of the velocity, we can find the acceleration, or the rate of change of velocity. It is also important to introduce the idea of **speed**, which is the magnitude of velocity. Thus, we can state the following mathematical definitions.

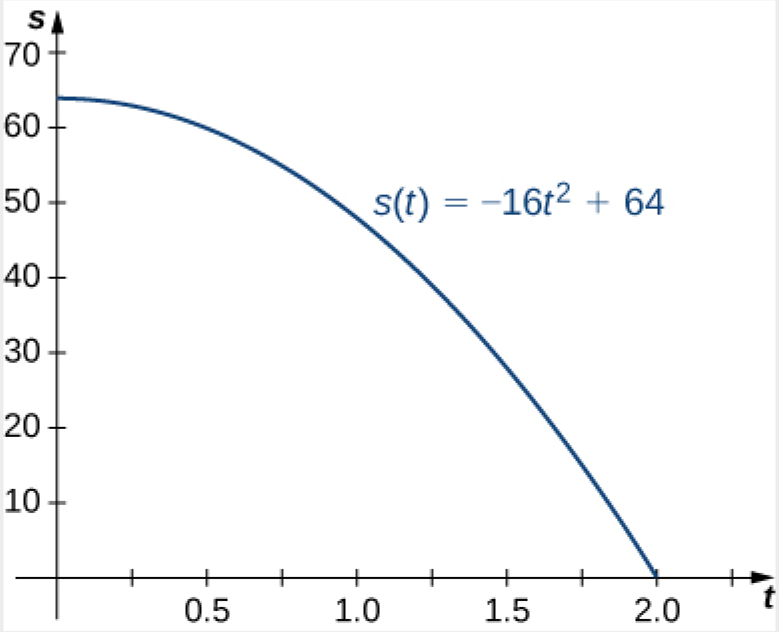
Note:
Definition
Let $s(t)$ be a function giving the position of an object at time t.
The velocity of the object at time t is given by $v(t) = s'(t)$.

The speed of the object at time t is given by $|v(t)|$.
The acceleration of the object at t is given by $a(t) = v'(t) = s''(t)$.

Example:
Exercise:

Problem:
Comparing Instantaneous Velocity and Average Velocity

A ball is dropped from a height of 64 feet. Its height above ground (in feet) t seconds later is given by $s(t) = -16t^2 + 64$.



a. What is the instantaneous velocity of the ball when it hits the ground?
b. What is the average velocity during its fall?

Solution:

The first thing to do is determine how long it takes the ball to reach the ground. To do this, set $s(t) = 0$. Solving $-16t^2 + 64 = 0$, we get $t = 2$, so it take 2 seconds for the ball to reach the ground.

a. The instantaneous velocity of the ball as it strikes the ground is $v(2)$. Since $v(t) = s'(t) = -32t$, we obtain $v(t) = -64 \text{ ft/s}$.
b. The average velocity of the ball during its fall is
Equation:

$$v_{ave} = \frac{s(2) - s(0)}{2 - 0} = \frac{0 - 64}{2} = -32 \text{ ft/s}.$$

Example:
Exercise:

Problem:
Interpreting the Relationship between $v(t)$ and $a(t)$

A particle moves along a coordinate axis in the positive direction to the right. Its position at time t is given by $s(t) = t^3 - 4t + 2$. Find $v(1)$ and $a(1)$ and use these values to answer the following questions.

a. Is the particle moving from left to right or from right to left at time $t = 1$?
b. Is the particle speeding up or slowing down at time $t = 1$?

Solution:

Begin by finding $v(t)$ and $a(t)$.

$v(t) = s'(t) = 3t^2 - 4$ and $a(t) = v'(t) = s\prime\prime(t) = 6t$.

Evaluating these functions at $t = 1$, we obtain $v(1) = -1$ and $a(1) = 6$.

a. Because $v(1) < 0$, the particle is moving from right to left.
b. Because $v(1) < 0$ and $a(1) > 0$, velocity and acceleration are acting in opposite directions. In other words, the particle is being accelerated in the direction opposite the direction in which it is traveling, causing $|v(t)|$ to decrease. The particle is slowing down.

Example:
Exercise:

Problem:
Position and Velocity

The position of a particle moving along a coordinate axis is given by $s(t) = t^3 - 9t^2 + 24t + 4, t \geq 0$.

a. Find $v(t)$.
b. At what time(s) is the particle at rest?
c. On what time intervals is the particle moving from left to right? From right to left?
d. Use the information obtained to sketch the path of the particle along a coordinate axis.

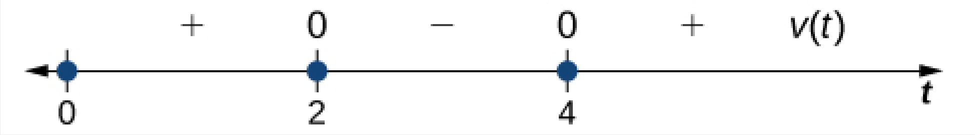
Solution:

a. The velocity is the derivative of the position function:
Equation:

$$v(t) = s'(t) = 3t^2 - 18t + 24.$$

b. The particle is at rest when $v(t) = 0$, so set $3t^2 - 18t + 24 = 0$. Factoring the left-hand side of the equation produces $3(t-2)(t-4) = 0$. Solving, we find that the particle is at rest at $t = 2$ and $t = 4$.

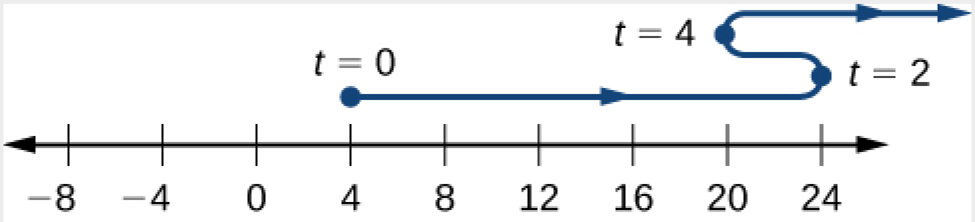
c. The particle is moving from left to right when $v(t) > 0$ and from right to left when $v(t) < 0$. [link] gives the analysis of the sign of $v(t)$ for $t \geq 0$, but it does not represent the axis along which the particle is moving.



The sign of v(t) determines the direction of the particle.

Since $3t^2 - 18t + 24 > 0$ on $[0,2) \cup (2,+\infty)$, the particle is moving from left to right on these intervals.
Since $3t^2 - 18t + 24 < 0$ on $(2,4)$, the particle is moving from right to left on this interval.

d. Before we can sketch the graph of the particle, we need to know its position at the time it starts moving $(t = 0)$ and at the times that it changes direction $(t = 2, 4)$. We have $s(0) = 4$, $s(2) = 24$, and $s(4) = 20$. This means that the particle begins on the coordinate axis at 4 and changes direction at 0 and 20 on the coordinate axis. The path of the particle is shown on a coordinate axis in [link].



The path of the particle can be determined by analyzing v(t).

Note:
Exercise:

Problem:

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = t^2 - 5t + 1$. Is the particle moving from right to left or from left to right at time $t = 3$?

Solution:

left to right

Hint

Find $v(3)$ and look at the sign.

Population Change

In addition to analyzing velocity, speed, acceleration, and position, we can use derivatives to analyze various types of populations, including those as diverse as bacteria colonies and cities. We can use a current population, together with a growth rate, to estimate the size of a population in the future. The population growth rate is the rate of change of a population and consequently can be represented by the derivative of the size of the population.

Note:
Definition
If $P(t)$ is the number of entities present in a population, then the population growth rate of $P(t)$ is defined to be $P'(t)$.

Example:

Exercise:

Problem:
Estimating a Population

The population of a city is tripling every 5 years. If its current population is 10,000, what will be its approximate population 2 years from now?

Solution:

Let $P\left(t\right)$ be the population (in thousands) t years from now. Thus, we know that $P\left(0\right)=10$ and based on the information, we anticipate $P\left(5\right)=30$. Now estimate $P'\left(0\right)$, the current growth rate, using
Equation:

$$P'\left(0\right)\approx\frac{P\left(5\right)-P\left(0\right)}{5-0}=\frac{30-10}{5}=4.$$

By applying [link] to $P\left(t\right)$, we can estimate the population 2 years from now by writing
Equation:

$$P\left(2\right)\approx P\left(0\right)+(2)P'\left(0\right)\approx 10+2\left(4\right)=18;$$

thus, in 2 years the population will be 18,000.

Note:
Exercise:

Problem:

The current population of a mosquito colony is known to be 3,000; that is, $P\left(0\right)=3{,}000$. If $P'\left(0\right)=100$, estimate the size of the population in 3 days, where t is measured in days.

Solution:

3,300

Hint

Use $P(3) \approx P(0) + 3P'(0)$.

Changes in Cost and Revenue

In addition to analyzing motion along a line and population growth, derivatives are useful in analyzing changes in cost, revenue, and profit. The concept of a marginal function is common in the fields of business and economics and implies the use of derivatives. The marginal cost is the derivative of the cost function. The marginal revenue is the derivative of the revenue function. The marginal profit is the derivative of the profit function, which is based on the cost function and the revenue function.

Note:
Definition
If $C(x)$ is the cost of producing *x* items, then the **marginal cost** $MC(x)$ is $MC(x) = C'(x)$.
If $R(x)$ is the revenue obtained from selling x items, then the marginal revenue $MR(x)$ is $MR(x) = R'(x)$.
If $P(x) = R(x) - C(x)$ is the profit obtained from selling *x* items, then the **marginal profit** $MP(x)$ is defined to be
$MP(x) = P'(x) = MR(x) - MC(x) = R'(x) - C'(x)$.

We can roughly approximate
Equation:

$$MC(x) = C'(x) = \lim_{h \to 0} \frac{C(x+h) - C(x)}{h}$$

by choosing an appropriate value for h. Since x represents objects, a reasonable and small value for h is 1. Thus, by substituting $h = 1$, we get the approximation $MC(x) = C'(x) \approx C(x+1) - C(x)$. Consequently, $C'(x)$ for a given value of x can be thought of as the change in cost associated with producing one additional item. In a similar way, $MR(x) = R'(x)$ approximates the revenue obtained by selling one additional item, and $MP(x) = P'(x)$ approximates the profit obtained by producing and selling one additional item.

Example:
Exercise:

Problem:
Applying Marginal Revenue

Assume that the number of barbeque dinners that can be sold, x, can be related to the price charged, p, by the equation $p(x) = 9 - 0.03x, 0 \leq x \leq 300$.

In this case, the revenue in dollars obtained by selling x barbeque dinners is given by

Equation:

$$R(x) = xp(x) = x(9 - 0.03x) = -0.03x^2 + 9x \text{ for } 0 \leq x \leq 300.$$

Use the marginal revenue function to estimate the revenue obtained from selling the 101st barbeque dinner. Compare this to the actual revenue obtained from the sale of this dinner.

Solution:

First, find the marginal revenue function:
$MR(x) = R'(x) = -0.06x + 9.$

Next, use $R'(100)$ to approximate $R(101) - R(100)$, the revenue obtained from the sale of the 101st dinner. Since $R'(100) = 3$, the revenue obtained from the sale of the 101st dinner is approximately \$3.

The actual revenue obtained from the sale of the 101st dinner is

Equation:

$$R(101) - R(100) = 602.97 - 600 = 2.97, \text{ or } \$2.97.$$

The marginal revenue is a fairly good estimate in this case and has the advantage of being easy to compute.

Note:

Exercise:

Problem:

Suppose that the profit obtained from the sale of x fish-fry dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Use the marginal profit function to estimate the profit from the sale of the 101st fish-fry dinner.

Solution:

\$2

Hint

Use $P'(100)$ to approximate $P(101) - P(100)$.

Key Concepts

- Using $f\left(a+h\right)\approx f\left(a\right)+f'\left(a\right)h$, it is possible to estimate $f\left(a+h\right)$ given $f'\left(a\right)$ and $f\left(a\right)$.
- The rate of change of position is velocity, and the rate of change of velocity is acceleration. Speed is the absolute value, or magnitude, of velocity.
- The population growth rate and the present population can be used to predict the size of a future population.
- Marginal cost, marginal revenue, and marginal profit functions can be used to predict, respectively, the cost of producing one more item, the revenue obtained by selling one more item, and the profit obtained by producing and selling one more item.

For the following exercises, the given functions represent the position of a particle traveling along a horizontal line.

a. Find the velocity and acceleration functions.
b. Determine the time intervals when the object is slowing down or speeding up.

Exercise:

Problem: $s\left(t\right)=2t^3-3t^2-12t+8$

Exercise:

Problem: $s\left(t\right)=2t^3-15t^2+36t-10$

Solution:

a. $v\left(t\right)=6t^2-30t+36, a\left(t\right)=12t-30$; b. speeds up $\left(2,2.5\right)\cup\left(3,\infty\right)$, slows down $\left(0,2\right)\cup\left(2.5,3\right)$

Exercise:

Problem: $s\left(t\right)=\frac{t}{1+t^2}$

Exercise:

Problem:

A rocket is fired vertically upward from the ground. The distance s in feet that the rocket travels from the ground after t seconds is given by $s\left(t\right)=-16t^2+560t$.

a. Find the velocity of the rocket 3 seconds after being fired.
b. Find the acceleration of the rocket 3 seconds after being fired.

Solution:

a. $464\ \text{ft/s}^2$ b. $-32\ \text{ft/s}^2$

Exercise:

Problem:

A ball is thrown downward with a speed of 8 ft/s from the top of a 64-foot-tall building. After *t* seconds, its height above the ground is given by $s\left(t\right)=-16t^2-8t+64$.

a. Determine how long it takes for the ball to hit the ground.
b. Determine the velocity of the ball when it hits the ground.

Exercise:

Problem:

The position function $s\left(t\right)=t^2-3t-4$ represents the position of the back of a car backing out of a driveway and then driving in a straight line, where s is in feet and t is in seconds. In this case, $s\left(t\right)=0$ represents the time at which the back of the car is at the garage door, so $s\left(0\right)=-4$ is the starting position of the car, 4 feet inside the garage.

a. Determine the velocity of the car when $s\left(t\right)=0$.
b. Determine the velocity of the car when $s\left(t\right)=14$.

Solution:

a. 5 ft/s b. 9 ft/s

Exercise:

Problem:

The position of a hummingbird flying along a straight line in t seconds is given by $s(t) = 3t^3 - 7t$ meters.

a. Determine the velocity of the bird at $t = 1$ sec.
b. Determine the acceleration of the bird at $t = 1$ sec.
c. Determine the acceleration of the bird when the velocity equals 0.

Exercise:

Problem:

A potato is launched vertically upward with an initial velocity of 100 ft/s from a potato gun at the top of an 85-foot-tall building. The distance in feet that the potato travels from the ground after t seconds is given by $s(t) = -16t^2 + 100t + 85$.

a. Find the velocity of the potato after 0.5 s and 5.75 s.
b. Find the speed of the potato at 0.5 s and 5.75 s.
c. Determine when the potato reaches its maximum height.
d. Find the acceleration of the potato at 0.5 s and 1.5 s.
e. Determine how long the potato is in the air.
f. Determine the velocity of the potato upon hitting the ground.

Solution:

a. 84 ft/s, −84 ft/s b. 84 ft/s c. $\frac{25}{8}$ s d. $-32\ \text{ft/s}^2$ in both cases e. $\frac{1}{8}\left(25 + \sqrt{965}\right)$ s f. $-4\sqrt{965}\ \text{ft/s}$

Exercise:

Problem:

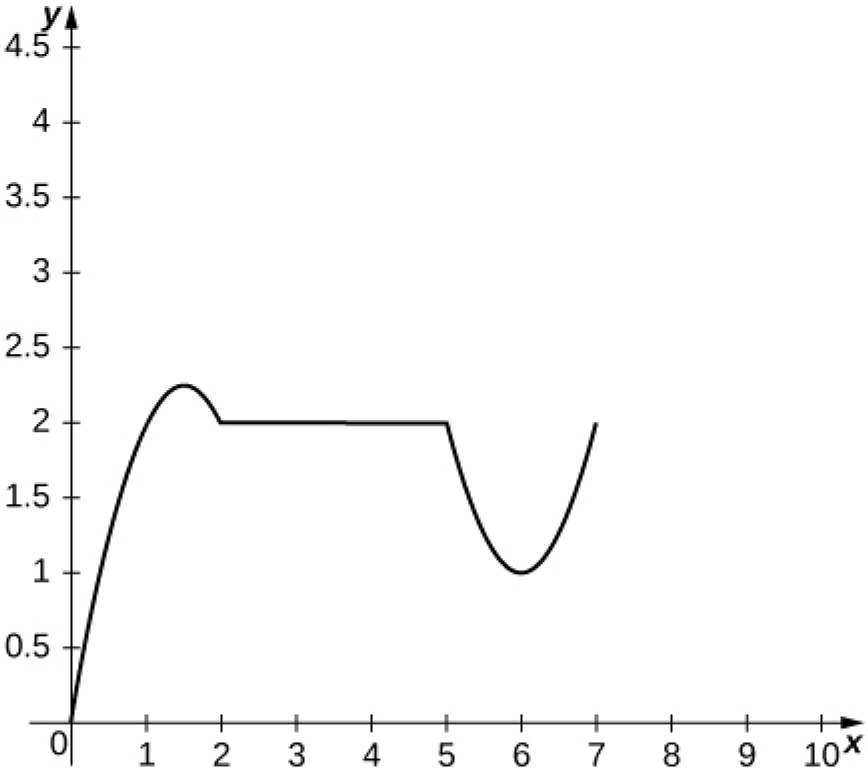
The position function $s\left(t\right)={t}^{3}-8t$ gives the position in miles of a freight train where east is the positive direction and t is measured in hours.

a. Determine the direction the train is traveling when $s\left(t\right)=0.$
b. Determine the direction the train is traveling when $a\left(t\right)=0.$
c. Determine the time intervals when the train is slowing down or speeding up.

Exercise:

Problem:

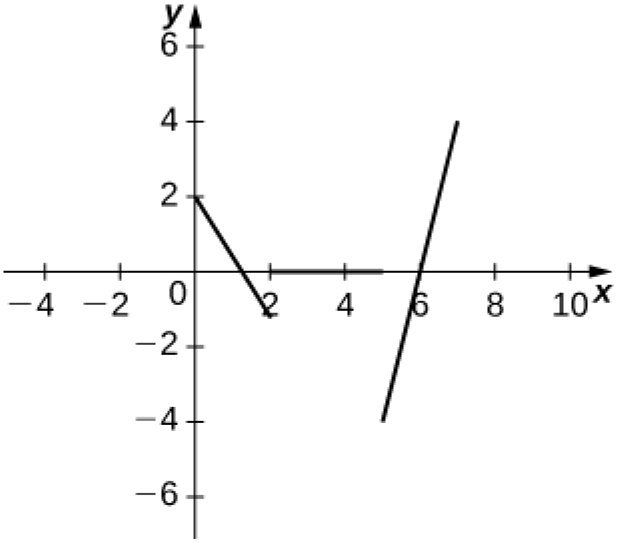
The following graph shows the position $y=s\left(t\right)$ of an object moving along a straight line.



a. Use the graph of the position function to determine the time intervals when the velocity is positive, negative, or zero.
b. Sketch the graph of the velocity function.
c. Use the graph of the velocity function to determine the time intervals when the acceleration is positive, negative, or zero.
d. Determine the time intervals when the object is speeding up or slowing down.

Solution:

a. Velocity is positive on $(0, 1.5) \cup (6, 7)$, negative on $(1.5, 2) \cup (5, 6)$, and zero on $(2, 5)$. b.



c. Acceleration is positive on $(5, 7)$, negative on $(0, 2)$, and zero on $(2, 5)$. d. The object is speeding up on $(6, 7) \cup (1.5, 2)$ and slowing down on $(0, 1.5) \cup (5, 6)$.

Exercise:

Problem:

The cost function, in dollars, of a company that manufactures food processors is given by $C\left(x\right)=200+\frac{7}{x}+\frac{x^2}{7}$, where x is the number of food processors manufactured.

a. Find the marginal cost function.
b. Use the marginal cost function to estimate the cost of manufacturing the thirteenth food processor.
c. Find the actual cost of manufacturing the thirteenth food processor.

Exercise:

Problem:

The price p (in dollars) and the demand x for a certain digital clock radio is given by the price–demand function $p=10-0.001x$.

a. Find the revenue function $R\left(x\right)$.
b. Find the marginal revenue function.
c. Find the marginal revenue at $x=2000$ and 5000.

Solution:

a. $R\left(x\right)=10x-0.001x^2$ b. $R'\left(x\right)=10-0.002x$ c. $6 per item, $0 per item

Exercise:

Problem:

[T] A profit is earned when revenue exceeds cost. Suppose the profit function for a skateboard manufacturer is given by $P\left(x\right)=30x-0.3x^2-250$, where x is the number of skateboards sold.

a. Find the exact profit from the sale of the thirtieth skateboard.

b. Find the marginal profit function and use it to estimate the profit from the sale of the thirtieth skateboard.

Exercise:

Problem:

[T] In general, the profit function is the difference between the revenue and cost functions: $P(x) = R(x) - C(x)$.

Suppose the price-demand and cost functions for the production of cordless drills is given respectively by $p = 143 - 0.03x$ and $C(x) = 75{,}000 + 65x$, where x is the number of cordless drills that are sold at a price of p dollars per drill and $C(x)$ is the cost of producing x cordless drills.

a. Find the marginal cost function.
b. Find the revenue and marginal revenue functions.
c. Find $R'(1000)$ and $R'(4000)$. Interpret the results.
d. Find the profit and marginal profit functions.
e. Find $P'(1000)$ and $P'(4000)$. Interpret the results.

Solution:

a. $C'(x) = 65$ b. $R(x) = 143x - 0.03x^2, R'(x) = 143 - 0.06x$ c. $83, -97$. At a production level of 1000 cordless drills, revenue is increasing at a rate of \$83 per drill; at a production level of 4000 cordless drills, revenue is decreasing at a rate of \$97 per drill. d. $P(x) = -0.03x^2 + 78x - 75000, P'(x) = -0.06x + 78$ e. $18, -162$. At a production level of 1000 cordless drills, profit is increasing at a rate of \$18 per drill; at a production level of 4000 cordless drills, profit is decreasing at a rate of \$162 per drill.

Exercise:

Problem:

A small town in Ohio commissioned an actuarial firm to conduct a study that modeled the rate of change of the town's population. The study found that the town's population (measured in thousands of people) can be modeled by the function $P\left(t\right)=-\frac{1}{3}t^{3}+64t+3000$, where t is measured in years.

a. Find the rate of change function $P'\left(t\right)$ of the population function.
b. Find $P'\left(1\right),P'\left(2\right),P'\left(3\right)$, and $P'\left(4\right)$. Interpret what the results mean for the town.
c. Find $P''\left(1\right),P''\left(2\right),P''\left(3\right)$, and $P''\left(4\right)$. Interpret what the results mean for the town's population.

Exercise:

Problem:

[T] A culture of bacteria grows in number according to the function $N\left(t\right)=3000\left(1+\frac{4t}{t^{2}+100}\right)$, where t is measured in hours.

a. Find the rate of change of the number of bacteria.
b. Find $N'\left(0\right),N'\left(10\right),N'\left(20\right)$, and $N'\left(30\right)$.
c. Interpret the results in (b).
d. Find $N''\left(0\right),N''\left(10\right),N''\left(20\right)$, and $N''\left(30\right)$. Interpret what the answers imply about the bacteria population growth.

Solution:

a. $N'\left(t\right)=3000\left(\frac{-4t^{2}+400}{\left(t^{2}+100\right)^{2}}\right)$ b. $120,0,-14.4,-9.6$ c. The bacteria population increases from time 0 to 10 hours; afterwards, the bacteria population decreases. d. $0,-6,0.384,0.432$. The rate at which the bacteria is increasing is decreasing during the first 10 hours. Afterwards, the bacteria population is decreasing at a decreasing rate.

Exercise:

Problem:

The centripetal force of an object of mass m is given by $F\left(r\right)=\frac{mv^2}{r}$, where v is the speed of rotation and r is the distance from the center of rotation.

a. Find the rate of change of centripetal force with respect to the distance from the center of rotation.
b. Find the rate of change of centripetal force of an object with mass 1000 kilograms, velocity of 13.89 m/s, and a distance from the center of rotation of 200 meters.

The following questions concern the population (in millions) of London by decade in the 19th century, which is listed in the following table.

Years since 1800	**Population (millions)**
1	0.8795
11	1.040
21	1.264
31	1.516
41	1.661
51	2.000

Years since 1800	**Population (millions)**
61	2.634
71	3.272
81	3.911
91	4.422

Population of LondonSource: http://en.wikipedia.org/wiki/Demographics_of_London.

Exercise:

Problem: [T]

a. Using a calculator or a computer program, find the best-fit linear function to measure the population.
b. Find the derivative of the equation in a. and explain its physical meaning.
c. Find the second derivative of the equation and explain its physical meaning.

Solution:

a. $P\left(t\right)=0.03983+0.4280$ b. $P'\left(t\right)=0.03983.$ The population is increasing. c. $P\prime\prime\left(t\right)=0.$ The rate at which the population is increasing is constant.

Exercise:

Problem: [T]

a. Using a calculator or a computer program, find the best-fit quadratic curve through the data.

b. Find the derivative of the equation and explain its physical meaning.
c. Find the second derivative of the equation and explain its physical meaning.

For the following exercises, consider an astronaut on a large planet in another galaxy. To learn more about the composition of this planet, the astronaut drops an electronic sensor into a deep trench. The sensor transmits its vertical position every second in relation to the astronaut’s position. The summary of the falling sensor data is displayed in the following table.

Time after dropping (s)	**Position (m)**
0	0
1	−1
2	−2
3	−5
4	−7
5	−14

Exercise:

Problem: **[T]**

a. Using a calculator or computer program, find the best-fit quadratic curve to the data.
b. Find the derivative of the position function and explain its physical meaning.
c. Find the second derivative of the position function and explain its physical meaning.

Solution:

a. $p(t) = -0.6071x^2 + 0.4357x - 0.3571$ b. $p'(t) = -1.214x + 0.4357$. This is the velocity of the sensor. c. $p\prime\prime(t) = -1.214$. This is the acceleration of the sensor; it is a constant acceleration downward.

Exercise:

Problem: **[T]**

a. Using a calculator or computer program, find the best-fit cubic curve to the data.
b. Find the derivative of the position function and explain its physical meaning.
c. Find the second derivative of the position function and explain its physical meaning.
d. Using the result from c. explain why a cubic function is not a good choice for this problem.

The following problems deal with the Holling type I, II, and III equations. These equations describe the ecological event of growth of a predator population given the amount of prey available for consumption.

Exercise:

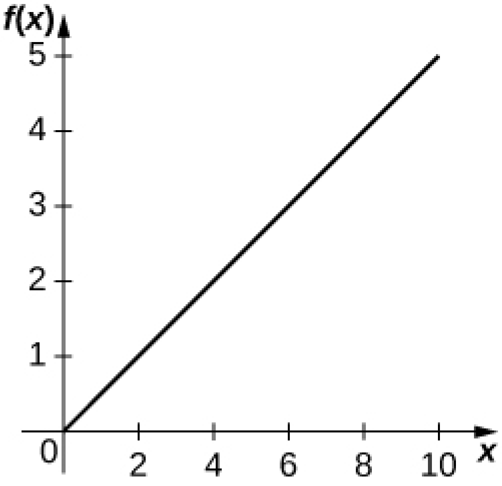
Problem:

[T] The Holling type I equation is described by $f(x) = ax$, where x is the amount of prey available and $a > 0$ is the rate at which the predator meets the prey for consumption.

a. Graph the Holling type I equation, given $a = 0.5$.
b. Determine the first derivative of the Holling type I equation and explain physically what the derivative implies.
c. Determine the second derivative of the Holling type I equation and explain physically what the derivative implies.
d. Using the interpretations from b. and c. explain why the Holling type I equation may not be realistic.

Solution:

a.



b. $f'(x) = a$. The more increase in prey, the more growth for predators. c. $f''(x) = 0$. As the amount of prey increases, the rate at which the predator population growth increases is constant. d. This equation assumes that if there is more prey, the predator is able to increase consumption linearly. This assumption is unphysical because we would expect there to be some saturation point at which there is too much prey for the predator to consume adequately.

Exercise:

Problem:

[T] The Holling type II equation is described by $f\left(x\right)=\frac{ax}{n+x}$, where x is the amount of prey available and $a>0$ is the maximum consumption rate of the predator.

a. Graph the Holling type II equation given $a=0.5$ and $n=5.$ What are the differences between the Holling type I and II equations?
b. Take the first derivative of the Holling type II equation and interpret the physical meaning of the derivative.
c. Show that $f\left(n\right)=\frac{1}{2}a$ and interpret the meaning of the parameter $n.$
d. Find and interpret the meaning of the second derivative. What makes the Holling type II function more realistic than the Holling type I function?

Exercise:

Problem:

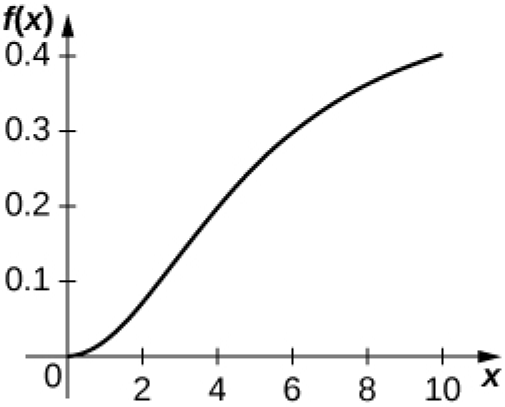
[T] The Holling type III equation is described by $f\left(x\right)=\frac{ax^2}{n^2+x^2}$, where x is the amount of prey available and $a>0$ is the maximum consumption rate of the predator.

a. Graph the Holling type III equation given $a=0.5$ and $n=5.$ What are the differences between the Holling type II and III equations?
b. Take the first derivative of the Holling type III equation and interpret the physical meaning of the derivative.
c. Find and interpret the meaning of the second derivative (it may help to graph the second derivative).

d. What additional ecological phenomena does the Holling type III function describe compared with the Holling type II function?

Solution:

a.



b. $f'(x) = \frac{2axn^2}{(n^2+x^2)^2}$. When the amount of prey increases, the predator growth increases. c. $f\prime\prime(x) = \frac{2an^2(n^2-3x^2)}{(n^2+x^2)^3}$. When the amount of prey is extremely small, the rate at which predator growth is increasing is increasing, but when the amount of prey reaches above a certain threshold, the rate at which predator growth is increasing begins to decrease. d. At lower levels of prey, the prey is more easily able to avoid detection by the predator, so fewer prey individuals are consumed, resulting in less predator growth.

Exercise:

Problem:

[T] The populations of the snowshoe hare (in thousands) and the lynx (in hundreds) collected over 7 years from 1937 to 1943 are shown in the following table. The snowshoe hare is the primary prey of the lynx.

Population of snowshoe hare (thousands)	**Population of lynx (hundreds)**
20	10
55	15
65	55
95	60

Snowshoe Hare and Lynx PopulationsSource: http://www.biotopics.co.uk/newgcse/predatorprey.html.

a. Graph the data points and determine which Holling-type function fits the data best.
b. Using the meanings of the parameters a and n, determine values for those parameters by examining a graph of the data. Recall that n measures what prey value results in the half-maximum of the predator value.
c. Plot the resulting Holling-type I, II, and III functions on top of the data. Was the result from part a. correct?

Glossary

acceleration
: is the rate of change of the velocity, that is, the derivative of velocity

amount of change
: the amount of a function $f\left(x\right)$ over an interval $\left[x,x+h\right]$ is $f\left(x+h\right)-f\left(x\right)$

average rate of change
: is a function $f\left(x\right)$ over an interval $\left[x,x+h\right]$ is $\frac{f(x+h)-f(a)}{b-a}$

marginal cost

is the derivative of the cost function, or the approximate cost of producing one more item

marginal revenue

is the derivative of the revenue function, or the approximate revenue obtained by selling one more item

marginal profit

is the derivative of the profit function, or the approximate profit obtained by producing and selling one more item

population growth rate

is the derivative of the population with respect to time

speed

is the absolute value of velocity, that is, $|v(t)|$ is the speed of an object at time t whose velocity is given by $v(t)$

Derivatives of Trigonometric Functions

- Find the derivatives of the sine and cosine function.
- Find the derivatives of the standard trigonometric functions.
- Calculate the higher-order derivatives of the sine and cosine.

One of the most important types of motion in physics is simple harmonic motion, which is associated with such systems as an object with mass oscillating on a spring. Simple harmonic motion can be described by using either sine or cosine functions. In this section we expand our knowledge of derivative formulas to include derivatives of these and other trigonometric functions. We begin with the derivatives of the sine and cosine functions and then use them to obtain formulas for the derivatives of the remaining four trigonometric functions. Being able to calculate the derivatives of the sine and cosine functions will enable us to find the velocity and acceleration of simple harmonic motion.

Derivatives of the Sine and Cosine Functions

We begin our exploration of the derivative for the sine function by using the formula to make a reasonable guess at its derivative. Recall that for a function $f(x)$,

Equation:

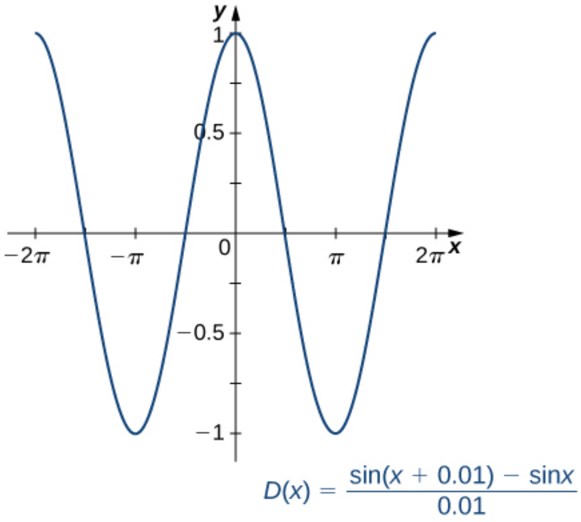
$$f'\left(x\right)=\lim_{h\to 0}\frac{f\left(x+h\right)-f(x)}{h}.$$

Consequently, for values of h very close to 0, $f'\left(x\right)\approx\frac{f(x+h)-f(x)}{h}$. We see that by using $h=0.01$,

Equation:

$$\frac{d}{dx}(\sin x)\approx\frac{\sin\left(x+0.01\right)-\sin x}{0.01}$$

By setting $D\left(x\right)=\frac{\sin\left(x+0.01\right)-\sin x}{0.01}$ and using a graphing utility, we can get a graph of an approximation to the derivative of $\sin x$ ([link]).



The graph of the function $D\left(x\right)$ looks a lot like a cosine curve.

Upon inspection, the graph of $D(x)$ appears to be very close to the graph of the cosine function. Indeed, we will show that

Equation:

$$\frac{d}{dx}(\sin x) = \cos x.$$

If we were to follow the same steps to approximate the derivative of the cosine function, we would find that

Equation:

$$\frac{d}{dx}(\cos x) = -\sin \mathrm{x}.$$

Note:

The Derivatives of sin *x* and cos *x*

The derivative of the sine function is the cosine and the derivative of the cosine function is the negative sine.

Equation:

$$\frac{d}{dx}(\sin x) = \cos x$$

Equation:

$$\frac{d}{dx}(\cos x) = -\sin x$$

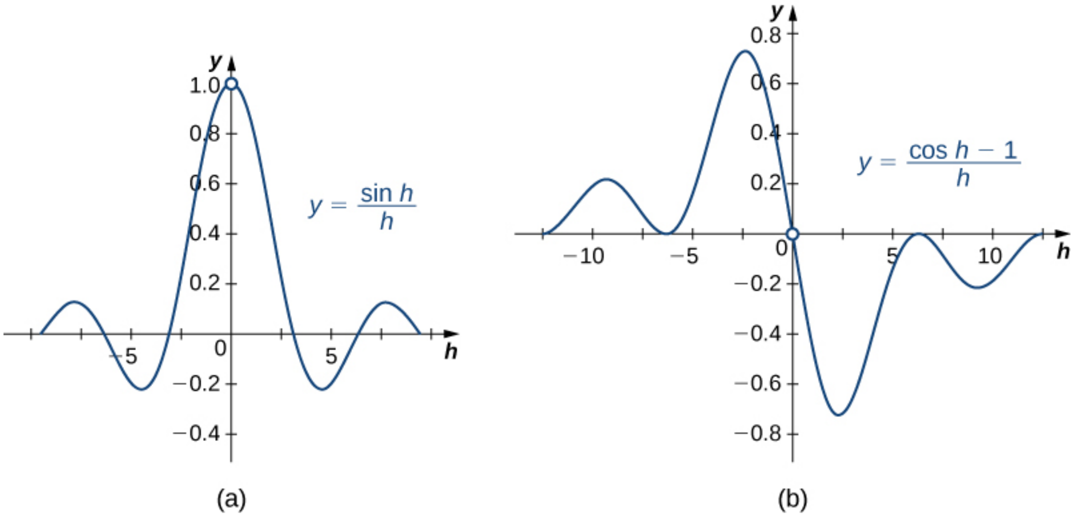
Proof

Because the proofs for $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ use similar techniques, we provide only the proof for $\frac{d}{dx}(\sin x) = \cos x$. Before beginning, recall two important trigonometric limits we learned in Introduction to Limits:

Equation:

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \text{ and } \lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

The graphs of $y = \frac{(\sin h)}{h}$ and $y = \frac{(\cos h - 1)}{h}$ are shown in [link].



These graphs show two important limits needed to establish the derivative formulas for the sine and cosine functions.

We also recall the following trigonometric identity for the sine of the sum of two angles:
Equation:

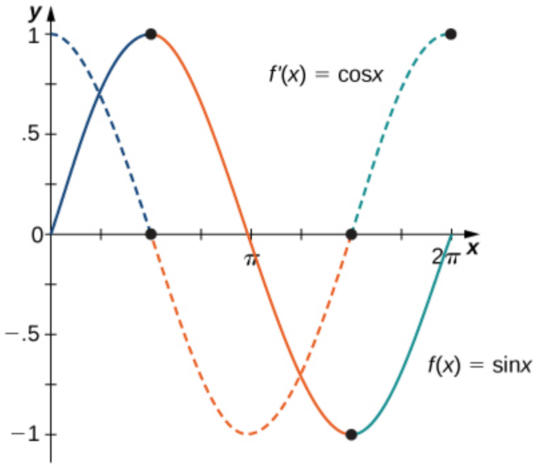
$$\sin\left(x+h\right) = \sin x \cos h + \cos x \sin h.$$

Now that we have gathered all the necessary equations and identities, we proceed with the proof.
Equation:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin\left(x+h\right)-\sin x}{h} && \text{Apply the definition of the derivative.}\\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} && \text{Use trig identity for the sine of the sum of two angles}\\
&= \lim_{h\to 0}\left(\frac{\sin x\cos h-\sin x}{h}+\frac{\cos x\sin h}{h}\right) && \text{Regroup.}\\
&= \lim_{h\to 0}\left(\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right) && \text{Factor out } \sin x \text{ and } \cos x.\\
&= \sin x\cdot 0+\cos x\cdot 1 && \text{Apply trig limit formulas.}\\
&= \cos x && \text{Simplify.}
\end{aligned}$$

□

[link] shows the relationship between the graph of $f\left(x\right) = \sin x$ and its derivative $f'\left(x\right) = \cos x$. Notice that at the points where $f\left(x\right) = \sin x$ has a horizontal tangent, its derivative $f'\left(x\right) = \cos x$ takes on the value zero. We also see that where $f\left(x\right) = \sin x$ is increasing, $f'\left(x\right) = \cos x > 0$ and where $f\left(x\right) = \sin x$ is decreasing, $f'\left(x\right) = \cos x < 0$.



Where $f(x)$ has a maximum or a minimum, $f\prime(x) = 0$ that is, $f\prime(x) = 0$ where $f(x)$ has a horizontal tangent. These points are noted with dots on the graphs.

Example:
Exercise:

Problem:
Differentiating a Function Containing sin *x*

Find the derivative of $f\left(x\right) = 5x^3\sin x$.

Solution:

Using the product rule, we have
Equation:

$$\begin{aligned} f\prime(x) &= \frac{d}{dx}\left(5x^3\right) \cdot \sin x + \frac{d}{dx}(\sin x) \cdot 5x^3 \\ &= 15x^2 \cdot \sin x + \cos x \cdot 5x^3. \end{aligned}$$

After simplifying, we obtain
Equation:

$$f'\left(x\right) = 15x^2\sin x + 5x^3\cos x.$$

Note:
Exercise:

Problem: Find the derivative of $f\left(x\right) = \sin x\cos x$.

Solution:

$f'(x) = \cos^2 x - \sin^2 x$

Hint

Don't forget to use the product rule.

Example:
Exercise:

Problem:
Finding the Derivative of a Function Containing cos *x*

Find the derivative of $g(x) = \frac{\cos x}{4x^2}$.

Solution:

By applying the quotient rule, we have
Equation:

$$g'(x) = \frac{(-\sin x)4x^2 - 8x(\cos x)}{(4x^2)^2}.$$

Simplifying, we obtain
Equation:

$$\begin{aligned} g'(x) &= \frac{-4x^2 \sin x - 8x \cos x}{16x^4} \\ &= \frac{-x \sin x - 2\cos x}{4x^3}. \end{aligned}$$

Note:
Exercise:

Problem: Find the derivative of $f(x) = \frac{x}{\cos x}$.

Solution:

$\frac{\cos x + x \sin x}{\cos^2 x}$

Hint

Use the quotient rule.

Example:
Exercise:

Problem:
An Application to Velocity

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2\sin t - t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Solution:

To determine when the particle is at rest, set $s'(t) = v(t) = 0$. Begin by finding $s'(t)$. We obtain

Equation:

$$s'(t) = 2\cos t - 1,$$

so we must solve

Equation:

$$2\cos t - 1 = 0 \text{ for } 0 \le t \le 2\pi.$$

The solutions to this equation are $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$. Thus the particle is at rest at times $t = \frac{\pi}{3}$ and $t = \frac{5\pi}{3}$.

Note:

Exercise:

Problem:

A particle moves along a coordinate axis. Its position at time t is given by $s(t) = \sqrt{3}t + 2\cos t$ for $0 \le t \le 2\pi$. At what times is the particle at rest?

Solution:

$t = \frac{\pi}{3}, t = \frac{2\pi}{3}$

Hint

Use the previous example as a guide.

Derivatives of Other Trigonometric Functions

Since the remaining four trigonometric functions may be expressed as quotients involving sine, cosine, or both, we can use the quotient rule to find formulas for their derivatives.

Example:

Exercise:

Problem:

The Derivative of the Tangent Function

Find the derivative of $f(x) = \tan x$.

Solution:

Start by expressing $\tan x$ as the quotient of $\sin x$ and $\cos x$:

Equation:

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Now apply the quotient rule to obtain
Equation:

$$f'(x)=\frac{\cos x\cos x-(-\sin x)\sin x}{(\cos x)^2}.$$

Simplifying, we obtain
Equation:

$$f'(x)=\frac{\cos^2 x+\ \sin^2 x}{\cos^2 x}.$$

Recognizing that $\cos^2 x+\sin^2 x=1$, by the Pythagorean theorem, we now have
Equation:

$$f'(x)=\frac{1}{\cos^2 x}.$$

Finally, use the identity $\sec x=\frac{1}{\cos x}$ to obtain
Equation:

$$f'(x)=\sec^2 x.$$

Note:
Exercise:

Problem: Find the derivative of $f(x)=\cot x$.

Solution:

$f'(x)=-\csc^2 x$

Hint

Rewrite $\cot x$ as $\frac{\cos x}{\sin x}$ and use the quotient rule.

The derivatives of the remaining trigonometric functions may be obtained by using similar techniques. We provide these formulas in the following theorem.

Note:
Derivatives of $\tan x, \cot x, \sec x$, and $\csc x$
The derivatives of the remaining trigonometric functions are as follows:
Equation:

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

Equation:

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Equation:

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Equation:

$$\frac{d}{dx}(\csc x) = -\csc x \cot \mathrm{x}.$$

Example:
Exercise:

Problem:
Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $f(x) = \cot x$ at $x = \frac{\pi}{4}$.

Solution:

To find the equation of the tangent line, we need a point and a slope at that point. To find the point, compute
Equation:

$$f\left(\frac{\pi}{4}\right) = \cot \frac{\pi}{4} = 1.$$

Thus the tangent line passes through the point $\left(\frac{\pi}{4}, 1\right)$. Next, find the slope by finding the derivative of $f(x) = \cot x$ and evaluating it at $\frac{\pi}{4}$:
Equation:

$$f'(x) = -\csc^2 x \text{ and } f'\left(\frac{\pi}{4}\right) = -\csc^2\left(\frac{\pi}{4}\right) = -2.$$

Using the point-slope equation of the line, we obtain
Equation:

$$y - 1 = -2\left(x - \frac{\pi}{4}\right)$$

or equivalently,
Equation:

$$y = -2x + 1 + \frac{\pi}{2}.$$

Example:
Exercise:

Problem:
Finding the Derivative of Trigonometric Functions

Find the derivative of $f(x) = \csc x + x \tan x$.

Solution:

To find this derivative, we must use both the sum rule and the product rule. Using the sum rule, we find
Equation:

$$f'(x) = \frac{d}{dx}(\csc x) + \frac{d}{dx}(x \tan x).$$

In the first term, $\frac{d}{dx}(\csc x) = -\csc x \cot x$, and by applying the product rule to the second term we obtain
Equation:

$$\frac{d}{dx}(x \tan x) = (1)(\tan x) + (\sec^2 x)(x).$$

Therefore, we have
Equation:

$$f'(x) = -\csc x \cot x + \tan x + x \sec^2 x.$$

Note:
Exercise:

Problem: Find the derivative of $f(x) = 2 \tan x - 3 \cot x$.

Solution:

$f'(x) = 2 \sec^2 x + 3 \csc^2 x$

Hint

Use the rule for differentiating a constant multiple and the rule for differentiating a difference of two functions.

Note:
Exercise:

Problem: Find the slope of the line tangent to the graph of $f(x) = \tan x$ at $x = \frac{\pi}{6}$.

Solution:

$\frac{4}{3}$

Hint

Evaluate the derivative at $x = \frac{\pi}{6}$.

Higher-Order Derivatives

The higher-order derivatives of $\sin x$ and $\cos x$ follow a repeating pattern. By following the pattern, we can find any higher-order derivative of $\sin x$ and $\cos x$.

Example:
Exercise:

Problem:
Finding Higher-Order Derivatives of $y = \sin x$

Find the first four derivatives of $y = \sin x$.

Solution:

Each step in the chain is straightforward:
Equation:

$$\begin{aligned} y &= \sin x \\ \frac{dy}{dx} &= \cos x \\ \frac{d^2y}{dx^2} &= -\sin x \\ \frac{d^3y}{dx^3} &= -\cos x \\ \frac{d^4y}{dx^4} &= \sin x. \end{aligned}$$

Analysis

Once we recognize the pattern of derivatives, we can find any higher-order derivative by determining the step in the pattern to which it corresponds. For example, every fourth derivative of sin *x* equals sin *x*, so
Equation:

$$\frac{d^4}{dx^4}(\sin x) = \frac{d^8}{dx^8}(\sin x) = \frac{d^{12}}{dx^{12}}(\sin x) = \ldots = \frac{d^{4n}}{dx^{4n}}(\sin x) = \sin x$$
$$\frac{d^5}{dx^5}(\sin x) = \frac{d^9}{dx^9}(\sin x) = \frac{d^{13}}{dx^{13}}(\sin x) = \ldots = \frac{d^{4n+1}}{dx^{4n+1}}(\sin x) = \cos x.$$

Note:
Exercise:

Problem: For $y = \cos x$, find $\frac{d^4y}{dx^4}$.

Solution:

$\cos x$

Hint

See the previous example.

Example:
Exercise:

Problem:
Using the Pattern for Higher-Order Derivatives of $y = \sin x$

Find $\frac{d^{74}}{dx^{74}}(\sin x)$.

Solution:

We can see right away that for the 74th derivative of $\sin x$, $74 = 4(18) + 2$, so
Equation:

$$\frac{d^{74}}{dx^{74}}(\sin x) = \frac{d^{72+2}}{dx^{72+2}}(\sin x) = \frac{d^2}{dx^2}(\sin x) = -\sin x.$$

Note:
Exercise:

Problem: For $y = \sin x$, find $\frac{d^{59}}{dx^{59}}(\sin x)$.

Solution:

$-\cos x$

Hint

$\frac{d^{59}}{dx^{59}}(\sin x) = \frac{d^{4\cdot 14+3}}{dx^{4\cdot 14+3}}(\sin x)$

Example:
Exercise:

Problem:
An Application to Acceleration

A particle moves along a coordinate axis in such a way that its position at time t is given by $s(t) = 2 - \sin t$. Find $v(\pi/4)$ and $a(\pi/4)$. Compare these values and decide whether the particle is speeding up or slowing down.

Solution:

First find $v(t) = s'(t)$:
Equation:

$$v(t) = s'(t) = -\cos t.$$

Thus,
Equation:

$$v\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$

Next, find $a\left(t\right)=v'(t)$. Thus, $a\left(t\right)=v'\left(t\right)=\sin t$ and we have
Equation:

$$a\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}.$$

Since $v\left(\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}<0$ and $a\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}>0$, we see that velocity and acceleration are acting in opposite directions; that is, the object is being accelerated in the direction opposite to the direction in which it is travelling. Consequently, the particle is slowing down.

Note:
Exercise:

Problem:

A block attached to a spring is moving vertically. Its position at time t is given by $s\left(t\right)=2\sin t$. Find $v\left(\frac{5\pi}{6}\right)$ and $a\left(\frac{5\pi}{6}\right)$. Compare these values and decide whether the block is speeding up or slowing down.

Solution:

$v\left(\frac{5\pi}{6}\right)=-\sqrt{3}<0$ and $a\left(\frac{5\pi}{6}\right)=-1<0$. The block is speeding up.

Hint

Use [link] as a guide.

Key Concepts

- We can find the derivatives of sin *x* and cos *x* by using the definition of derivative and the limit formulas found earlier. The results are
  Equation:

$$\frac{d}{dx}\sin x=\cos x\ \frac{d}{dx}\cos x=-\sin x.$$

- With these two formulas, we can determine the derivatives of all six basic trigonometric functions.

Key Equations

- **Derivative of sine function**
  $\frac{d}{dx}(\sin x)=\cos x$
- **Derivative of cosine function**
  $\frac{d}{dx}(\cos x)=-\sin x$
- **Derivative of tangent function**
  $\frac{d}{dx}(\tan x)=\sec^2 x$
- **Derivative of cotangent function**
  $\frac{d}{dx}(\cot x)=-\csc^2 x$
- **Derivative of secant function**
  $\frac{d}{dx}(\sec x)=\sec x\tan x$

- **Derivative of cosecant function**
  $\frac{d}{dx}(\csc x) = -\csc x \cot x$

For the following exercises, find $\frac{dy}{dx}$ for the given functions.

Exercise:

Problem: $y = x^2 - \sec x + 1$

Solution:

$\frac{dy}{dx} = 2x - \sec x \tan x$

Exercise:

Problem: $y = 3\csc x + \frac{5}{x}$

Exercise:

Problem: $y = x^2 \cot x$

Solution:

$\frac{dy}{dx} = 2x \cot x - x^2 \csc^2 x$

Exercise:

Problem: $y = x - x^3 \sin x$

Exercise:

Problem: $y = \frac{\sec x}{x}$

Solution:

$\frac{dy}{dx} = \frac{x \sec x \tan x - \sec x}{x^2}$

Exercise:

Problem: $y = \sin x \tan x$

Exercise:

Problem: $y = (x + \cos x)(1 - \sin x)$

Solution:

$\frac{dy}{dx} = (1 - \sin x)(1 - \sin x) - \cos x (x + \cos x)$

Exercise:

Problem: $y = \frac{\tan x}{1 - \sec x}$

Exercise:

Problem: $y = \frac{1 - \cot x}{1 + \cot x}$

Solution:

$\frac{dy}{dx} = \frac{2\csc^2 x}{(1+\cot x)^2}$

Exercise:

Problem: $y = \cos x\,(1 + \csc x)$

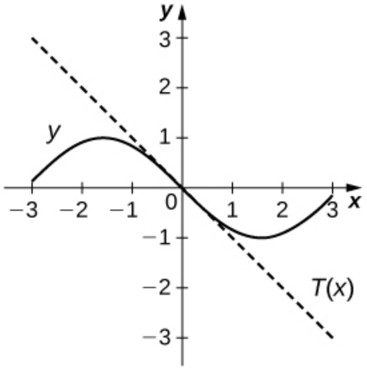
For the following exercises, find the equation of the tangent line to each of the given functions at the indicated values of x. Then use a calculator to graph both the function and the tangent line to ensure the equation for the tangent line is correct.

Exercise:

Problem: **[T]** $f(x) = -\sin x, x = 0$

Solution:

$y = -x$



Exercise:

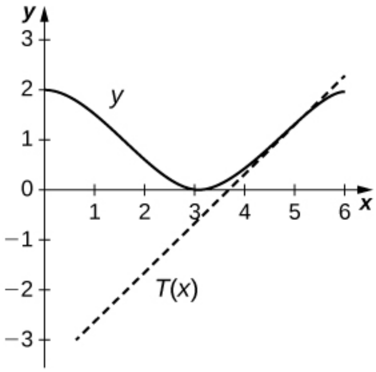
Problem: **[T]** $f(x) = \csc x, x = \frac{\pi}{2}$

Exercise:

Problem: **[T]** $f(x) = 1 + \cos x, x = \frac{3\pi}{2}$

Solution:

$y = x + \frac{2-3\pi}{2}$



Exercise:

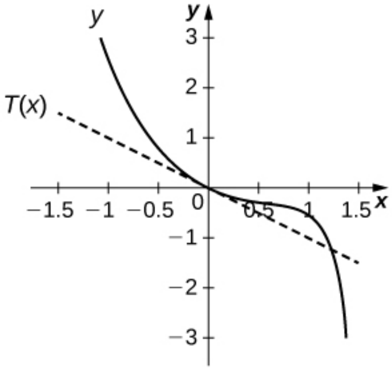
Problem: **[T]** $f\left(x\right)=\sec x,x=\frac{\pi }{4}$

Exercise:

Problem: **[T]** $f\left(x\right)={x}^{2}-\tan x,\;x=0$

Solution:

$y=-x$



Exercise:

Problem: **[T]** $f\left(x\right)=5\cot x,\;x=\frac{\pi }{4}$

For the following exercises, find $\frac{{d}^{2}y}{d{x}^{2}}$ for the given functions.

Exercise:

Problem: $y=x\sin x-\cos x$

Solution:

$3\cos x-x\sin x$

Exercise:

Problem: $y = \sin x \cos x$

Exercise:

Problem: $y = x - \frac{1}{2} \sin x$

Solution:

$\frac{1}{2} \sin x$

Exercise:

Problem: $y = \frac{1}{x} + \tan x$

Exercise:

Problem: $y = 2 \csc x$

Solution:

$2\csc x \left(\csc^2 x + \cot^2 x\right)$

Exercise:

Problem: $y = \sec^2 x$

Exercise:

Problem: Find all x values on the graph of $f(x) = -3 \sin x \cos x$ where the tangent line is horizontal.

Solution:

$\frac{(2n+1)\pi}{4}$, where n is an integer

Exercise:

Problem:

Find all x values on the graph of $f(x) = x - 2 \cos x$ for $0 < x < 2\pi$ where the tangent line has slope 2.

Exercise:

Problem:

Let $f(x) = \cot x$. Determine the points on the graph of f for $0 < x < 2\pi$ where the tangent line(s) is (are) parallel to the line $y = -2x$.

Solution:

$\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right)$

Exercise:

Problem:

[T] A mass on a spring bounces up and down in simple harmonic motion, modeled by the function $s(t) = -6 \cos t$ where s is measured in inches and t is measured in seconds. Find the rate at which the spring is oscillating at $t = 5$ s.

Exercise:

Problem:

Let the position of a swinging pendulum in simple harmonic motion be given by $s\left(t\right)=a\cos t+b\sin t$ where a and b are constants, t measures time in seconds, and s measures position in centimeters. If the position is 0 cm and the velocity is 3 cm/s when $t=0$, find the values of a and b.

Solution:

$a=0,b=3$

Exercise:

Problem:

After a diver jumps off a diving board, the edge of the board oscillates with position given by $s(t)=-5\cos t$ cm at t seconds after the jump.

a. Sketch one period of the position function for $t\ge 0$.
b. Find the velocity function.
c. Sketch one period of the velocity function for $t\ge 0$.
d. Determine the times when the velocity is 0 over one period.
e. Find the acceleration function.
f. Sketch one period of the acceleration function for $t\ge 0$.

Exercise:

Problem:

The number of hamburgers sold at a fast-food restaurant in Pasadena, California, is given by $y=10+5\sin x$ where y is the number of hamburgers sold and x represents the number of hours after the restaurant opened at 11 a.m. until 11 p.m., when the store closes. Find $y\prime$ and determine the intervals where the number of burgers being sold is increasing.

Solution:

$y'=5\cos\left(x\right)$, increasing on $\left(0,\frac{\pi}{2}\right)$, $\left(\frac{3\pi}{2},\frac{5\pi}{2}\right)$, and $\left(\frac{7\pi}{2},12\right)$

Exercise:

Problem:

[T] The amount of rainfall per month in Phoenix, Arizona, can be approximated by $y\left(t\right)=0.5+0.3\cos t$, where t is months since January. Find y' and use a calculator to determine the intervals where the amount of rain falling is decreasing.

For the following exercises, use the quotient rule to derive the given equations.

Exercise:

Problem: $\frac{d}{dx}\left(\cot x\right)=-\csc^2 x$

Exercise:

Problem: $\frac{d}{dx}\left(\sec x\right)=\sec x\tan x$

Exercise:

Problem: $\frac{d}{dx}\left(\csc x\right)=-\csc x\cot x$

Exercise:

Problem: Use the definition of derivative and the identity

$\cos\,(x+h)=\cos x\cos h-\sin x\sin h$ to prove that $\frac{d(\cos x)}{dx}=-\sin x$.

For the following exercises, find the requested higher-order derivative for the given functions.

Exercise:

Problem: $\frac{d^3y}{dx^3}$ of $y=3\cos x$

Solution:

$3\sin x$

Exercise:

Problem: $\frac{d^2y}{dx^2}$ of $y=3\sin x+x^2\cos x$

Exercise:

Problem: $\frac{d^4y}{dx^4}$ of $y=5\cos x$

Solution:

$5\cos x$

Exercise:

Problem: $\frac{d^2y}{dx^2}$ of $y=\sec x+\cot x$

Exercise:

Problem: $\frac{d^3y}{dx^3}$ of $y=x^{10}-\sec x$

Solution:

$720x^7-5\tan\,(x)\sec^3\,(x)-\tan^3\,(x)\sec\,(x)$

The Chain Rule

- State the chain rule for the composition of two functions.
- Apply the chain rule together with the power rule.
- Apply the chain rule and the product/quotient rules correctly in combination when both are necessary.
- Recognize the chain rule for a composition of three or more functions.
- Describe the proof of the chain rule.

We have seen the techniques for differentiating basic functions $\left(x^n, \sin x, \cos x, \text{etc.}\right)$ as well as sums, differences, products, quotients, and constant multiples of these functions. However, these techniques do not allow us to differentiate compositions of functions, such as $h\left(x\right)=\sin\left(x^3\right)$ or $k\left(x\right)=\sqrt{3x^2+1}$. In this section, we study the rule for finding the derivative of the composition of two or more functions.

Deriving the Chain Rule

When we have a function that is a composition of two or more functions, we could use all of the techniques we have already learned to differentiate it. However, using all of those techniques to break down a function into simpler parts that we are able to differentiate can get cumbersome. Instead, we use the **chain rule**, which states that the derivative of a composite function is the derivative of the outer function evaluated at the inner function times the derivative of the inner function.

To put this rule into context, let's take a look at an example: $h\left(x\right)=\sin\ \left(x^3\right)$. We can think of the derivative of this function with respect to x as the rate of change of $\sin\left(x^3\right)$ relative to the change in x. Consequently, we want to know how $\sin\left(x^3\right)$ changes as x changes. We can think of this event as a chain reaction: As x changes, x^3 changes, which leads to a change in $\sin\ \left(x^3\right)$. This chain reaction gives us hints as to what is involved in computing the derivative of $\sin\left(x^3\right)$. First of all, a change in x forcing a change in x^3 suggests that somehow the derivative of x^3 is involved. In addition, the change in x^3 forcing a change in $\sin\left(x^3\right)$ suggests that the derivative of $\sin(u)$ with respect to u, where $u=x^3$, is also part of the final derivative.

We can take a more formal look at the derivative of $h\left(x\right)=\sin\ \left(x^3\right)$ by setting up the limit that would give us the derivative at a specific value a in the domain of $h\left(x\right)=\sin\ \left(x^3\right)$.

Equation:

$$h'\left(a\right)=\lim_{x\to a}\frac{\sin\ \left(x^3\right)-\sin\left(a^3\right)}{x-a}.$$

This expression does not seem particularly helpful; however, we can modify it by multiplying and dividing by the expression x^3-a^3 to obtain

Equation:

$$h'\left(a\right)=\lim_{x\to a}\frac{\sin\ \left(x^3\right)-\sin\left(a^3\right)}{x^3-a^3}\cdot\frac{x^3-a^3}{x-a}.$$

From the definition of the derivative, we can see that the second factor is the derivative of x^3 at $x=a$. That is,

Equation:

$$\lim_{x\to a}\frac{x^3-a^3}{x-a}=\frac{d}{dx}\left(x^3\right)_{x=a}=3a^2.$$

However, it might be a little more challenging to recognize that the first term is also a derivative. We can see this by letting $u=x^3$ and observing that as $x\to a, u\to a^3$:

Equation:

$$\begin{aligned}\lim_{x\to a}\frac{\sin\left(x^3\right)-\sin\left(a^3\right)}{x^3-a^3} &= \lim_{u\to a^3}\frac{\sin u-\sin\left(a^3\right)}{u-a^3}\\ &= \frac{d}{du}(\sin u)_{u=a^3}\\ &= \cos\left(a^3\ \right).\end{aligned}$$

Thus, $h'\left(a\right)=\cos\left(a^3\right)\cdot 3a^2$.

In other words, if $h\left(x\right)=\sin\ \left(x^3\right)$, then $h'\left(x\right)=\cos\ \left(x^3\right)\cdot 3x^2$. Thus, if we think of $h\left(x\right)=\sin\left(x^3\right)$ as the composition $\left(f\circ g\right)\left(x\right)=f\left(g\left(x\right)\right)$ where $f\left(x\right)=\sin x$ and $g\left(x\right)=x^3$, then the derivative of $h\left(x\right)=\sin\left(x^3\right)$ is the product of the derivative of $g\left(x\right)=x^3$ and the derivative of the function $f\left(x\right)=\sin x$ evaluated at the function $g\left(x\right)=x^3$. At this point, we anticipate that for $h\left(x\right)=\sin\ \left(g\left(x\right)\right)$, it is quite likely that $h'(x)=\cos(g(x))g'(x)$. As we determined above, this is the case for $h\left(x\right)=\sin\ \left(x^3\right)$.

Now that we have derived a special case of the chain rule, we state the general case and then apply it in a general form to other composite functions. An informal proof is provided at the end of the section.

Note:
Rule: The Chain Rule
Let f and g be functions. For all x in the domain of g for which g is differentiable at x and f is differentiable at $g\left(x\right)$, the derivative of the composite function
Equation:

$$h\left(x\right)=(f\circ g)(x)=f\left(g\left(x\right)\right)$$

is given by
Equation:

$$h'\left(x\right)=f'\left(g\left(x\right)\right)g'\left(x\right).$$

Alternatively, if y is a function of u, and u is a function of x, then
Equation:

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

Note:
Watch an animation of the chain rule.

Note:
Problem-Solving Strategy: Applying the Chain Rule

1. To differentiate $h\left(x\right)=f\left(g\left(x\right)\right)$, begin by identifying $f(x)$ and $g(x)$.
2. Find $f\prime(x)$ and evaluate it at $g\left(x\right)$ to obtain $f'\left(g\left(x\right)\right)$.
3. Find $g'(x)$.
4. Write $h'\left(x\right)=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$.

Note: When applying the chain rule to the composition of two or more functions, keep in mind that we work our way from the outside function in. It is also useful to remember that the derivative of the composition of two functions can be thought of as having two parts; the derivative of the composition of three functions has three parts; and so on. Also, remember that we never evaluate a derivative at a derivative.

The Chain and Power Rules Combined

We can now apply the chain rule to composite functions, but note that we often need to use it with other rules. For example, to find derivatives of functions of the form $h\left(x\right)=\left(g\left(x\right)\right)^{n}$, we need to use the chain rule combined with the power rule. To do so, we can think of $h\left(x\right)=\left(g\left(x\right)\right)^{n}$ as $f\left(g\left(x\right)\right)$ where $f\left(x\right)=x^{n}$. Then $f'\left(x\right)=nx^{n-1}$. Thus, $f'\left(g\left(x\right)\right)=n\left(g\left(x\right)\right)^{n-1}$. This leads us to the derivative of a power function using the chain rule,
Equation:

$$h'\left(x\right)=n\left(g\left(x\right)\right)^{n-1}g'\left(x\right)$$

Note:
Rule: Power Rule for Composition of Functions
For all values of *x* for which the derivative is defined, if
Equation:

$$h\left(x\right)=\left(g(x)\right)^{n}.$$

Then
Equation:

$$h'\left(x\right)=n\left(g\left(x\right)\right)^{n-1}g'(x\ ).$$

Example:
Exercise:

Problem:
Using the Chain and Power Rules

Find the derivative of $h\left(x\right)=\frac{1}{\left(3x^{2}+1\right)^{2}}$.

Solution:

First, rewrite $h\left(x\right)=\frac{1}{\left(3x^{2}+1\right)^{2}}=\left(3x^{2}+1\right)^{-2}$.

Applying the power rule with $g\left(x\right)=3x^{2}+1$, we have
Equation:

$$h'\left(x\right)=-2\left(3x^{2}+1\right)^{-3}\left(6x\right).$$

Rewriting back to the original form gives us
Equation:

$$h'(x) = \frac{-12x}{(3x^2+1)^3}.$$

Note:
Exercise:

Problem: Find the derivative of $h(x) = \left(2x^3 + 2x - 1\right)^4$.

Solution:

$h'(x) = 4\left(2x^3 + 2x - 1\right)^3 \left(6x^2 + 2\right) = 8\left(3x^2 + 1\right)\left(2x^3 + 2x - 1\right)^3$

Hint

Use [link] with $g(x) = 2x^3 + 2x - 1$

Example:
Exercise:

Problem:
Using the Chain and Power Rules with a Trigonometric Function

Find the derivative of $h(x) = \sin^3 x$.

Solution:

First recall that $\sin^3 x = (\sin x)^3$, so we can rewrite $h(x) = \sin^3 x$ as $h(x) = (\sin x)^3$.

Applying the power rule with $g(x) = \sin x$, we obtain
Equation:

$$h'(x) = 3(\sin x)^2 \cos x = 3\sin^2 x \cos x.$$

Example:
Exercise:

Problem:
Finding the Equation of a Tangent Line

Find the equation of a line tangent to the graph of $h(x) = \frac{1}{(3x-5)^2}$ at $x = 2$.

Solution:

Because we are finding an equation of a line, we need a point. The x-coordinate of the point is 2. To find the y-coordinate, substitute 2 into $h(x)$. Since $h(2) = \frac{1}{(3(2)-5)^2} = 1$, the point is $(2, 1)$.

For the slope, we need $h'(2)$. To find $h'(x)$, first we rewrite $h(x)=(3x-5)^{-2}$ and apply the power rule to obtain

Equation:

$$h'(x)=-2(3x-5)^{-3}(3)=-6(3x-5)^{-3}.$$

By substituting, we have $h'(2)=-6(3(2)-5)^{-3}=-6$. Therefore, the line has equation $y-1=-6(x-2)$. Rewriting, the equation of the line is $y=-6x+13$.

Note:

Exercise:

Problem: Find the equation of the line tangent to the graph of $f(x)=\left(x^2-2\right)^3$ at $x=-2$.

Solution:

$y=-48x-88$

Hint

Use the preceding example as a guide.

Combining the Chain Rule with Other Rules

Now that we can combine the chain rule and the power rule, we examine how to combine the chain rule with the other rules we have learned. In particular, we can use it with the formulas for the derivatives of trigonometric functions or with the product rule.

Example:

Exercise:

Problem:

Using the Chain Rule on a General Cosine Function

Find the derivative of $h(x)=\cos(g(x))$.

Solution:

Think of $h(x)=\cos(g(x))$ as $f(g(x))$ where $f(x)=\cos x$. Since $f'(x)=-\sin x$. we have $f'(g(x))=-\sin(g(x))$. Then we do the following calculation.

Equation:

$$\begin{aligned} h'(x) &= f'(g(x))g'(x) && \text{Apply the chain rule.} \\ &= -\sin(g(x))g'(x) && \text{Substitute } f'(g(x))=-\sin(g(x)). \end{aligned}$$

Thus, the derivative of $h(x)=\cos(g(x))$ is given by $h'(x)=-\sin(g(x))g'(x)$.

In the following example we apply the rule that we have just derived.

Example:
Exercise:

Problem:
Using the Chain Rule on a Cosine Function

Find the derivative of $h\left(x\right)=\text{cos}\left(5{x}^{2}\right)$.

Solution:

Let $g\left(x\right)=5{x}^{2}$. Then ${g}^{\prime }\left(x\right)=10x$. Using the result from the previous example,
Equation:

$$\begin{array}{cc}{h}^{\prime }\left(x\right)\hfill & =-\text{sin}\left(5{x}^{2}\right)\cdot 10x\hfill \\ & =-10x\,\text{sin}\left(5{x}^{2}\right).\hfill \end{array}$$

Example:
Exercise:

Problem:
Using the Chain Rule on Another Trigonometric Function

Find the derivative of $h\left(x\right)=\text{sec}\left(4{x}^{5}+2x\right)$.

Solution:

Apply the chain rule to $h\left(x\right)=\text{sec}\left(g\left(x\right)\right)$ to obtain
Equation:

$${h}^{\prime }\left(x\right)=\text{sec}(g\left(x\right)\text{tan}\left(g\left(x\right)\right){g}^{\prime }\left(x\right).$$

In this problem, $g\left(x\right)=4{x}^{5}+2x$, so we have ${g}^{\prime }\left(x\right)=20{x}^{4}+2$. Therefore, we obtain
Equation:

$$\begin{array}{cc}{h}^{\prime }\left(x\right)\hfill & =\text{sec}\left(4{x}^{5}+2x\right)\text{tan}\left(4{x}^{5}+2x\right)\left(20{x}^{4}+2\right)\hfill \\ & =(20{x}^{4}+2)\text{sec}\left(4{x}^{5}+2x\right)\text{tan}\left(4{x}^{5}+2x\right).\hfill \end{array}$$

Note:
Exercise:

Problem: Find the derivative of $h\left(x\right)=\text{sin}(7x+2)$.

Solution:

$h'(x) = 7\cos(7x+2)$

Hint

Apply the chain rule to $h(x) = \sin g(x)$ first and then use $g(x) = 7x+2$.

At this point we provide a list of derivative formulas that may be obtained by applying the chain rule in conjunction with the formulas for derivatives of trigonometric functions. Their derivations are similar to those used in [link] and [link]. For convenience, formulas are also given in Leibniz's notation, which some students find easier to remember. (We discuss the chain rule using Leibniz's notation at the end of this section.) It is not absolutely necessary to memorize these as separate formulas as they are all applications of the chain rule to previously learned formulas.

Note:
Using the Chain Rule with Trigonometric Functions
For all values of x for which the derivative is defined,
Equation:

$$\begin{aligned}
\frac{d}{dx}(\sin(g(x)) &= \cos(g(x))g'(x) & \frac{d}{dx}\sin u &= \cos u\frac{du}{dx} \\
\frac{d}{dx}(\cos(g(x)) &= -\sin(g(x))g'(x) & \frac{d}{dx}\cos u &= -\sin u\frac{du}{dx} \\
\frac{d}{dx}(\tan(g(x)) &= \sec^2(g(x))g'(x) & \frac{d}{dx}\tan u &= \sec^2 u\frac{du}{dx} \\
\frac{d}{dx}(\cot(g(x)) &= -\csc^2(g(x))g'(x) & \frac{d}{dx}\cot u &= -\csc^2 u\frac{du}{dx} \\
\frac{d}{dx}(\sec(g(x)) &= \sec(g(x)\tan(g(x))g'(x) & \frac{d}{dx}\sec u &= \sec u\tan u\frac{du}{dx} \\
\frac{d}{dx}(\csc(g(x)) &= -\csc(g(x))\cot(g(x))g'(x) & \frac{d}{dx}\csc u &= -\csc u\cot u\frac{du}{dx}.
\end{aligned}$$

Example:
Exercise:

Problem:
Combining the Chain Rule with the Product Rule

Find the derivative of $h(x) = (2x+1)^5(3x-2)^7$.

Solution:

First apply the product rule, then apply the chain rule to each term of the product.
Equation:

$$\begin{aligned}
h'(x) &= \frac{d}{dx}\left((2x+1)^5\right)\cdot(3x-2)^7 + \frac{d}{dx}\left((3x-2)^7\right)\cdot(2x+1)^5 && \text{Apply the product rule.} \\
&= 5(2x+1)^4\cdot 2\cdot(3x-2)^7 + 7(3x-2)^6\cdot 3\cdot(2x+1)^5 && \text{Apply the chain rule.} \\
&= 10(2x+1)^4(3x-2)^7 + 21(3x-2)^6(2x+1)^5 && \text{Simplify.} \\
&= (2x+1)^4(3x-2)^6(10(3x-2)+21(2x+1)) && \text{Factor out } (2x+1)^4(3x-2)^6. \\
&= (2x+1)^4(3x-2)^6(72x+1) && \text{Simplify.}
\end{aligned}$$

Note:
Exercise:

Problem: Find the derivative of $h\left(x\right)=\frac{x}{{\left(2x+3\right)}^{3}}$.

Solution:

$h\prime \left(x\right)=\frac{3-4x}{{\left(2x+3\right)}^{4}}$

Hint

Start out by applying the quotient rule. Remember to use the chain rule to differentiate the denominator.

Composites of Three or More Functions

We can now combine the chain rule with other rules for differentiating functions, but when we are differentiating the composition of three or more functions, we need to apply the chain rule more than once. If we look at this situation in general terms, we can generate a formula, but we do not need to remember it, as we can simply apply the chain rule multiple times.

In general terms, first we let
Equation:

$$k\left(x\right)=h\left(f\left(g\left(x\right)\right)\right).$$

Then, applying the chain rule once we obtain
Equation:

$$k\prime \left(x\right)=\frac{d}{dx}(h(f\left(g\left(x\right)\right))=h\prime(f\left(g\left(x\right)\right))\cdot \frac{d}{dx}f\left(\left(g\left(x\right)\right)\right).$$

Applying the chain rule again, we obtain
Equation:

$$k\prime \left(x\right)=h\prime \left(f\left(g\left(x\right)\right)f\prime \left(g\left(x\right)\right)g\prime \left(x\right)\right).$$

Note:
Rule: Chain Rule for a Composition of Three Functions
For all values of *x* for which the function is differentiable, if
Equation:

$$k\left(x\right)=h\left(f\left(g\left(x\right)\right)\right),$$

then
Equation:

$$k\prime \left(x\right)=h\prime \left(f\left(g\left(x\right)\right)\right)f\prime \left(g\left(x\right)\right)g\prime \left(x\right).$$

In other words, we are applying the chain rule twice.

Notice that the derivative of the composition of three functions has three parts. (Similarly, the derivative of the composition of four functions has four parts, and so on.) Also, *remember, we can always work from the outside in, taking one derivative at a time.*

Example:
Exercise:

Problem:
Differentiating a Composite of Three Functions

Find the derivative of $k\left(x\right)=\cos^4\left(7x^2+1\right)$.

Solution:

First, rewrite $k\left(x\right)$ as
Equation:

$$k\left(x\right)=\left(\cos\left(7x^2+1\right)\right)^4.$$

Then apply the chain rule several times.
Equation:

$$\begin{aligned} k'\left(x\right) &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(\tfrac{d}{dx}\cos\left(7x^2+1\right)\right) && \text{Apply the chain rule.}\\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)\left(\tfrac{d}{dx}\left(7x^2+1\right)\right) && \text{Apply the chain rule.}\\ &= 4\left(\cos\left(7x^2+1\right)\right)^3\left(-\sin\left(7x^2+1\right)\right)\left(14x\right) && \text{Apply the chain rule.}\\ &= -56x\sin(7x^2+1)\cos^3(7x^2+1) && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Find the derivative of $h\left(x\right)=\sin^6\left(x^3\right)$.

Solution:

$h'\left(x\right)=18x^2\sin^5\left(x^3\right)\cos\left(x^3\right)$

Hint

Rewrite $h\left(x\right)=\sin^6(x^3)=\left(\sin(x^3)\right)^6$ and use [link] as a guide.

Example:
Exercise:

Problem:
Using the Chain Rule in a Velocity Problem

A particle moves along a coordinate axis. Its position at time t is given by $s\left(t\right)=\text{sin}\left(2t\right)+\text{cos}\left(3t\right)$. What is the velocity of the particle at time $t=\frac{\pi }{6}$?

Solution:

To find $v(t)$, the velocity of the particle at time t, we must differentiate $s\left(t\right)$. Thus,
Equation:

$$v\left(t\right)=s'\left(t\right)=2\text{cos}\left(2t\right)-3\text{sin}\left(3t\right).$$

Substituting $t=\frac{\pi }{6}$ into $v(t)$, we obtain $v\left(\frac{\pi }{6}\right)=-2$.

Note:
Exercise:

Problem:

A particle moves along a coordinate axis. Its position at time t is given by $s\left(t\right)=\text{sin}(4t)$. Find its acceleration at time t.

Solution:

$a\left(t\right)=-16\text{sin}(4t)$

Hint

Acceleration is the second derivative of position.

Proof

At this point, we present a very informal proof of the chain rule. For simplicity's sake we ignore certain issues: For example, we assume that $g\left(x\right)\ne g(a)$ for $x\ne a$ in some open interval containing a. We begin by applying the limit definition of the derivative to the function $h(x)$ to obtain $h'(a)$:
Equation:

$$h'\left(a\right)=\underset{x\to a}{\text{lim}}\frac{f\left(g(x)\right)-f\left(g\left(a\right)\right)}{x-a}.$$

Rewriting, we obtain
Equation:

$$h'\left(a\right)=\underset{x\to a}{\text{lim}}\frac{f\left(g\left(x\right)\right)-f\left(g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}\cdot \frac{g\left(x\right)-g\left(a\right)}{x-a}.$$

Although it is clear that
Equation:

$$\underset{x\to a}{\text{lim}}\frac{g\left(x\right)-g\left(a\right)}{x-a}=g'(a),$$

it is not obvious that

Equation:

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a)).$$

To see that this is true, first recall that since g is differentiable at a, g is also continuous at a. Thus,

Equation:

$$\lim_{x \to a} g(x) = g(a).$$

Next, make the substitution $y = g(x)$ and $b = g(a)$ and use change of variables in the limit to obtain

Equation:

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = \lim_{y \to b} \frac{f(y) - f(b)}{y - b} = f'(b) = f'(g(a)).$$

Finally,

Equation:

$$h'(a) = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} = f'(g(a))g'(a).$$

□

Example:
Exercise:

Problem:
Using the Chain Rule with Functional Values

Let $h(x) = f(g(x))$. If $g(1) = 4$, $g'(1) = 3$, and $f'(4) = 7$, find $h'(1)$.

Solution:

Use the chain rule, then substitute.

Equation:

$$\begin{array}{lll} h'(1) & = f'(g(1))g'(1) & \text{Apply the chain rule.} \\ & = f'(4) \cdot 3 & \text{Substitute } g(1) = 4 \text{ and } g'(1) = 3. \\ & = 7 \cdot 3 & \text{Substitute } f\prime(4) = 7. \\ & = 21 & \text{Simplify.} \end{array}$$

Note:
Exercise:

Problem: Given $h(x) = f(g(x))$. If $g(2) = -3, g'(2) = 4$, and $f'(-3) = 7$, find $h'(2)$.

Solution:

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Hint

Follow [link].

The Chain Rule Using Leibniz’s Notation

As with other derivatives that we have seen, we can express the chain rule using Leibniz’s notation. This notation for the chain rule is used heavily in physics applications.

For $h(x) = f(g(x))$, let $u = g(x)$ and $y = h(x) = f(u)$. Thus,

Equation:

$$h'(x) = \frac{dy}{dx}, f'(g(x)) = f'(u) = \frac{dy}{du} \text{ and } g'(x) = \frac{du}{dx}.$$

Consequently,

Equation:

$$\frac{dy}{dx} = h'(x) = f'(g(x))g'(x) = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Note:

Rule: Chain Rule Using Leibniz’s Notation

If y is a function of u, and u is a function of x, then

Equation:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example:

Exercise:

Problem:

Taking a Derivative Using Leibniz’s Notation, Example 1

Find the derivative of $y = \left(\frac{x}{3x+2}\right)^5$.

Solution:

First, let $u = \frac{x}{3x+2}$. Thus, $y = u^5$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$. Using the quotient rule,

Equation:

$$\frac{du}{dx} = \frac{2}{(3x+2)^2}$$

and
Equation:

$$\frac{dy}{du} = 5u^4.$$

Finally, we put it all together.
Equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= 5u^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } \frac{dy}{du} = 5u^4 \text{ and } \frac{du}{dx} = \frac{2}{(3x+2)^2}. \\ &= 5\left(\frac{x}{3x+2}\right)^4 \cdot \frac{2}{(3x+2)^2} && \text{Substitute } u = \frac{x}{3x+2}. \\ &= \frac{10x^4}{(3x+2)^6} && \text{Simplify.} \end{aligned}$$

It is important to remember that, when using the Leibniz form of the chain rule, the final answer must be expressed entirely in terms of the original variable given in the problem.

Example:
Exercise:

Problem:
Taking a Derivative Using Leibniz’s Notation, Example 2

Find the derivative of $y = \tan\left(4x^2 - 3x + 1\right)$.

Solution:

First, let $u = 4x^2 - 3x + 1$. Then $y = \tan u$. Next, find $\frac{du}{dx}$ and $\frac{dy}{du}$:
Equation:

$$\frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u.$$

Finally, we put it all together.
Equation:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{Apply the chain rule.} \\ &= \sec^2 u \cdot (8x - 3) && \text{Use } \frac{du}{dx} = 8x - 3 \text{ and } \frac{dy}{du} = \sec^2 u. \\ &= \sec^2(4x^2 - 3x + 1) \cdot (8x - 3) && \text{Substitute } u = 4x^2 - 3x + 1. \end{aligned}$$

Note:
Exercise:

Problem:

Use Leibniz's notation to find the derivative of $y = \cos\left(x^3\right)$. Make sure that the final answer is expressed entirely in terms of the variable x.

Solution:

$\frac{dy}{dx} = -3x^2\sin\left(x^3\right)$

Hint

Let $u = x^3$.

Key Concepts

- The chain rule allows us to differentiate compositions of two or more functions. It states that for $h\left(x\right) = f\left(g\left(x\right)\right)$,
  Equation:
  $$h'\left(x\right) = f'\left(g\left(x\right)\right)g'\left(x\right).$$
  In Leibniz's notation this rule takes the form
  Equation:
  $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$
- We can use the chain rule with other rules that we have learned, and we can derive formulas for some of them.
- The chain rule combines with the power rule to form a new rule:
  Equation:
  $$\text{If } h\left(x\right) = \left(g\left(x\right)\right)^n, \text{then } h'\left(x\right) = n\left(g\left(x\right)\right)^{n-1}g'\left(x\right).$$
- When applied to the composition of three functions, the chain rule can be expressed as follows: If $h\left(x\right) = f\left(g\left(k\left(x\right)\right)\right)$, then $h'\left(x\right) = f'(g\left(k\left(x\right)\right)g'\left(k\left(x\right)\right)k'\left(x\right)$.

Key Equations

- **The chain rule**
  $h'\left(x\right) = f'\left(g\left(x\right)\right)g'\left(x\right)$
- **The power rule for functions**
  $h'\left(x\right) = n\left(g\left(x\right)\right)^{n-1}g'\left(x\right)$

For the following exercises, given $y = f(u)$ and $u = g\left(x\right)$, find $\frac{dy}{dx}$ by using Leibniz's notation for the chain rule: $\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$.
Exercise:

Problem: $y = 3u - 6, u = 2x^2$

Exercise:

Problem: $y = 6u^3, u = 7x - 4$

Solution:

$18u^2 \cdot 7 = 18(7x-4)^2 \cdot 7$

Exercise:

Problem: $y = \sin u, u = 5x - 1$

Exercise:

Problem: $y = \cos u, u = \frac{-x}{8}$

Solution:

$-\sin u \cdot \frac{-1}{8} = -\sin\left(\frac{-x}{8}\right) \cdot \frac{-1}{8}$

Exercise:

Problem: $y = \tan u, u = 9x + 2$

Exercise:

Problem: $y = \sqrt{4u+3}, u = x^2 - 6x$

Solution:

$\frac{8x-24}{2\sqrt{4u+3}} = \frac{4x-12}{\sqrt{4x^2-24x+3}}$

For each of the following exercises,

a. decompose each function in the form $y = f(u)$ and $u = g(x)$, and
b. find $\frac{dy}{dx}$ as a function of x.

Exercise:

Problem: $y = (3x-2)^6$

Exercise:

Problem: $y = \left(3x^2+1\right)^3$

Solution:

a. $u = 3x^2 + 1$; b. $18x\left(3x^2+1\right)^2$

Exercise:

Problem: $y = \sin^5(x)$

Exercise:

Problem: $y = \left(\frac{x}{7} + \frac{7}{x}\right)^7$

Solution:

a. $f\left(u\right) = u^7, u = \frac{x}{7} + \frac{7}{x}$; b. $7\left(\frac{x}{7} + \frac{7}{x}\right)^6 \cdot \left(\frac{1}{7} - \frac{7}{x^2}\right)$

Exercise:

Problem: $y = \tan\left(\sec x\right)$

Exercise:

Problem: $y = \csc\left(\pi x + 1\right)$

Solution:

a. $f\left(u\right) = \csc u, u = \pi x + 1$; b. $-\pi\csc\left(\pi x + 1\right) \cdot \cot\left(\pi x + 1\right)$

Exercise:

Problem: $y = \cot^2 x$

Exercise:

Problem: $y = -6\left(\sin\right)^{-3}x$

Solution:

a. $f\left(u\right) = -6u^{-3}, u = \sin x$, b. $18\left(\sin\right)^{-4}x \cdot \cos x$

For the following exercises, find $\frac{dy}{dx}$ for each function.

Exercise:

Problem: $y = \left(3x^2 + 3x - 1\right)^4$

Exercise:

Problem: $y = \left(5 - 2x\right)^{-2}$

Solution:

$\frac{4}{\left(5-2x\right)^3}$

Exercise:

Problem: $y = \cos^3\left(\pi x\right)$

Exercise:

Problem: $y = \left(2x^3 - x^2 + 6x + 1\right)^3$

Solution:

$6\left(2x^3-x^2+6x+1\right)^2\left(3x^2-x+3\right)$

Exercise:

Problem: $y=\frac{1}{\sin^2(x)}$

Exercise:

Problem: $y=(\tan x+\sin x)^{-3}$

Solution:

$-3(\tan x+\sin x)^{-4}\cdot\left(\sec^2 x+\cos x\right)$

Exercise:

Problem: $y=x^2\cos^4 x$

Exercise:

Problem: $y=\sin\,(\cos 7x)$

Solution:

$-7\cos\,(\cos 7x)\cdot\sin 7x$

Exercise:

Problem: $y=\sqrt{6+\sec\pi x^2}$

Exercise:

Problem: $y=\cot^3(4x+1)$

Solution:

$-12\cot^2(4x+1)\cdot\csc^2(4x+1)$

Exercise:

Problem: Let $y=[f(x)]^3$ and suppose that $f'(1)=4$ and $\frac{dy}{dx}=10$ for $x=1$. Find $f(1)$.

Exercise:

Problem: Let $y=\left(f(x)+5x^2\right)^4$ and suppose that $f(-1)=-4$ and $\frac{dy}{dx}=3$ when $x=-1$. Find $f'(-1)$

Solution:

$10\frac{3}{4}$

Exercise:

Problem: Let $y=(f(u)+3x)^2$ and $u=x^3-2x$. If $f(4)=6$ and $\frac{dy}{dx}=18$ when $x=2$, find $f'(4)$.

Exercise:

Problem:

[T] Find the equation of the tangent line to $y=-\sin\left(\frac{x}{2}\right)$ at the origin. Use a calculator to graph the function and the tangent line together.

Solution:

$y=\frac{-1}{2}x$

Exercise:

Problem:

[T] Find the equation of the tangent line to $y=\left(3x+\frac{1}{x}\right)^2$ at the point $(1,16)$. Use a calculator to graph the function and the tangent line together.

Exercise:

Problem: Find the x-coordinates at which the tangent line to $y=\left(x-\frac{6}{x}\right)^8$ is horizontal.

Solution:

$x=\pm\sqrt{6}$

Exercise:

Problem:

[T] Find an equation of the line that is normal to $g(\theta)=\sin^2(\pi\theta)$ at the point $\left(\frac{1}{4},\frac{1}{2}\right)$. Use a calculator to graph the function and the normal line together.

For the following exercises, use the information in the following table to find $h'(a)$ at the given value for a.

x	$f(x)$	$f\prime(x)$	$g(x)$	$g\prime(x)$
0	2	5	0	2
1	1	−2	3	0
2	4	4	1	−1
3	3	−3	2	3

Exercise:

Problem: $h(x)=f(g(x));a=0$

Solution:

10

Exercise:

Problem: $h\left(x\right)=g\left(f\left(x\right)\right);a=0$

Exercise:

Problem: $h\left(x\right)={\left({x}^{4}+g(x)\right)}^{-2};a=1$

Solution:

$-\frac{1}{8}$

Exercise:

Problem: $h(x)={\left(\frac{f(x)}{g(x)}\right)}^{2};a=3$

Exercise:

Problem: $h\left(x\right)=f\left(x+f\left(x\right)\right);a=1$

Solution:

-4

Exercise:

Problem: $h\left(x\right)={\left(1+g(x)\right)}^{3};a=2$

Exercise:

Problem: $h\left(x\right)=g\left(2+f\left({x}^{2}\right)\right);a=1$

Solution:

-12

Exercise:

Problem: $h\left(x\right)=f\left(g\left(\sin x\right)\right);a=0$

Exercise:

Problem:

[T] The position function of a freight train is given by $s\left(t\right)=100{(t+1)}^{-2}$, with s in meters and t in seconds. At time $t=6$ s, find the train’s

a. velocity and
b. acceleration.
c. Using a. and b. is the train speeding up or slowing down?

Solution:

a. $-\frac{200}{343}$ m/s, b. $\frac{600}{2401}$ m/s^2, c. The train is slowing down since velocity and acceleration have opposite signs.

Exercise:

Problem:

[T] A mass hanging from a vertical spring is in simple harmonic motion as given by the following position function, where t is measured in seconds and s is in inches:

$s\left(t\right)=-3\cos\left(\pi t+\frac{\pi}{4}\right).$

a. Determine the position of the spring at $t=1.5$ s.
b. Find the velocity of the spring at $t=1.5$ s.

Exercise:

Problem:

[T] The total cost to produce x boxes of Thin Mint Girl Scout cookies is C dollars, where $C=0.0001x^3-0.02x^2+3x+300$. In t weeks production is estimated to be $x=1600+100t$ boxes.

a. Find the marginal cost $C'\left(x\right)$.
b. Use Leibniz's notation for the chain rule, $\frac{dC}{dt}=\frac{dC}{dx}\cdot\frac{dx}{dt}$, to find the rate with respect to time t that the cost is changing.
c. Use b. to determine how fast costs are increasing when $t=2$ weeks. Include units with the answer.

Solution:

a. $C'\left(x\right)=0.0003x^2-0.04x+3$ b. $\frac{dC}{dt}=100\cdot\left(0.0003x^2-0.04x+3\right)$ c. Approximately $90,300 per week

Exercise:

Problem:

[T] The formula for the area of a circle is $A=\pi r^2$, where r is the radius of the circle. Suppose a circle is expanding, meaning that both the area A and the radius r (in inches) are expanding.

a. Suppose $r=2-\frac{100}{\left(t+7\right)^2}$ where t is time in seconds. Use the chain rule $\frac{dA}{dt}=\frac{dA}{dr}\cdot\frac{dr}{dt}$ to find the rate at which the area is expanding.
b. Use a. to find the rate at which the area is expanding at $t=4$ s.

Exercise:

Problem:

[T] The formula for the volume of a sphere is $S=\frac{4}{3}\pi r^3$, where r (in feet) is the radius of the sphere. Suppose a spherical snowball is melting in the sun.

a. Suppose $r=\frac{1}{\left(t+1\right)^2}-\frac{1}{12}$ where t is time in minutes. Use the chain rule $\frac{dS}{dt}=\frac{dS}{dr}\cdot\frac{dr}{dt}$ to find the rate at which the snowball is melting.
b. Use a. to find the rate at which the volume is changing at $t=1$ min.

Solution:

a. $\frac{dS}{dt}=-\frac{8\pi r^2}{\left(t+1\right)^3}$ b. The volume is decreasing at a rate of $-\frac{\pi}{36}$ ft^3/min.

Exercise:

Problem:

[T] The daily temperature in degrees Fahrenheit of Phoenix in the summer can be modeled by the function $T\left(x\right)=94-10\cos\left[\frac{\pi}{12}\left(x-2\right)\right]$, where x is hours after midnight. Find the rate at which the temperature is changing at 4 p.m.

Exercise:

Problem:

[T] The depth (in feet) of water at a dock changes with the rise and fall of tides. The depth is modeled by the function $D\left(t\right)=5\sin\left(\frac{\pi}{6}t-\frac{7\pi}{6}\right)+8$, where t is the number of hours after midnight. Find the rate at which the depth is changing at 6 a.m.

Solution:

~2.3 ft/hr

Glossary

chain rule

the chain rule defines the derivative of a composite function as the derivative of the outer function evaluated at the inner function times the derivative of the inner function

Implicit Differentiation

- Find the derivative of a complicated function by using implicit differentiation.
- Use implicit differentiation to determine the equation of a tangent line.

We have already studied how to find equations of tangent lines to functions and the rate of change of a function at a specific point. In all these cases we had the explicit equation for the function and differentiated these functions explicitly. Suppose instead that we want to determine the equation of a tangent line to an arbitrary curve or the rate of change of an arbitrary curve at a point. In this section, we solve these problems by finding the derivatives of functions that define y implicitly in terms of x.

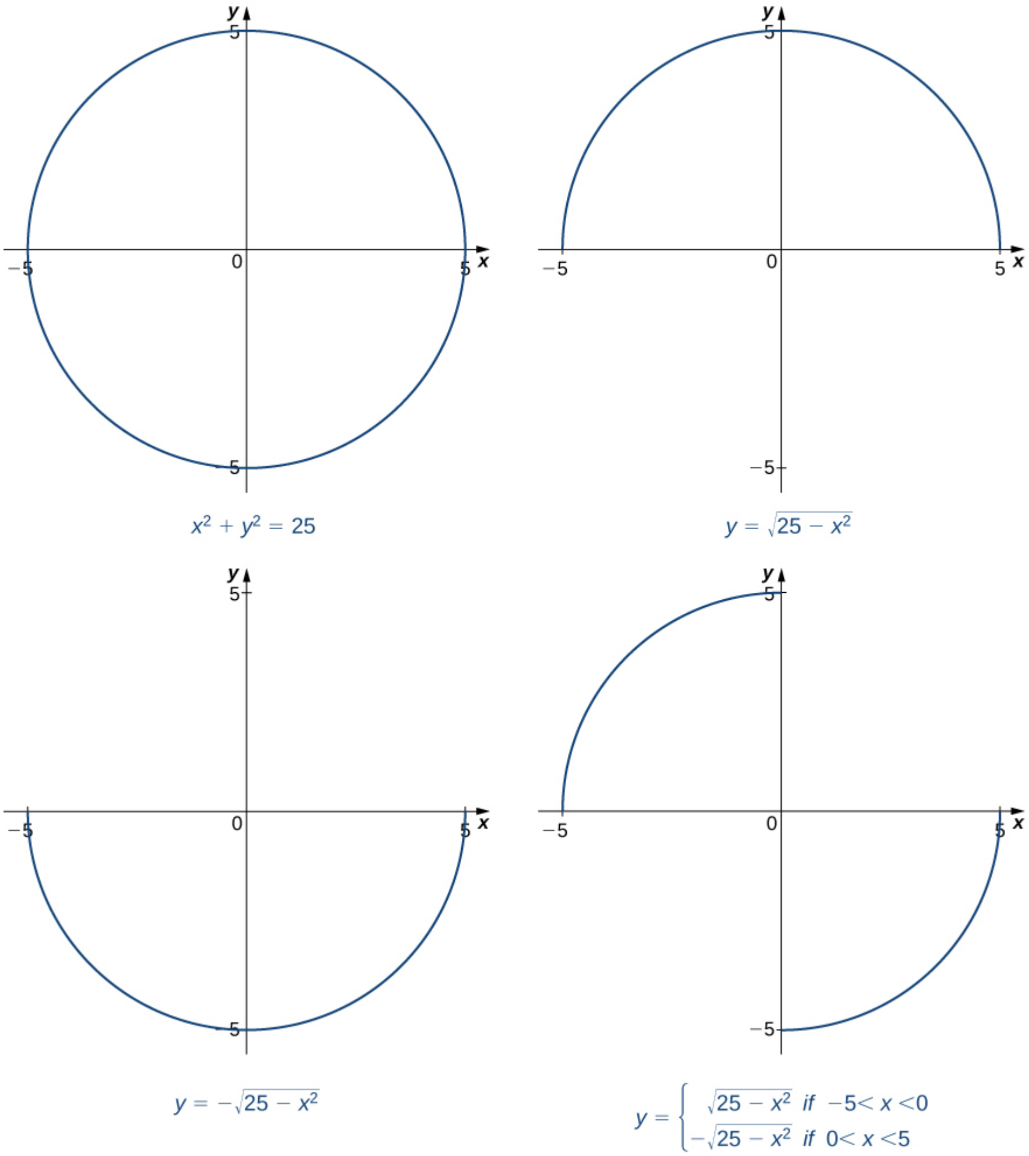
Implicit Differentiation

In most discussions of math, if the dependent variable y is a function of the independent variable x, we express y in terms of x. If this is the case, we say that y is an explicit function of x. For example, when we write the equation $y = x^2 + 1$, we are defining y explicitly in terms of x. On the other hand, if the relationship between the function y and the variable x is expressed by an equation where y is not expressed entirely in terms of x, we say that the equation defines y implicitly in terms of x. For example, the equation $y - x^2 = 1$ defines the function $y = x^2 + 1$ implicitly.

Implicit differentiation allows us to find slopes of tangents to curves that are clearly not functions (they fail the vertical line test). We are using the idea that portions of y are functions that satisfy the given equation, but that y is not actually a function of x.

In general, an equation defines a function implicitly if the function satisfies that equation. An equation may define many different functions implicitly. For example, the functions

$y = \sqrt{25 - x^2}$ and $y = \begin{cases} \sqrt{25 - x^2} \text{ if } -5 < x < 0 \\ -\sqrt{25 - x^2} \text{ if } 0 < x < 25 \end{cases}$, which are illustrated in [link], are just three of the many functions defined implicitly by the equation $x^2 + y^2 = 25$.



The equation $x^2 + y^2 = 25$ defines many functions implicitly.

If we want to find the slope of the line tangent to the graph of $x^2 + y^2 = 25$ at the point $(3, 4)$, we could evaluate the derivative of the function $y = \sqrt{25 - x^2}$ at $x = 3$. On the other hand, if we want the slope of the tangent line at the point $(3, -4)$, we could use the derivative of $y = -\sqrt{25 - x^2}$. However, it is not always easy to solve for a function defined implicitly by an equation. Fortunately, the technique of **implicit differentiation** allows us to find the derivative of an implicitly defined function without ever solving for the function explicitly. The process of finding $\frac{dy}{dx}$ using implicit differentiation is described in the following problem-solving strategy.

Note:

Problem-Solving Strategy: Implicit Differentiation

To perform implicit differentiation on an equation that defines a function y implicitly in terms of a variable x, use the following steps:

1. Take the derivative of both sides of the equation. Keep in mind that y is a function of x. Consequently, whereas $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\sin y) = \cos y \frac{dy}{dx}$ because we must use the chain rule to differentiate $\sin y$ with respect to x.
2. Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and all terms that do not contain $\frac{dy}{dx}$ are on the right.
3. Factor out $\frac{dy}{dx}$ on the left.
4. Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by an appropriate algebraic expression.

Example:
Exercise:

Problem:
Using Implicit Differentiation

Assuming that y is defined implicitly by the equation $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

Solution:

Follow the steps in the problem-solving strategy.
Equation:

$$\begin{aligned}
\frac{d}{dx}\left(x^2+y^2\right) &= \frac{d}{dx}(25) && \text{Step 1. Differentiate both sides of the equation.} \\
\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}\left(y^2\right) &= 0 && \text{Step 1.1. Use the sum rule on the left.} \\
&&& \text{On the right } \frac{d}{dx}(25)=0. \\
2x+2y\frac{dy}{dx} &= 0 && \text{Step 1.2. Take the derivatives, so } \frac{d}{dx}\left(x^2\right)=2x \\
&&& \text{and } \frac{d}{dx}\left(y^2\right)=2y\frac{dy}{dx}. \\
2y\frac{dy}{dx} &= -2x && \text{Step 2. Keep the terms with } \frac{dy}{dx} \text{ on the left.} \\
&&& \text{Move the remaining terms to the right.} \\
\frac{dy}{dx} &= -\frac{x}{y} && \text{Step 4. Divide both sides of the equation by} \\
&&& 2y. \text{ (Step 3 does not apply in this case.)}
\end{aligned}$$

Analysis

Note that the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.

Example:
Exercise:

Problem:
Using Implicit Differentiation and the Product Rule

Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$.

Solution:
Equation:

$\frac{d}{dx}\left(x^3\sin y + y\right) = \frac{d}{dx}(4x+3)$	Step 1: Differentiate both sides of the equa
$\frac{d}{dx}\left(x^3\sin y\right) + \frac{d}{dx}(y) = 4$	Step 1.1: Apply the sum rule on the left. On the right, $\frac{d}{dx}(4x+3)=4$.
$\left(\frac{d}{dx}\left(x^3\right)\cdot\sin y + \frac{d}{dx}(\sin y)\cdot x^3\right) + \frac{dy}{dx} = 4$	Step 1.2: Use the product rule to find $\frac{d}{dx}\left(x^3\sin y\right)$. Observe that $\frac{d}{dx}(y)=\frac{dy}{dx}$.
$3x^2\sin y + \left(\cos y\frac{dy}{dx}\right)\cdot x^3 + \frac{dy}{dx} = 4$	Step 1.3: We know $\frac{d}{dx}\left(x^3\right)=3x^2$. Use the chain rule to obtain $\frac{d}{dx}(\sin y)=\cos y\frac{dy}{dx}$.
$\mathrm{x}^3\cos y\frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2\sin y$	Step 2: Keep all terms containing $\frac{dy}{dx}$ on th left. Move all other terms to the right.
$\frac{dy}{dx}\left(\mathrm{x}^3\cos y + 1\right) = 4 - 3x^2\sin y$	Step 3: Factor out $\frac{dy}{dx}$ on the left.
$\frac{dy}{dx} = \frac{4-3x^2\sin y}{x^3\cos y+1}$	Step 4: Solve for $\frac{dy}{dx}$ by dividing both sides the equation by $\mathrm{x}^3\cos y + 1$.

Example:
Exercise:

Problem:
Using Implicit Differentiation to Find a Second Derivative

Find $\frac{d^2y}{dx^2}$ if $x^2+y^2=25$.

Solution:

In [link], we showed that $\frac{dy}{dx}=-\frac{x}{y}$. We can take the derivative of both sides of this equation to find $\frac{d^2y}{dx^2}$.

Equation:

$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(-\frac{x}{y}\right)$	Differentiate both sides of $\frac{dy}{dx}=-\frac{x}{y}$.
$= -\frac{\left(1\cdot y - x\frac{dy}{dx}\right)}{y^2}$	Use the quotient rule to find $\frac{d}{dy}\left(-\frac{x}{y}\right)$.
$= \frac{-y+x\frac{dy}{dx}}{y^2}$	Simplify.
$= \frac{-y+x\left(-\frac{x}{y}\right)}{y^2}$	Substitute $\frac{dy}{dx}=-\frac{x}{y}$.
$= \frac{-y^2-x^2}{y^3}$	Simplify.

At this point we have found an expression for $\frac{d^2y}{dx^2}$. If we choose, we can simplify the expression further by recalling that $x^2+y^2=25$ and making this substitution in the numerator to obtain $\frac{d^2y}{dx^2}=-\frac{25}{y^3}$.

Note:
Exercise:

Problem: Find $\frac{dy}{dx}$ for y defined implicitly by the equation $4x^5 + \tan y = y^2 + 5x$.

Solution:

$$\frac{dy}{dx} = \frac{5-20x^4}{\sec^2 y - 2y}$$

Hint

Follow the problem solving strategy, remembering to apply the chain rule to differentiate $\tan$ and y^2.

Finding Tangent Lines Implicitly

Now that we have seen the technique of implicit differentiation, we can apply it to the problem of finding equations of tangent lines to curves described by equations.

Example:
Exercise:

Problem:
Finding a Tangent Line to a Circle

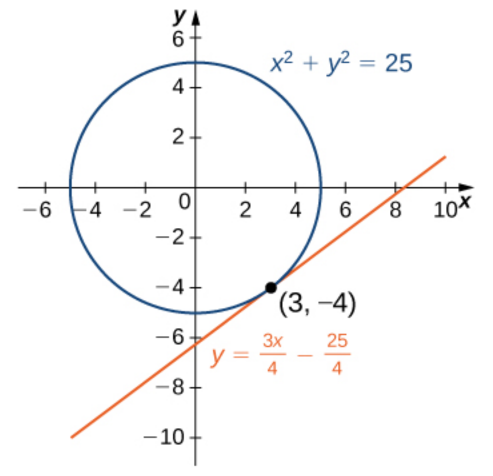
Find the equation of the line tangent to the curve $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution:

Although we could find this equation without using implicit differentiation, using that method makes it much easier. In [link], we found $\frac{dy}{dx} = -\frac{x}{y}$.

The slope of the tangent line is found by substituting $(3, -4)$ into this expression. Consequently, the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Using the point $(3, -4)$ and the slope $\frac{3}{4}$ in the point-slope equation of the line, we obtain the equation $y = \frac{3}{4}x - \frac{25}{4}$ ([link]).

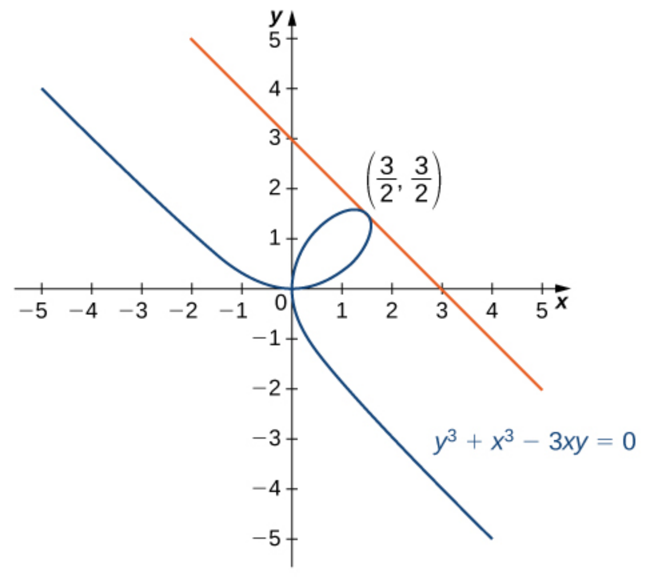


The line $y = \frac{3}{4}x - \frac{25}{4}$ is tangent to $x^2 + y^2 = 25$ at the point (3, −4).

Example:
Exercise:

Problem:
Finding the Equation of the Tangent Line to a Curve

Find the equation of the line tangent to the graph of $y^3+x^3-3xy=0$ at the point $\left(\frac{3}{2},\frac{3}{2}\right)$ ([link]). This curve is known as the folium (or leaf) of Descartes.



Finding the tangent line to the folium of Descartes at $\left(\frac{3}{2},\frac{3}{2}\right)$.

Solution:

Begin by finding $\frac{dy}{dx}$.

Equation:

$$\begin{aligned}\frac{d}{dx}\left(y^3+x^3-3xy\right)&=\frac{d}{dx}(0)\\3y^2\frac{dy}{dx}+3x^2-\left(3y+\frac{dy}{dx}3x\right)&=0\\\frac{dy}{dx}&=\frac{3y-3x^2}{3y^2-3x}.\end{aligned}$$

Next, substitute $\left(\frac{3}{2},\frac{3}{2}\right)$ into $\frac{dy}{dx}=\frac{3y-3x^2}{3y^2-3x}$ to find the slope of the tangent line:

Equation:

$$\left.\frac{dy}{dx}\right|_{\left(\frac{3}{2},\frac{3}{2}\right)} = -1.$$

Finally, substitute into the point-slope equation of the line to obtain

Equation:

$$y = -x + 3.$$

Example:

Exercise:

Problem:

Applying Implicit Differentiation

In a simple video game, a rocket travels in an elliptical orbit whose path is described by the equation $4x^2 + 25y^2 = 100$. The rocket can fire missiles along lines tangent to its path. The object of the game is to destroy an incoming asteroid traveling along the positive *x*-axis toward $(0, 0)$. If the rocket fires a missile when it is located at $\left(3, \frac{8}{5}\right)$, where will it intersect the *x*-axis?

Solution:

To solve this problem, we must determine where the line tangent to the graph of

$4x^2 + 25y^2 = 100$ at $\left(3, \frac{8}{5}\right)$ intersects the *x*-axis. Begin by finding $\frac{dy}{dx}$ implicitly.

Differentiating, we have

Equation:

$$8x + 50y\frac{dy}{dx} = 0.$$

Solving for $\frac{dy}{dx}$, we have

Equation:

$$\frac{dy}{dx} = -\frac{4x}{25y}.$$

The slope of the tangent line is $\left.\frac{dy}{dx}\right|_{\left(3,\frac{8}{5}\right)} = -\frac{3}{10}$. The equation of the tangent line is $y = -\frac{3}{10}x + \frac{5}{2}$. To determine where the line intersects the *x*-axis, solve $0 = -\frac{3}{10}x + \frac{5}{2}$. The solution is $x = \frac{25}{3}$. The missile intersects the *x*-axis at the point $\left(\frac{25}{3}, 0\right)$.

Note:

Exercise:

Problem: Find the equation of the line tangent to the hyperbola $x^2 - y^2 = 16$ at the point $(5, 3)$.

Solution:

$y = \frac{5}{3}x - \frac{16}{3}$

Hint

$\frac{dy}{dx} = \frac{x}{y}$

Key Concepts

- We use implicit differentiation to find derivatives of implicitly defined functions (functions defined by equations).
- By using implicit differentiation, we can find the equation of a tangent line to the graph of a curve.

For the following exercises, use implicit differentiation to find $\frac{dy}{dx}$.

Exercise:

Problem: $x^2 - y^2 = 4$

Exercise:

Problem: $6x^2 + 3y^2 = 12$

Solution:

$\frac{dy}{dx} = \frac{-2x}{y}$

Exercise:

Problem: $x^2y = y - 7$

Exercise:

Problem: $3x^3 + 9xy^2 = 5x^3$

Solution:

$\frac{dy}{dx} = \frac{x}{3y} - \frac{y}{2x}$

Exercise:

Problem: $xy - \cos(xy) = 1$

Exercise:

Problem: $y\sqrt{x+4} = xy + 8$

Solution:

$\frac{dy}{dx} = \frac{y - \frac{y}{2\sqrt{x+4}}}{\sqrt{x+4} - x}$

Exercise:

Problem: $-xy - 2 = \frac{x}{7}$

Exercise:

Problem: $y\sin(xy) = y^2 + 2$

Solution:

$$\frac{dy}{dx} = \frac{y^2\cos(xy)}{2y - \sin(xy) - xy\cos xy}$$

Exercise:

Problem: $(xy)^2 + 3x = y^2$

Exercise:

Problem: $x^3y + xy^3 = -8$

Solution:

$$\frac{dy}{dx} = \frac{-3x^2y - y^3}{x^3 + 3xy^2}$$

For the following exercises, find the equation of the tangent line to the graph of the given equation at the indicated point. Use a calculator or computer software to graph the function and the tangent line.

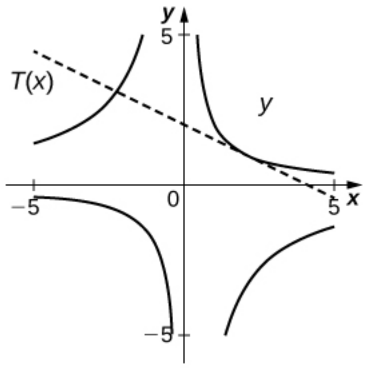
Exercise:

Problem: **[T]** $x^4y - xy^3 = -2, (-1, -1)$

Exercise:

Problem: **[T]** $x^2y^2 + 5xy = 14, (2, 1)$

Solution:



$y = \frac{-1}{2}x + 2$

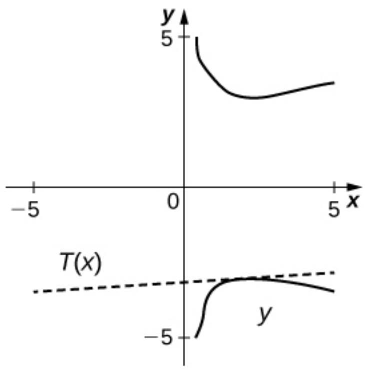
Exercise:

Problem: **[T]** $\tan(xy) = y, \left(\frac{\pi}{4}, 1\right)$

Exercise:

Problem: **[T]** $xy^2 + \sin(\pi y) - 2x^2 = 10, (2, -3)$

Solution:



$y = \frac{1}{\pi+12}x - \frac{3\pi+38}{\pi+12}$

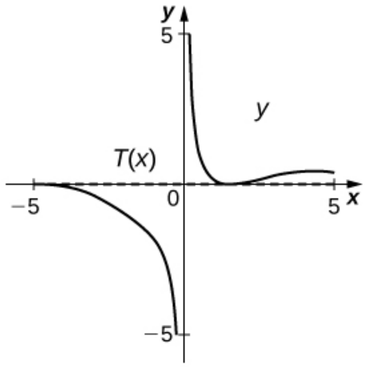
Exercise:

Problem: **[T]** $\frac{x}{y} + 5x - 7 = -\frac{3}{4}y, (1, 2)$

Exercise:

Problem: **[T]** $xy + \sin(x) = 1, \left(\frac{\pi}{2}, 0\right)$

Solution:

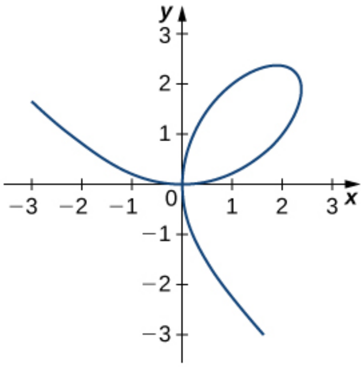


$y = 0$

Exercise:

Problem:

[T] The graph of a folium of Descartes with equation $2x^3 + 2y^3 - 9xy = 0$ is given in the following graph.



a. Find the equation of the tangent line at the point $(2, 1)$. Graph the tangent line along with the folium.
b. Find the equation of the normal line to the tangent line in a. at the point $(2, 1)$.

Exercise:

Problem: For the equation $x^2 + 2xy - 3y^2 = 0$,

a. Find the equation of the normal to the tangent line at the point $(1, 1)$.
b. At what other point does the normal line in a. intersect the graph of the equation?

Solution:

a. $y = -x + 2$ b. $(3, -1)$

Exercise:

Problem: Find all points on the graph of $y^3 - 27y = x^2 - 90$ at which the tangent line is vertical.

Exercise:

Problem: For the equation $x^2 + xy + y^2 = 7$,

a. Find the x-intercept(s).
b. Find the slope of the tangent line(s) at the x-intercept(s).
c. What does the value(s) in b. indicate about the tangent line(s)?

Solution:

a. $\left(\pm\sqrt{7}, 0\right)$ b. -2 c. They are parallel since the slope is the same at both intercepts.

Exercise:

Problem:

Find the equation of the tangent line to the graph of the equation $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{6}$ at the point $\left(0, \frac{1}{2}\right)$.

Exercise:

Problem:

Find the equation of the tangent line to the graph of the equation $\tan^{-1}(x + y) = x^2 + \frac{\pi}{4}$ at the point $(0, 1)$.

Solution:

$y=-x+1$

Exercise:

Problem: Find y' and y'' for $x^2+6xy-2y^2=3$.

Exercise:

Problem:

[T] The number of cell phones produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $60x^{3/4}y^{1/4}=3240$.

a. Find $\frac{dy}{dx}$ and evaluate at the point $(81,16)$.
b. Interpret the result of a.

Solution:

a. -0.5926 b. When \$81 is spent on labor and \$16 is spent on capital, the amount spent on capital is decreasing by \$0.5926 per \$1 spent on labor.

Exercise:

Problem:

[T] The number of cars produced when x dollars is spent on labor and y dollars is spent on capital invested by a manufacturer can be modeled by the equation $30x^{1/3}y^{2/3}=360$.

(Both x and y are measured in thousands of dollars.)

a. Find $\frac{dy}{dx}$ and evaluate at the point $(27,8)$.
b. Interpret the result of a.

Exercise:

Problem:

The volume of a right circular cone of radius x and height y is given by $V=\frac{1}{3}\pi x^2y$. Suppose that the volume of the cone is $85\pi\,\text{cm}^3$. Find $\frac{dy}{dx}$ when $x=4$ and $y=16$.

Solution:

-8

For the following exercises, consider a closed rectangular box with a square base with side x and height y.

Exercise:

Problem: Find an equation for the surface area of the rectangular box, $S(x,y)$.

Exercise:

Problem: If the surface area of the rectangular box is 78 square feet, find $\frac{dy}{dx}$ when $x=3$ feet and $y=5$ feet.

Solution:

-2.67

For the following exercises, use implicit differentiation to determine y'. Does the answer agree with the formulas we have previously determined?

Exercise:

Problem: $x = \sin y$

Exercise:

Problem: $x = \cos y$

Solution:

$y' = -\frac{1}{\sqrt{1-x^2}}$

Exercise:

Problem: $x = \tan y$

Glossary

implicit differentiation

is a technique for computing $\frac{dy}{dx}$ for a function defined by an equation, accomplished by differentiating both sides of the equation (remembering to treat the variable y as a function) and solving for $\frac{dy}{dx}$

Derivatives of Exponential and Logarithmic Functions

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in Introduction to Functions and Graphs, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B\left(x\right)=b^x, b>0$, is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0=1, b^{-n}=\frac{1}{b^n}$, for a positive integer n, and $b^{s/t}=\left(\sqrt[t]{b}\right)^s$ for positive integers s and t. These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the *continuity* of $B\left(x\right)=b^x, b>0$, we may interpret b^r as $\lim\limits_{x\to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

Equation:

$$\begin{array}{c}4^3<4^\pi<4^4, 4^{3.1}<4^\pi<4^{3.2}, 4^{3.14}<4^\pi<4^{3.15},\\ 4^{3.141}<4^\pi<4^{3.142}, 4^{3.1415}<4^\pi<4^{3.1416},\ldots.\end{array}$$

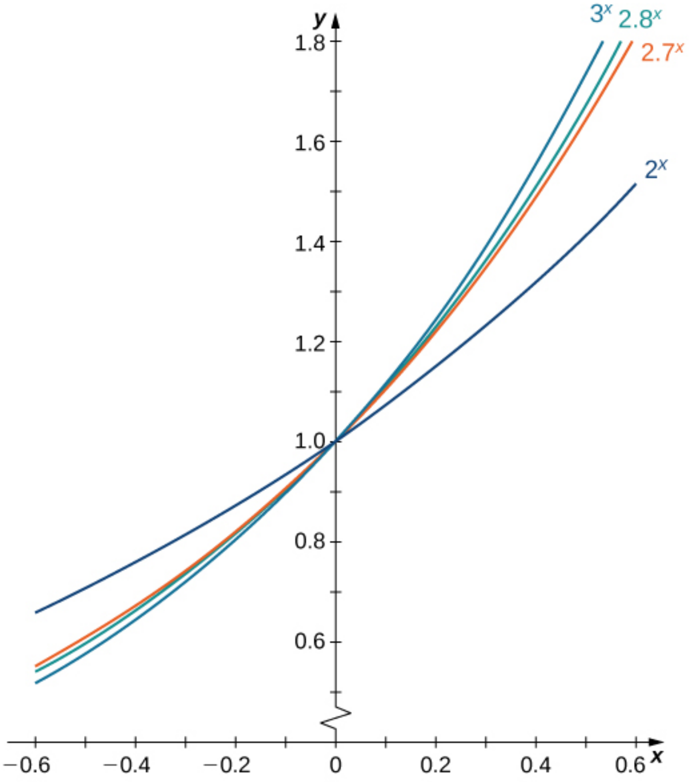
As we see in the following table, $4^\pi\approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x, b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. [link] provides graphs of the functions $y = 2^x, y = 3^x, y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function**.



The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

Equation:

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$ to obtain the estimate

Equation:

$$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}.$$

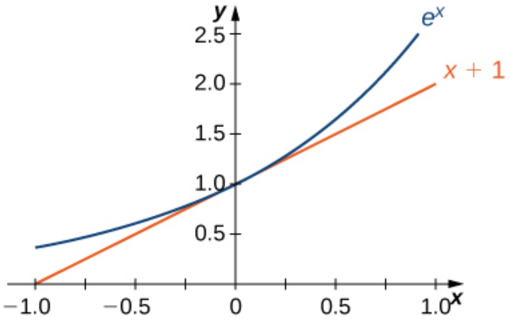
See the following table.

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

Estimating a Value of e

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in [link]. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.



The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x, b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

Equation:

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.$$

Turning to $B'(x)$, we obtain the following.
Equation:

$$\begin{aligned} B'(x) &= \lim_{h\to 0}\frac{b^{x+h}-b^x}{h} && \text{Apply the limit definition of the derivative.} \\ &= \lim_{h\to 0}\frac{b^x b^h-b^x}{h} && \text{Note that } b^{x+h}=b^x b^h. \\ &= \lim_{h\to 0}\frac{b^x(b^h-1)}{h} && \text{Factor out } b^x. \\ &= b^x \lim_{h\to 0}\frac{b^h-1}{h} && \text{Apply a property of limits.} \\ &= b^x B'(0) && \text{Use } B'(0)=\lim_{h\to 0}\frac{b^{0+h}-b^0}{h}=\lim_{h\to 0}\frac{b^h-1}{h}. \end{aligned}$$

We see that on the basis of the assumption that $B(x)=b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is
Equation:

$$B'(x)=b^x B'(0).$$

For $E(x)=e^x$, $E'(0)=1$. Thus, we have $E'(x)=e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x)=b^x$, $b>0$, will be derived later.)

Note:
Derivative of the Natural Exponential Function
Let $E(x)=e^x$ be the natural exponential function. Then
Equation:

$$E'(x)=e^x.$$

In general,
Equation:

$$\frac{d}{dx}\left(e^{g(x)}\right)=e^{g(x)}g'(x).$$

Example:
Exercise:

Problem:
Derivative of an Exponential Function

Find the derivative of $f(x)=e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,
Equation:

$$\begin{aligned} f'(x) &= e^{\tan(2x)}\frac{d}{dx}(\tan(2x)) \\ &= e^{\tan(2x)}\sec^2(2x)\cdot 2. \end{aligned}$$

Example:
Exercise:

Problem:
Combining Differentiation Rules

Find the derivative of $y = \frac{e^{x^2}}{x}$.

Solution:

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.
Equation:

$$\begin{aligned} y' &= \frac{\left(e^{x^2} \cdot 2\right) x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\ &= \frac{e^{x^2}\left(2x^2 - 1\right)}{x^2} && \text{Simplify.} \end{aligned}$$

Note:
Exercise:

Problem: Find the derivative of $h\left(x\right) = xe^{2x}$.

Solution:

$h'\left(x\right) = e^{2x} + 2xe^{2x}$

Hint

Don’t forget to use the product rule.

Example:
Exercise:

Problem:
Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A\left(t\right) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'\left(t\right)$, to the population, $A\left(t\right)$ is constant.

Solution:

First find $A'\left(t\right)$. By using the chain rule, we have $A'\left(t\right) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by
Equation:

$$A'\left(t\right) = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Note:
Exercise:

Problem:

If $A\left(t\right)=1000{e}^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A\left(t\right)$ after 4 days?

Solution:

996

Hint

Find $A'\left(4\right)$.

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Note:
The Derivative of the Natural Logarithmic Function
If $x>0$ and $y=\ln x$, then
Equation:

$$\frac{dy}{dx}=\frac{1}{x}.$$

More generally, let $g\left(x\right)$ be a differentiable function. For all values of x for which $g'\left(x\right)>0$, the derivative of $h\left(x\right)=\ln\left(g\left(x\right)\right)$ is given by
Equation:

$$h'\left(x\right)=\frac{1}{g\left(x\right)}g'\left(x\right).$$

Proof

If $x>0$ and $y=\ln x$, then $e^y=x$. Differentiating both sides of this equation results in the equation
Equation:

$$e^y\frac{dy}{dx}=1.$$

Solving for $\frac{dy}{dx}$ yields

Equation:

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

Equation:

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f\left(x\right) = e^x$, by applying the inverse function theorem we have

Equation:

$$\frac{dy}{dx} = \frac{1}{f'\left(g\left(x\right)\right)} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

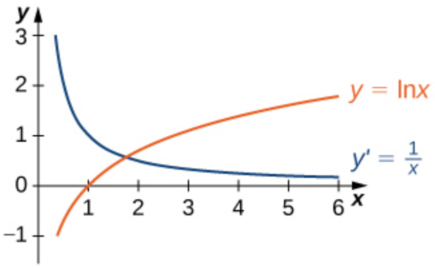
Using this result and applying the chain rule to $h\left(x\right) = \ln\left(g\left(x\right)\right)$ yields

Equation:

$$h'\left(x\right) = \frac{1}{g\left(x\right)} g'\left(x\right).$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in [link].



The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$.

Example:

Exercise:

Problem:

Taking a Derivative of a Natural Logarithm

Find the derivative of $f\left(x\right)=\ln\left(x^3+3x-4\right)$.

Solution:

Use [link] directly.
Equation:

$$\begin{array}{lll} f'\left(x\right) & =\frac{1}{x^3+3x-4}\cdot\left(3x^2+3\right) & \text{Use } g\left(x\right)=x^3+3x-4 \text{ in } h'\left(x\right)=\frac{1}{g(x)}g'\left(x\right). \\ & =\frac{3x^2+3}{x^3+3x-4} & \text{Rewrite.} \end{array}$$

Example:
Exercise:

Problem:
Using Properties of Logarithms in a Derivative

Find the derivative of $f\left(x\right)=\ln\left(\frac{x^2\sin x}{2x+1}\right)$.

Solution:

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.
Equation:

$$\begin{array}{lll} f\left(x\right) & = & \ln\left(\frac{x^2\sin x}{2x+1}\right)=2\ln x+\ln\left(\sin x\right)-\ln\left(2x+1\right) & \text{Apply properties of logarithms.} \\ f'\left(x\right) & = & \frac{2}{x}+\cot x-\frac{2}{2x+1} & \text{Apply sum rule and } h'\left(x\right)=\frac{1}{g(x)}g'\left(x\right). \end{array}$$

Note:
Exercise:

Problem: Differentiate: $f\left(x\right)=\ln(3x+2)^5$.

Solution:

$f'\left(x\right)=\frac{15}{3x+2}$

Hint

Use a property of logarithms to simplify before taking the derivative.

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y=log_bx$ and $y=b^x$ for $b>0,b\neq 1$.

Note:
Derivatives of General Exponential and Logarithmic Functions
Let $b > 0, b \neq 1$, and let $g\left(x\right)$ be a differentiable function.

i. If, $y = \log_{b}x$, then
   Equation:

$$\frac{dy}{dx} = \frac{1}{x\ln b}.$$

   More generally, if $h\left(x\right) = \log_{b}\left(g\left(x\right)\right)$, then for all values of x for which $g\left(x\right) > 0$,
   Equation:

$$h'\left(x\right) = \frac{g'\left(x\right)}{g\left(x\right)\ln b}.$$

ii. If $y = b^{x}$, then
   Equation:

$$\frac{dy}{dx} = b^{x}\ln b.$$

   More generally, if $h\left(x\right) = b^{g\left(x\right)}$, then
   Equation:

$$h'\left(x\right) = b^{g\left(x\right)}g\prime\left(x\right)\ln b.$$

Proof

If $y = \log_{b}x$, then $b^{y} = x$. It follows that $\ln\left(b^{y}\right) = \ln x$. Thus $y\ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that
Equation:

$$\frac{dy}{dx} = \frac{1}{x\ln b}.$$

The derivative in [link] now follows from the chain rule.

If $y = b^{x}$, then $\ln y = x\ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y}\frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^{x}$, we see that
Equation:

$$\frac{dy}{dx} = y\ln b = b^{x}\ln b.$$

The more general derivative ([link]) follows from the chain rule.

□

Example:
Exercise:

Problem:
Applying Derivative Formulas

Find the derivative of $h\left(x\right)=\frac{3^x}{3^x+2}$.

Solution:

Use the quotient rule and [link].
Equation:

$$\begin{array}{lll}h'\left(x\right) & = \frac{3^x \ln 3(3^x+2)-3^x \ln 3(3^x)}{(3^x+2)^2} & \text{Apply the quotient rule.} \\ & = \frac{2\cdot 3^x \ln 3}{(3^x+2)^2} & \text{Simplify.}\end{array}$$

Example:
Exercise:

Problem:
Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y=\log_2\left(3x+1\right)$ at $x=1$.

Solution:

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x=1$. Using [link], we see that
Equation:

$$\frac{dy}{dx}=\frac{3}{(3x+1)\ln 2}.$$

By evaluating the derivative at $x=1$, we see that the tangent line has slope
Equation:

$$\left.\frac{dy}{dx}\right|_{x=1}=\frac{3}{4\ln 2}=\frac{3}{\ln 16}.$$

Note:
Exercise:

Problem: Find the slope for the line tangent to $y=3^x$ at $x=2$.

Solution:

$9\ln\left(3\right)$

Hint

Evaluate the derivative at $x = 2$.

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g\left(x\right))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h\left(x\right) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Note:
Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h\left(x\right)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln\left(h\left(x\right)\right)$.
2. Use properties of logarithms to expand $\ln\left(h\left(x\right)\right)$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\ \frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h\left(x\right)$.

Example:
Exercise:

Problem:
Using Logarithmic Differentiation

Find the derivative of $y = \left(2x^4+1\right)^{\tan x}$.

Solution:

Use logarithmic differentiation to find this derivative.
Equation:

$$\begin{aligned}
\ln y &= \ln\left(2x^4+1\right)^{\tan x} && \text{Step 1. Take the natural logarithm o} \\
\ln y &= \tan x \ln\left(2x^4+1\right) && \text{Step 2. Expand using properties of lo} \\
\frac{1}{y}\frac{dy}{dx} &= \sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1} \cdot \tan x && \text{Step 3. Differentiate both sides. Use product rule on the right.} \\
\frac{dy}{dx} &= y \cdot \left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1} \cdot \tan x\right) && \text{Step 4. Multiply by } y \text{ on both sides.} \\
\frac{dy}{dx} &= \left(2x^4+1\right)^{\tan x}\left(\sec^2 x \ln\left(2x^4+1\right) + \frac{8x^3}{2x^4+1} \cdot \tan x\right) && \text{Step 5. Substitute } y = \left(2x^4+1\right)^{\tan x}.
\end{aligned}$$

Example:

Exercise:

Problem:
Using Logarithmic Differentiation

Find the derivative of $y=\frac{x\sqrt{2x+1}}{e^x\sin^3 x}$.

Solution:

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.
Equation:

$$\begin{array}{rcll}
\ln y & = & \ln\frac{x\sqrt{2x+1}}{e^x\sin^3 x} & \text{Step 1. Take the natural logarithm of both side} \\
\ln y & = & \ln x+\frac{1}{2}\ln\left(2x+1\right)-x\ln e-3\ln\sin x & \text{Step 2. Expand using properties of logarithms.} \\
\frac{1}{y}\frac{dy}{dx} & = & \frac{1}{x}+\frac{1}{2x+1}-1-3\frac{\cos x}{\sin x} & \text{Step 3. Differentiate both sides.} \\
\frac{dy}{dx} & = & y\left(\frac{1}{x}+\frac{1}{2x+1}-1-3\cot x\right) & \text{Step 4. Multiply by } y \text{ on both sides.} \\
\frac{dy}{dx} & = & \frac{x\sqrt{2x+1}}{e^x\sin^3 x}\left(\frac{1}{x}+\frac{1}{2x+1}-1-3\cot x\right) & \text{Step 5. Substitute } y=\frac{x\sqrt{2x+1}}{e^x\sin^3 x}.
\end{array}$$

Example:
Exercise:

Problem:
Extending the Power Rule

Find the derivative of $y=x^r$ where r is an arbitrary real number.

Solution:

The process is the same as in [link], though with fewer complications.
Equation:

$$\begin{array}{rcll}
\ln y & = & \ln x^r & \text{Step 1. Take the natural logarithm of both sides.} \\
\ln y & = & r\ln x & \text{Step 2. Expand using properties of logarithms.} \\
\frac{1}{y}\frac{dy}{dx} & = & r\frac{1}{x} & \text{Step 3. Differentiate both sides.} \\
\frac{dy}{dx} & = & y\frac{r}{x} & \text{Step 4. Multiply by } y \text{ on both sides.} \\
\frac{dy}{dx} & = & x^r\frac{r}{x} & \text{Step 5. Substitute } y=x^r. \\
\frac{dy}{dx} & = & rx^{r-1} & \text{Simplify.}
\end{array}$$

Note:
Exercise:

Problem: Use logarithmic differentiation to find the derivative of $y=x^x$.

Solution:

$\frac{dy}{dx} = x^x\left(1 + \ln x\right)$

Hint

Follow the problem solving strategy.

Note:
Exercise:

Problem: Find the derivative of $y = (\tan x)^{\pi}$.

Solution:

$y' = \pi(\tan x)^{\pi-1}\sec^2 x$

Hint

Use the result from [link].

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x, b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \frac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- **Derivative of the natural exponential function**
  $\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}g'\left(x\right)$
- **Derivative of the natural logarithmic function**
  $\frac{d}{dx}\left(\ln g\left(x\right)\right) = \frac{1}{g(x)}g'\left(x\right)$
- **Derivative of the general exponential function**
  $\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}g'\left(x\right)\ln b$
- **Derivative of the general logarithmic function**
  $\frac{d}{dx}\left(\log_b g\left(x\right)\right) = \frac{g'(x)}{g(x)\ln b}$

For the following exercises, find $f'\left(x\right)$ for each function.
Exercise:

Problem: $f\left(x\right) = x^2e^x$

Solution:

$2xe^x + x^2e^x$

Exercise:

Problem: $f\left(x\right)=\frac{e^{-x}}{x}$

Exercise:

Problem: $f\left(x\right)=e^{x^{3}\ln x}$

Solution:

$e^{x^{3}\ln x}\left(3x^{2}\ln x+x^{2}\right)$

Exercise:

Problem: $f\left(x\right)=\sqrt{e^{2x}+2x}$

Exercise:

Problem: $f\left(x\right)=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

Solution:

$\frac{4}{\left(e^{x}+e^{-x}\right)^{2}}$

Exercise:

Problem: $f\left(x\right)=\frac{10^{x}}{\ln 10}$

Exercise:

Problem: $f\left(x\right)=2^{4x}+4x^{2}$

Solution:

$2^{4x+2}\cdot\ln 2+8x$

Exercise:

Problem: $f\left(x\right)=3^{\sin 3\mathrm{x}}$

Exercise:

Problem: $f\left(x\right)=x^{\pi}\cdot\pi^{x}$

Solution:

$\pi x^{\pi-1}\cdot\pi^{x}+x^{\pi}\cdot\pi^{x}\ln\pi$

Exercise:

Problem: $f\left(x\right)=\ln\left(4x^{3}+x\right)$

Exercise:

Problem: $f\left(x\right)=\ln\sqrt{5x-7}$

Solution:

$\frac{5}{2(5x-7)}$

Exercise:

Problem: $f(x)=x^2 \ln 9x$

Exercise:

Problem: $f(x)=\log(\sec x)$

Solution:

$\frac{\tan x}{\ln 10}$

Exercise:

Problem: $f(x)=\log_7 \left(6x^4+3\right)^5$

Exercise:

Problem: $f(x)=2^x \cdot \log_3 7^{x^2-4}$

Solution:

$2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4}+2^x \cdot \frac{2x \ln 7}{\ln 3}$

For the following exercises, use logarithmic differentiation to find $\frac{dy}{dx}$.

Exercise:

Problem: $y=x^{\sqrt{x}}$

Exercise:

Problem: $y=(\sin 2x)^{4x}$

Solution:

$(\sin 2x)^{4x}\left[4 \cdot \ln(\sin 2x)+8x \cdot \cot 2x\right]$

Exercise:

Problem: $y=(\ln x)^{\ln x}$

Exercise:

Problem: $y=x^{\log_2 x}$

Solution:

$x^{\log_2 x} \cdot \frac{2 \ln x}{x \ln 2}$

Exercise:

Problem: $y=\left(x^2-1\right)^{\ln x}$

Exercise:

Problem: $y = x^{\cot x}$

Solution:

$x^{\cot x} \cdot \left[-\csc^2 x \cdot \ln x + \frac{\cot x}{x}\right]$

Exercise:

Problem: $y = \frac{x+11}{\sqrt[3]{x^2-4}}$

Exercise:

Problem: $y = x^{-1/2}\left(x^2+3\right)^{2/3}(3x-4)^4$

Solution:

$x^{-1/2}\left(x^2+3\right)^{2/3}(3x-4)^4 \cdot \left[\frac{-1}{2x} + \frac{4x}{3(x^2+3)} + \frac{12}{3x-4}\right]$

Exercise:

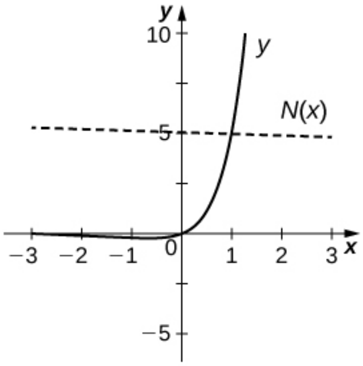
Problem: **[T]** Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where $x = -1$. Graph both the function and the tangent line.

Exercise:

Problem:

[T] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Solution:



$y = \frac{-1}{5+5\ln 5}x + \left(5 + \frac{1}{5+5\ln 5}\right)$

Exercise:

Problem:

[T] Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (*Hint*: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

Exercise:

Problem: Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.
b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Solution:

a. $x = e$~2.718 b. $(e, \infty), (0, e)$

Exercise:

Problem: The formula $I\left(t\right) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\frac{\pi}{2}$	(ii)
π	(iii)
$\frac{3\pi}{2}$	(iv)
2π	(v)
$\frac{5\pi}{2}$	(vi)
3π	(vii)
$\frac{7\pi}{2}$	(viii)
4π	(ix)

b. Using only the values in the table, determine where the tangent line to the graph of $I\left(t\right)$ is horizontal.

Exercise:

Problem:

[T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.
b. Use a. to determine the rate at which the population is increasing in t years.
c. Use b. to determine the rate at which the population is increasing in 10 years.

Solution:

a. $P = 500{,}000(1.05)^t$ individuals b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year c. $39{,}737$ individuals per year

Exercise:

Problem:

[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.
b. Use a. to determine the rate at which the substance is decaying in t hours.
c. Use b. to determine the rate of decay at $t = 4$ hours.

Exercise:

Problem:

[T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function

$N(t) = 5.3e^{0.093t^2 - 0.87t}$, $(0 \le t \le 4)$, where $N(t)$ gives the number of cases (in thousands) and *t* is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.
b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in New York City.

Solution:

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1964 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

Exercise:

Problem:

[T] The *relative rate of change* of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a, b, and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204, b = 0.0198$, and $c = 0.15$.
c. Briefly interpret what the result of b. means.

For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Years since 1790	**Population**
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.wikipedia.org/wiki/Largest_cities_in_the_United_States_by_population_by_decade.

Exercise:

Problem: **[T]** Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Solution:

$p = 35741(1.045)^t$

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

Exercise:

Problem:

[T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Solution:

Years since 1790	P''
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

Exercise:

Problem: **[T]** Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?
b. Estimate the population in 2010. Was the prediction correct from a.?

Chapter Review Exercises

True or False? Justify the answer with a proof or a counterexample.

Exercise:

Problem: Every function has a derivative.

Solution:

False.

Exercise:

Problem: A continuous function has a continuous derivative.

Exercise:

Problem: A continuous function has a derivative.

Solution:

False

Exercise:

Problem: If a function is differentiable, it is continuous.

Use the limit definition of the derivative to exactly evaluate the derivative.

Exercise:

Problem: $f\left(x\right)=\sqrt{x+4}$

Solution:

$\frac{1}{2\sqrt{x+4}}$

Exercise:

Problem: $f\left(x\right)=\frac{3}{x}$

Find the derivatives of the following functions.

Exercise:

Problem: $f\left(x\right)=3x^3-\frac{4}{x^2}$

Solution:

$9x^2+\frac{8}{x^3}$

Exercise:

Problem: $f\left(x\right)=\left(4-x^2\right)^3$

Exercise:

Problem: $f\left(x\right)=e^{\sin x}$

Solution:

$e^{\sin x}\cos x$

Exercise:

Problem: $f\left(x\right)=\ln\left(x+2\right)$

Exercise:

Problem: $f\left(x\right)=x^2\cos x+x\tan\left(x\right)$

Solution:

$x\sec^2\left(x\right)+2x\cos\left(x\right)+\tan\left(x\right)-x^2\sin\left(x\right)$

Exercise:

Problem: $f\left(x\right)=\sqrt{3x^2+2}$

Exercise:

Problem: $f\left(x\right)=\frac{x}{4}\sin^{-1}\left(x\right)$

Solution:

$\frac{1}{4}\left(\frac{x}{\sqrt{1-x^2}}+\sin^{-1}(x)\right)$

Exercise:

Problem: $x^2y=(y+2)+xy\sin(x)$

Find the following derivatives of various orders.

Exercise:

Problem: First derivative of $y=x\ln(x)\cos x$

Solution:

$\cos x\cdot(\ln x+1)-x\ln(x)\sin x$

Exercise:

Problem: Third derivative of $y=(3x+2)^2$

Exercise:

Problem: Second derivative of $y=4^x+x^2\sin(x)$

Solution:

$4^x(\ln 4)^2+2\sin x+4x\cos x-x^2\sin x$

Find the equation of the tangent line to the following equations at the specified point.

Exercise:

Problem: $y=\cos^{-1}(x)+x$ at $x=0$

Exercise:

Problem: $y=x+e^x-\frac{1}{x}$ at $x=1$

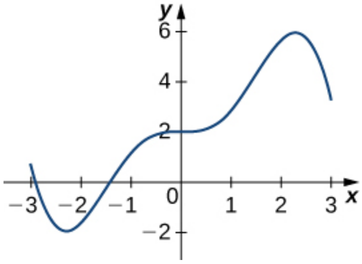
Solution:

$T=(2+e)x-2$

Draw the derivative for the following graphs.

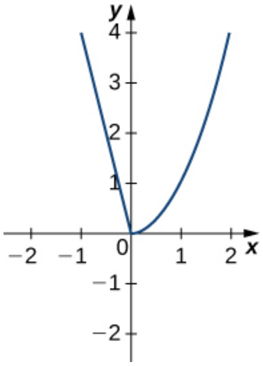
Exercise:

Problem:

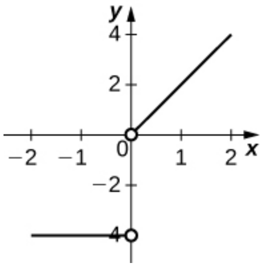


Exercise:

Problem:



Solution:



The following questions concern the water level in Ocean City, New Jersey, in January, which can be approximated by $w(t) = 1.9 + 2.9 \cos\left(\frac{\pi}{6} t\right)$, where t is measured in hours after midnight, and the height is measured in feet.

Exercise:

Problem: Find and graph the derivative. What is the physical meaning?

Exercise:

Problem: Find $w'(3)$. What is the physical meaning of this value?

Solution:

$w'(3) = -\frac{2.9\pi}{6}$. At 3 a.m. the tide is decreasing at a rate of 1.514 ft/hr.

The following questions consider the wind speeds of Hurricane Katrina, which affected New Orleans, Louisiana, in August 2005. The data are displayed in a table.

Hours after Midnight, August 26	Wind Speed (mph)
1	45
5	75
11	100
29	115
49	145
58	175
73	155
81	125
85	95
107	35

Wind Speeds of Hurricane KatrinaSource: http://news.nationalgeographic.com/news/2005/09/0914_050914_katrina_timeline.html.

Exercise:

Problem:

Using the table, estimate the derivative of the wind speed at hour 39. What is the physical meaning?

Exercise:

Problem: Estimate the derivative of the wind speed at hour 83. What is the physical meaning?

Solution:

-7.5. The wind speed is decreasing at a rate of 7.5 mph/hr

Glossary

logarithmic differentiation

is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

Related Rates

- Express changing quantities in terms of derivatives.
- Find relationships among the derivatives in a given problem.
- Use the chain rule to find the rate of change of one quantity that depends on the rate of change of other quantities.

We have seen that for quantities that are changing over time, the rates at which these quantities change are given by derivatives. If two related quantities are changing over time, the rates at which the quantities change are related. For example, if a balloon is being filled with air, both the radius of the balloon and the volume of the balloon are increasing. In this section, we consider several problems in which two or more related quantities are changing and we study how to determine the relationship between the rates of change of these quantities.

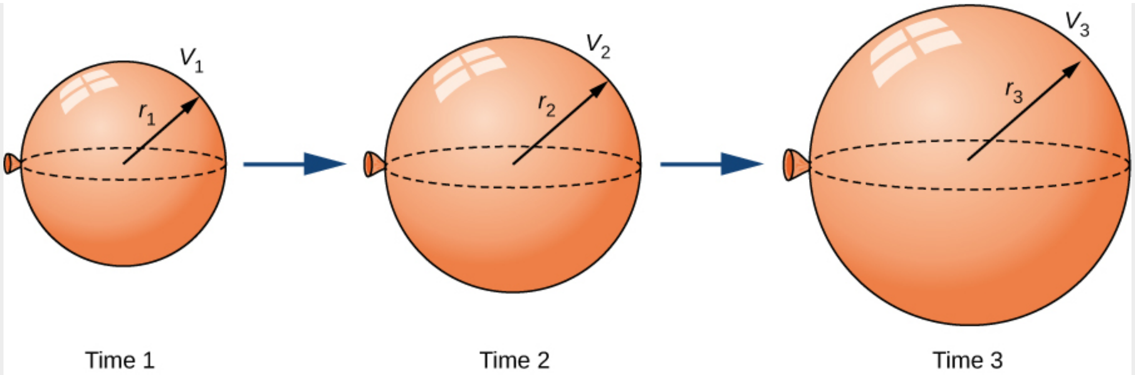
Setting up Related-Rates Problems

In many real-world applications, related quantities are changing with respect to time. For example, if we consider the balloon example again, we can say that the rate of change in the volume, V, is related to the rate of change in the radius, r. In this case, we say that $\frac{dV}{dt}$ and $\frac{dr}{dt}$ are **related rates** because V is related to r. Here we study several examples of related quantities that are changing with respect to time and we look at how to calculate one rate of change given another rate of change.

Example:
Exercise:

Problem:
Inflating a Balloon

A spherical balloon is being filled with air at the constant rate of $2\text{ cm}^3/\text{sec}$ ([link]). How fast is the radius increasing when the radius is 3 cm?



As the balloon is being filled with air, both the radius and the volume are increasing with respect to time.

Solution:

The volume of a sphere of radius r centimeters is

Equation:

$$V = \frac{4}{3}\pi r^3 \text{cm}^3.$$

Since the balloon is being filled with air, both the volume and the radius are functions of time. Therefore, t seconds after beginning to fill the balloon with air, the volume of air in the balloon is

Equation:

$$V(t) = \frac{4}{3}\pi [r(t)]^3 \text{cm}^3.$$

Differentiating both sides of this equation with respect to time and applying the chain rule, we see that the rate of change in the volume is related to the rate of change in the radius by the equation

Equation:

$$V\prime(t) = 4\pi [r\,(t)]^2 r'\,(t).$$

The balloon is being filled with air at the constant rate of 2 cm^3/sec, so $V\prime(t) = 2\text{ cm}^3/\text{sec}$. Therefore,

Equation:

$$2\text{cm}^3/\text{sec} = \left(4\pi[r(t)]^2\text{cm}^2\right) \cdot (r\prime(t)\text{cm/s}),$$

which implies

Equation:

$$r\prime(t) = \frac{1}{2\pi[r\,(t)]^2}\text{ cm/sec.}$$

When the radius $r = 3\text{ cm}$,

Equation:

$$r\prime(t) = \frac{1}{18\pi}\text{ cm/sec.}$$

Note:

Exercise:

Problem:

What is the instantaneous rate of change of the radius when $r = 6\text{ cm}$?

Solution:

$\frac{1}{72\pi}$ cm/sec, or approximately 0.0044 cm/sec

Hint

$$\frac{dr}{dt} = \frac{1}{2\pi r^2}$$

Before looking at other examples, let’s outline the problem-solving strategy we will be using to solve related-rates problems.

Note:
Problem-Solving Strategy: Solving a Related-Rates Problem

1. Assign symbols to all variables involved in the problem. Draw a figure if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1.
4. Using the chain rule, differentiate both sides of the equation found in step 3 with respect to the independent variable. This new equation will relate the derivatives.
5. Substitute all known values into the equation from step 4, then solve for the unknown rate of change.

Note that when solving a related-rates problem, it is crucial not to substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation found in step 4. We examine this potential error in the following example.

Examples of the Process

Let’s now implement the strategy just described to solve several related-rates problems. The first example involves a plane flying overhead. The relationship we are studying is between the speed of the plane and the rate at which the distance between the plane and a person on the ground is changing.

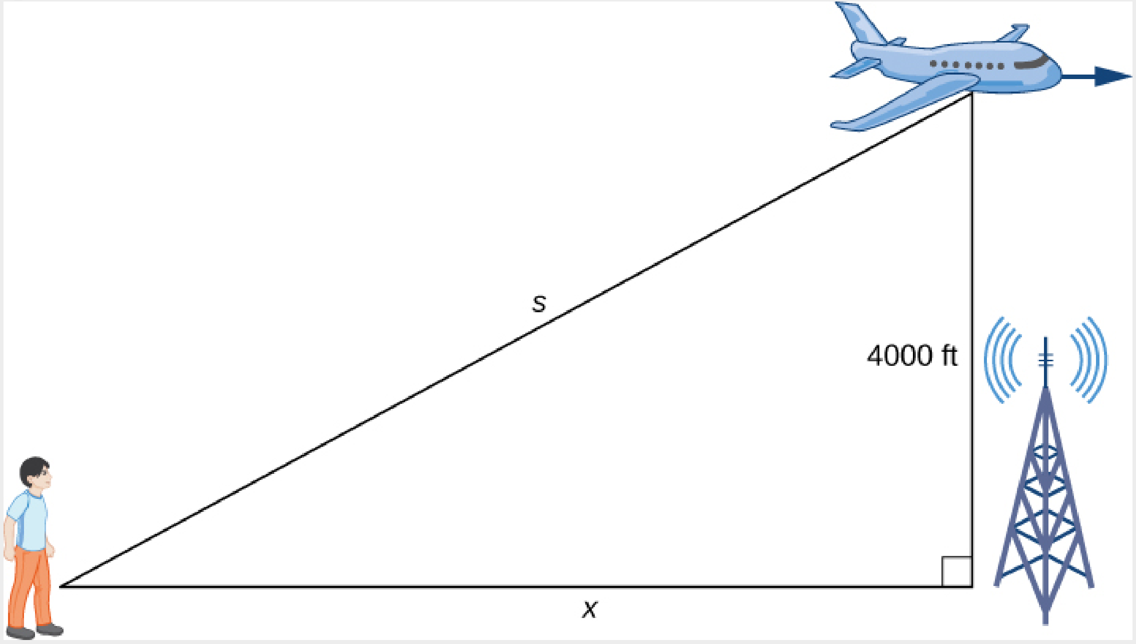
Example:
Exercise:

Problem:
An Airplane Flying at a Constant Elevation

An airplane is flying overhead at a constant elevation of 4000 ft. A man is viewing the plane from a position 3000 ft from the base of a radio tower. The airplane is flying horizontally away from the man. If the plane is flying at the rate of 600 ft/sec, at what rate is the distance between the man and the plane increasing when the plane passes over the radio tower?

Solution:

Step 1. Draw a picture, introducing variables to represent the different quantities involved.



An airplane is flying at a constant height of 4000 ft. The distance between the person and the airplane and the person and the place on the ground directly below the airplane are changing. We denote those quantities with the variables s and x, respectively.

As shown, x denotes the distance between the man and the position on the ground directly below the airplane. The variable s denotes the distance between the man and the plane. Note that both x and s are functions of time. We do not introduce a variable for the height of the plane because it remains at a constant elevation of $4000\,\text{ft}$. Since an object's height above the ground is measured as the shortest distance between the object and the ground, the line segment of length 4000 ft is perpendicular to the line segment of length x feet, creating a right triangle.

Step 2. Since x denotes the horizontal distance between the man and the point on the ground below the plane, dx/dt represents the speed of the plane. We are told the speed of the plane is 600 ft/sec. Therefore, $\frac{dx}{dt} = 600$ ft/sec. Since we are asked to find the rate of change in the distance between the man and the plane when the plane is directly above the radio tower, we need to find ds/dt when $x = 3000\,\text{ft}$.

Step 3. From the figure, we can use the Pythagorean theorem to write an equation relating x and s:

Equation:

$$[x(t)]^2 + 4000^2 = [s(t)]^2.$$

Step 4. Differentiating this equation with respect to time and using the fact that the derivative of a constant is zero, we arrive at the equation

Equation:

$$x\frac{dx}{dt} = s\frac{ds}{dt}.$$

Step 5. Find the rate at which the distance between the man and the plane is increasing when the plane is directly over the radio tower. That is, find $\frac{ds}{dt}$ when $x = 3000\,\text{ft}$. Since the speed of the plane is $600\,\text{ft/sec}$, we know that $\frac{dx}{dt} = 600\,\text{ft/sec}$. We are not given an explicit value for s; however, since we are trying to find $\frac{ds}{dt}$ when $x = 3000\,\text{ft}$, we can use the Pythagorean theorem to determine the distance s when $x = 3000$ and the height is $4000\,\text{ft}$. Solving the equation

Equation:

$$3000^2 + 4000^2 = s^2$$

for s, we have $s = 5000\,\text{ft}$ at the time of interest. Using these values, we conclude that ds/dt is a solution of the equation

Equation:

$$(3000)(600) = (5000) \cdot \frac{ds}{dt}.$$

Therefore,

Equation:

$$\frac{ds}{dt} = \frac{3000 \cdot 600}{5000} = 360\,\text{ft/sec}.$$

Note: When solving related-rates problems, it is important not to substitute values for the variables too soon. For example, in step 3, we related the variable quantities $x\left(t\right)$ and $s\left(t\right)$ by the equation

Equation:

$$\left[x\left(t\right)\right]^2 + 4000^2 = \left[s\left(t\right)\right]^2.$$

Since the plane remains at a constant height, it is not necessary to introduce a variable for the height, and we are allowed to use the constant 4000 to denote that quantity. However, the other two quantities are changing. If we mistakenly substituted $x(t) = 3000$ into the equation before differentiating, our equation would have been

Equation:

$$3000^2 + 4000^2 = [s(t)]^2.$$

After differentiating, our equation would become

Equation:

$$0 = s(t)\frac{ds}{dt}.$$

As a result, we would incorrectly conclude that $\frac{ds}{dt} = 0.$

Note:

Exercise:

Problem:

What is the speed of the plane if the distance between the person and the plane is increasing at the rate of $300\,\text{ft/sec}$?

Solution:

$500\,\text{ft/sec}$

Hint

$\frac{ds}{dt} = 300\,\text{ft/sec}$

We now return to the problem involving the rocket launch from the beginning of the chapter.

Example:
Exercise:

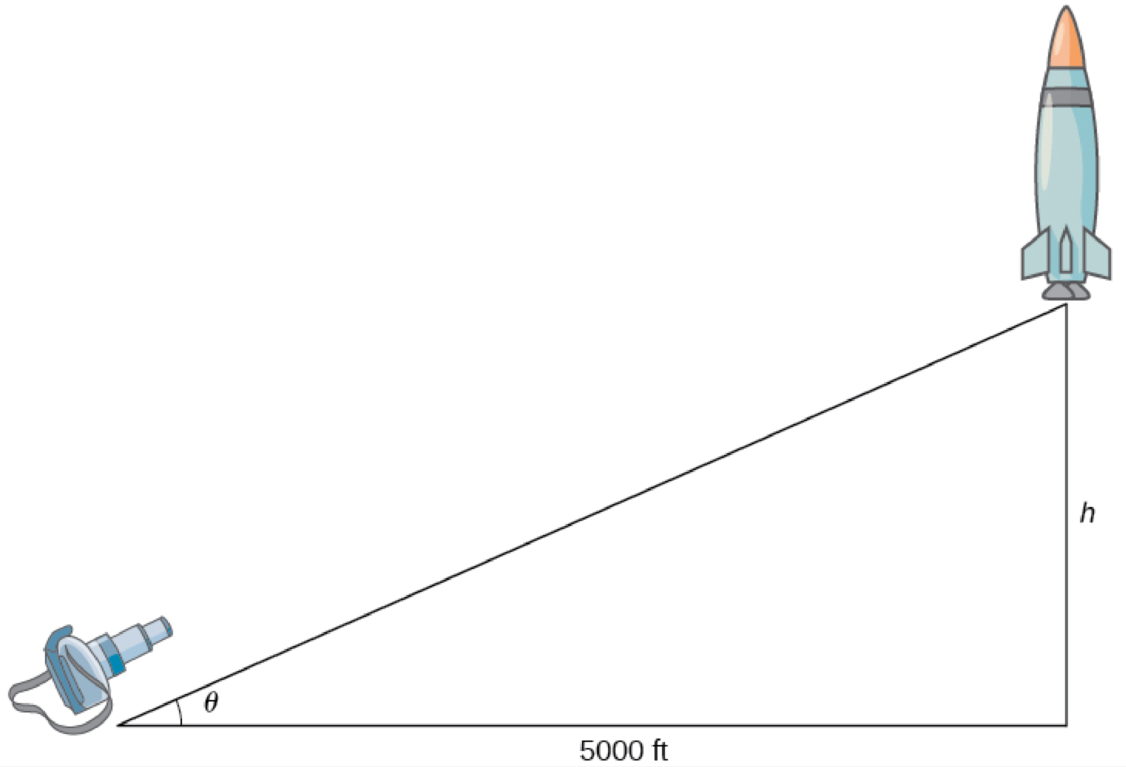
Problem:
Chapter Opener: A Rocket Launch

(credit: modification of work by Steve Jurvetson, Wikimedia Commons)

A rocket is launched so that it rises vertically. A camera is positioned 5000 ft from the launch pad. When the rocket is 1000 ft above the launch pad, its velocity is 600 ft/sec. Find the necessary rate of change of the camera’s angle as a function of time so that it stays focused on the rocket.

Solution:

Step 1. Draw a picture introducing the variables.



A camera is positioned 5000 ft from the launch pad of the rocket. The height of the rocket and the angle of the camera are changing with respect to time. We denote those quantities with the variables h and θ, respectively.

Let h denote the height of the rocket above the launch pad and θ be the angle between the camera lens and the ground.

Step 2. We are trying to find the rate of change in the angle of the camera with respect to time when the rocket is 1000 ft off the ground. That is, we need to find $\frac{d\theta}{dt}$ when $h = 1000\,\text{ft}$. At that time, we know the velocity of the rocket is $\frac{dh}{dt} = 600\,\text{ft/sec}$.

Step 3. Now we need to find an equation relating the two quantities that are changing with respect to time: h and θ. How can we create such an equation? Using the fact that we have drawn a right triangle, it is natural to think about trigonometric functions. Recall that $\tan\theta$ is the ratio of the length of the opposite side of the triangle to the length of the adjacent side. Thus, we have

Equation:

$$\tan\theta = \frac{h}{5000}.$$

This gives us the equation

Equation:

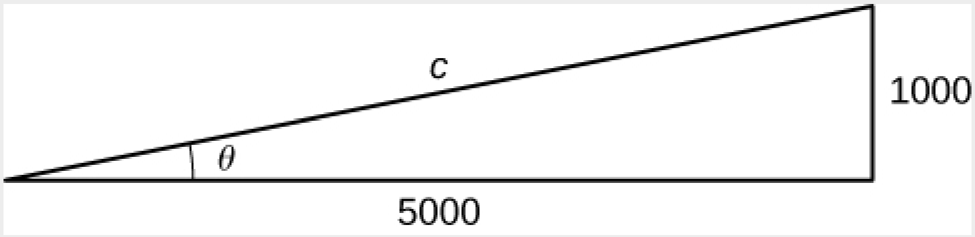
$$h = 5000\tan\theta.$$

Step 4. Differentiating this equation with respect to time t, we obtain

Equation:

$$\frac{dh}{dt} = 5000\sec^2\theta\,\frac{d\theta}{dt}.$$

Step 5. We want to find $\frac{d\theta}{dt}$ when $h = 1000\,\text{ft}$. At this time, we know that $\frac{dh}{dt} = 600\,\text{ft/sec}$. We need to determine $\sec^2\theta$. Recall that $\sec\theta$ is the ratio of the length of the hypotenuse to the length of the adjacent side. We know the length of the adjacent side is $5000\,\text{ft}$. To determine the length of the hypotenuse, we use the Pythagorean theorem, where the length of one leg is $5000\,\text{ft}$, the length of the other leg is $h = 1000\,\text{ft}$, and the length of the hypotenuse is c feet as shown in the following figure.



We see that

Equation:

$$1000^2 + 5000^2 = c^2$$

and we conclude that the hypotenuse is

Equation:

$$c = 1000\sqrt{26}\,\text{ft}.$$

Therefore, when $h = 1000$, we have

Equation:

$$\sec^2\theta = \left(\frac{1000\sqrt{26}}{5000}\right)^2 = \frac{26}{25}.$$

Recall from step 4 that the equation relating $\frac{d\theta}{dt}$ to our known values is

Equation:

$$\frac{dh}{dt} = 5000\sec^2\theta\,\frac{d\theta}{dt}.$$

When $h = 1000\,\text{ft}$, we know that $\frac{dh}{dt} = 600\,\text{ft/sec}$ and $\sec^2\theta = \frac{26}{25}$. Substituting these values into the previous equation, we arrive at the equation

Equation:

$$600 = 5000\left(\frac{26}{25}\right)\frac{d\theta}{dt}.$$

Therefore, $\frac{d\theta}{dt} = \frac{3}{26}\,\text{rad/sec}.$

Note:
Exercise:

Problem:

What rate of change is necessary for the elevation angle of the camera if the camera is placed on the ground at a distance of 4000 ft from the launch pad and the velocity of the rocket is 500 ft/sec when the rocket is 2000 ft off the ground?

Solution:

$\frac{1}{10}\text{ rad/sec}$

Hint

Find $\frac{d\theta}{dt}$ when $h = 2000\text{ ft}$. At that time, $\frac{dh}{dt} = 500\text{ ft/sec}$.

In the next example, we consider water draining from a cone-shaped funnel. We compare the rate at which the level of water in the cone is decreasing with the rate at which the volume of water is decreasing.

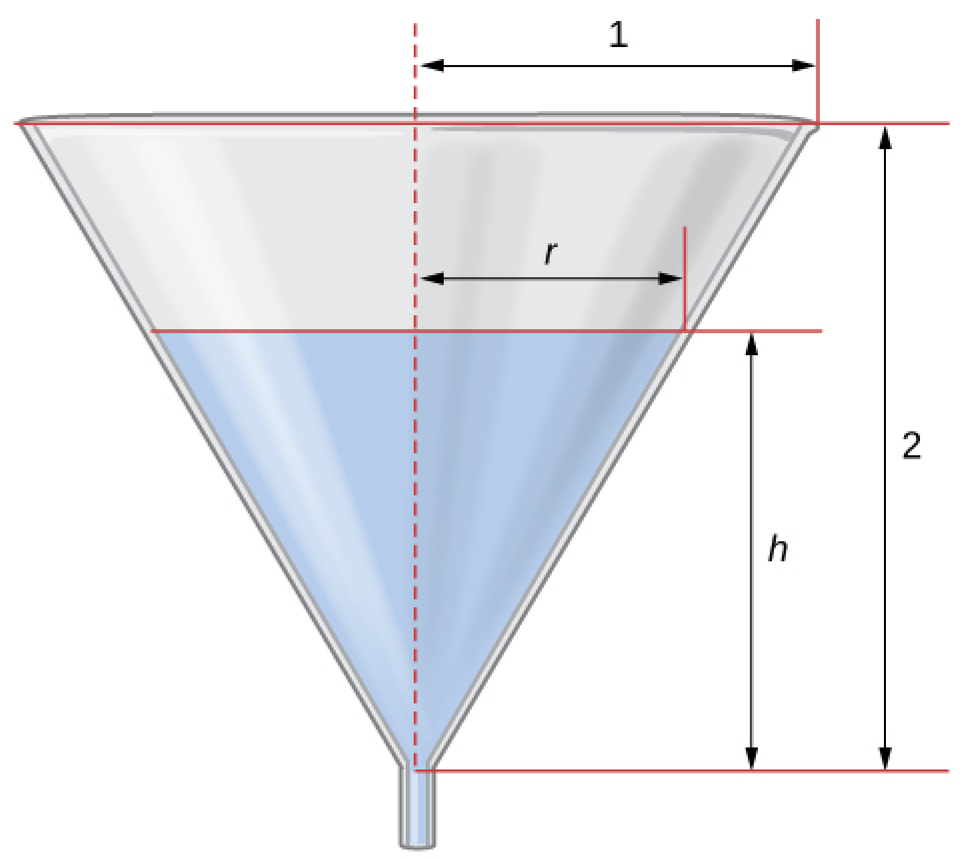
Example:
Exercise:

Problem:
Water Draining from a Funnel

Water is draining from the bottom of a cone-shaped funnel at the rate of $0.03\text{ ft}^3\text{/sec}$. The height of the funnel is 2 ft and the radius at the top of the funnel is 1 ft. At what rate is the height of the water in the funnel changing when the height of the water is $\frac{1}{2}\text{ ft}$?

Solution:

Step 1: Draw a picture introducing the variables.



Water is draining from a funnel of height 2 ft and radius 1 ft. The height of the water and the radius of water are changing over time. We denote these quantities with the variables h and r, respectively.

Let h denote the height of the water in the funnel, r denote the radius of the water at its surface, and V denote the volume of the water.

Step 2: We need to determine $\frac{dh}{dt}$ when $h = \frac{1}{2}\,\text{ft}$. We know that $\frac{dV}{dt} = -0.03\,\text{ft}/\text{sec}$.

Step 3: The volume of water in the cone is

Equation:

$$V = \frac{1}{3}\pi r^2 h.$$

From the figure, we see that we have similar triangles. Therefore, the ratio of the sides in the two triangles is the same. Therefore, $\frac{r}{h} = \frac{1}{2}$ or $r = \frac{h}{2}$. Using this fact, the equation for volume can be simplified to

Equation:

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3.$$

Step 4: Applying the chain rule while differentiating both sides of this equation with respect to time t, we obtain

Equation:

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt}.$$

Step 5: We want to find $\frac{dh}{dt}$ when $h = \frac{1}{2}$ ft. Since water is leaving at the rate of $0.03\,\text{ft}^3/\text{sec}$, we know that $\frac{dV}{dt} = -0.03\,\text{ft}^3/\text{sec}$. Therefore,

Equation:

$$-0.03 = \frac{\pi}{4}\left(\frac{1}{2}\right)^2\frac{dh}{dt},$$

which implies

Equation:

$$-0.03 = \frac{\pi}{16}\frac{dh}{dt}.$$

It follows that

Equation:

$$\frac{dh}{dt} = -\frac{0.48}{\pi} = -0.153\,\text{ft/sec.}$$

Note:

Exercise:

Problem:

At what rate is the height of the water changing when the height of the water is $\frac{1}{4}$ ft?

Solution:

$-0.61\,\text{ft/sec}$

Hint

We need to find $\frac{dh}{dt}$ when $h = \frac{1}{4}$.

Key Concepts

- To solve a related rates problem, first draw a picture that illustrates the relationship between the two or more related quantities that are changing with respect to time.
- In terms of the quantities, state the information given and the rate to be found.
- Find an equation relating the quantities.
- Use differentiation, applying the chain rule as necessary, to find an equation that relates the rates.

- Be sure not to substitute a variable quantity for one of the variables until after finding an equation relating the rates.

For the following exercises, find the quantities for the given equation.

Exercise:

Problem: Find $\frac{dy}{dt}$ at $x=1$ and $y=x^2+3$ if $\frac{dx}{dt}=4$.

Solution:

8

Exercise:

Problem: Find $\frac{dx}{dt}$ at $x=-2$ and $y=2x^2+1$ if $\frac{dy}{dt}=-1$.

Exercise:

Problem:

Find $\frac{dz}{dt}$ at $(x,y)=(1,3)$ and $z^2=x^2+y^2$ if $\frac{dx}{dt}=4$ and $\frac{dy}{dt}=3$.

Solution:

$\frac{13}{\sqrt{10}}$

For the following exercises, sketch the situation if necessary and used related rates to solve for the quantities.

Exercise:

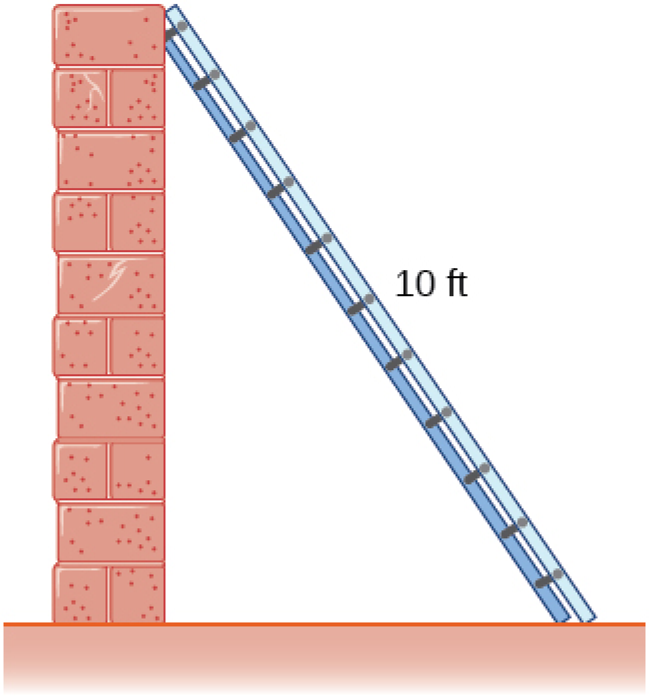
Problem:

[T] If two electrical resistors are connected in parallel, the total resistance (measured in ohms, denoted by the Greek capital letter omega, Ω) is given by the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at a rate of $0.5\ \Omega/\text{min}$ and R_2 decreases at a rate of $1.1\Omega/\text{min}$, at what rate does the total resistance change when $R_1 = 20\Omega$ and $R_2 = 50\Omega$?

Exercise:

Problem:

A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 ft/sec, how fast is the bottom moving along the ground when the bottom of the ladder is 5 ft from the wall?



Solution:

$2\sqrt{3}$ ft/sec

Exercise:

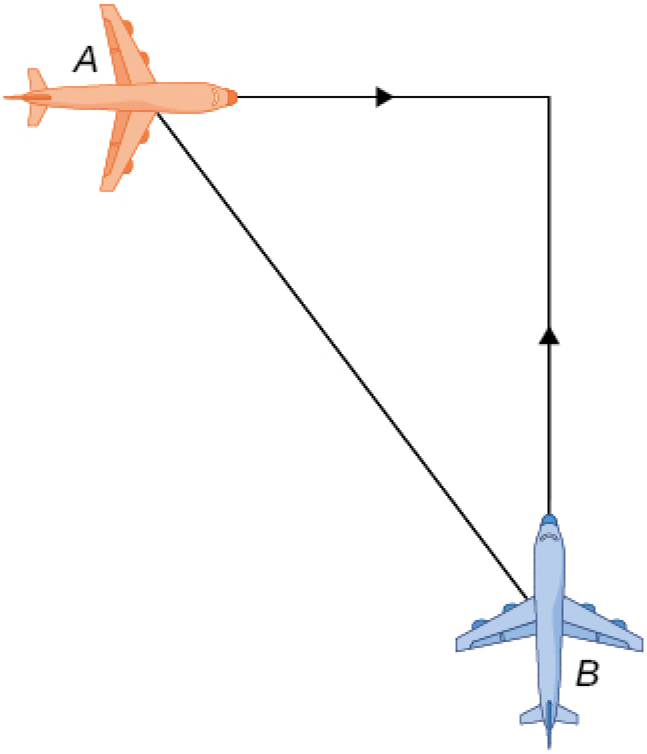
Problem:

A 25-ft ladder is leaning against a wall. If we push the ladder toward the wall at a rate of 1 ft/sec, and the bottom of the ladder is initially 20 ft away from the wall, how fast does the ladder move up the wall 5 sec after we start pushing?

Exercise:

Problem:

Two airplanes are flying in the air at the same height: airplane *A* is flying east at 250 mi/h and airplane *B* is flying north at $300\ \text{mi/h}$. If they are both heading to the same airport, located 30 miles east of airplane *A* and 40 miles north of airplane *B*, at what rate is the distance between the airplanes changing?



Solution:

The distance is decreasing at $390\ \mathrm{mi/h}$.

Exercise:

Problem:

You and a friend are riding your bikes to a restaurant that you think is east; your friend thinks the restaurant is north. You both leave from the same point, with you riding at 16 mph east and your friend riding $12\ \mathrm{mph}$ north. After you traveled $4\ \mathrm{mi}$, at what rate is the distance between you changing?

Exercise:

Problem:

Two buses are driving along parallel freeways that are $5\ \mathrm{mi}$ apart, one heading east and the other heading west. Assuming that each bus drives a constant $55\ \mathrm{mph}$, find the rate at which the distance between the buses is changing when they are $13\ \mathrm{mi}$ apart, heading toward each other.

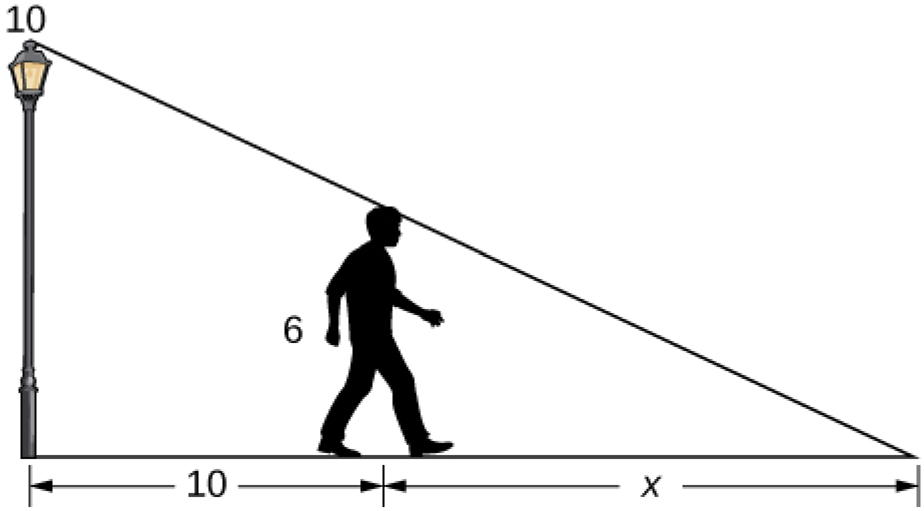
Solution:

The distance between them shrinks at a rate of $\frac{1320}{13} \approx 101.5\ \mathrm{mph}$.

Exercise:

Problem:

A 6-ft-tall person walks away from a 10-ft lamppost at a constant rate of $3\ \mathrm{ft/sec}$. What is the rate that the tip of the shadow moves away from the pole when the person is $10\ \mathrm{ft}$ away from the pole?



Exercise:

Problem:

Using the previous problem, what is the rate at which the tip of the shadow moves away from the person when the person is 10 ft from the pole?

Solution:

$\frac{9}{2}$ ft/sec

Exercise:

Problem:

A 5-ft-tall person walks toward a wall at a rate of 2 ft/sec. A spotlight is located on the ground 40 ft from the wall. How fast does the height of the person’s shadow on the wall change when the person is 10 ft from the wall?

Exercise:

Problem:

Using the previous problem, what is the rate at which the shadow changes when the person is 10 ft from the wall, if the person is walking away from the wall at a rate of 2 ft/sec?

Solution:

It grows at a rate $\frac{4}{9}$ ft/sec

Exercise:

Problem:

A helicopter starting on the ground is rising directly into the air at a rate of 25 ft/sec. You are running on the ground starting directly under the helicopter at a rate of 10 ft/sec. Find the rate of change of the distance between the helicopter and yourself after 5 sec.

Exercise:

Problem:

Using the previous problem, what is the rate at which the distance between you and the helicopter is changing when the helicopter has risen to a height of 60 ft in the air, assuming that, initially, it was 30 ft above you?

Solution:

The distance is increasing at $\frac{\left(135\sqrt{26}\right)}{26}$ ft/sec

For the following exercises, draw and label diagrams to help solve the related-rates problems.

Exercise:

Problem:

The side of a cube increases at a rate of $\frac{1}{2}$ m/sec. Find the rate at which the volume of the cube increases when the side of the cube is 4 m.

Exercise:

Problem:

The volume of a cube decreases at a rate of 10 m^3/s. Find the rate at which the side of the cube changes when the side of the cube is 2 m.

Solution:

$-\frac{5}{6}$ m/sec

Exercise:

Problem:

The radius of a circle increases at a rate of 2 m/sec. Find the rate at which the area of the circle increases when the radius is 5 m.

Exercise:

Problem:

The radius of a sphere decreases at a rate of 3 m/sec. Find the rate at which the surface area decreases when the radius is 10 m.

Solution:

240π m^2/sec

Exercise:

Problem:

The radius of a sphere increases at a rate of 1 m/sec. Find the rate at which the volume increases when the radius is 20 m.

Exercise:

Problem:

The radius of a sphere is increasing at a rate of 9 cm/sec. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate.

Solution:

$\frac{1}{2\sqrt{\pi}}$ cm

Exercise:

Problem:

The base of a triangle is shrinking at a rate of 1 cm/min and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when the height is 22 cm and the base is 10 cm.

Exercise:

Problem:

A triangle has two constant sides of length 3 ft and 5 ft. The angle between these two sides is increasing at a rate of 0.1 rad/sec. Find the rate at which the area of the triangle is changing when the angle between the two sides is $\pi/6$.

Solution:

The area is increasing at a rate $\frac{\left(3\sqrt{3}\right)}{8}\ \text{ft}^2/\text{sec}$.

Exercise:

Problem:

A triangle has a height that is increasing at a rate of 2 cm/sec and its area is increasing at a rate of 4 cm^2/sec. Find the rate at which the base of the triangle is changing when the height of the triangle is 4 cm and the area is 20 cm^2.

For the following exercises, consider a right cone that is leaking water. The dimensions of the conical tank are a height of 16 ft and a radius of 5 ft.

Exercise:

Problem:

How fast does the depth of the water change when the water is 10 ft high if the cone leaks water at a rate of 10 ft^3/min?

Solution:

The depth of the water decreases at $\frac{128}{125\pi}$ ft/min.

Exercise:

Problem:

Find the rate at which the surface area of the water changes when the water is 10 ft high if the cone leaks water at a rate of 10 ft^3/min.

Exercise:

Problem:

If the water level is decreasing at a rate of 3 in/min when the depth of the water is 8 ft, determine the rate at which water is leaking out of the cone.

Solution:

The volume is decreasing at a rate of $\frac{(25\pi)}{16}\text{ft}^3/\text{min}$.

Exercise:

Problem:

A vertical cylinder is leaking water at a rate of 1 ft^3/sec. If the cylinder has a height of 10 ft and a radius of 1 ft, at what rate is the height of the water changing when the height is 6 ft?

Exercise:

Problem:

A cylinder is leaking water but you are unable to determine at what rate. The cylinder has a height of 2 m and a radius of 2 m. Find the rate at which the water is leaking out of the cylinder if the rate at which the height is decreasing is 10 cm/min when the height is 1 m.

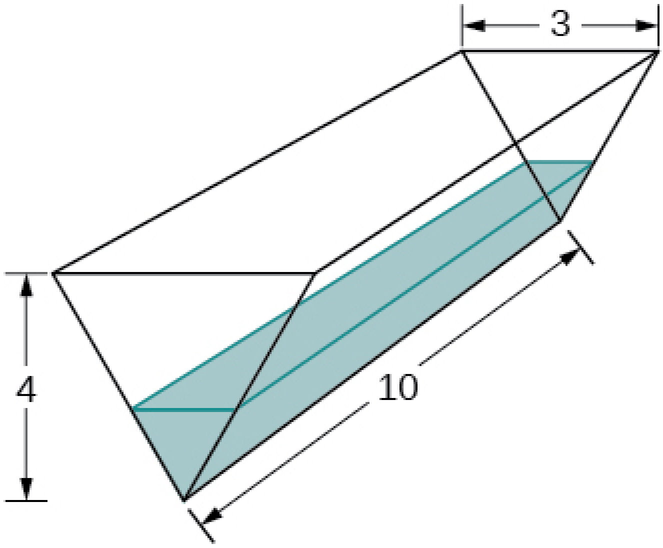
Solution:

The water flows out at rate $\frac{(2\pi)}{5}\ \mathrm{m}^3/\mathrm{min}$.

Exercise:

Problem:

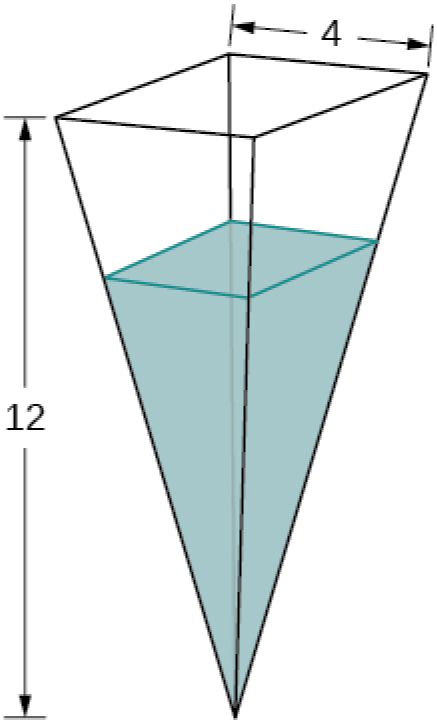
A trough has ends shaped like isosceles triangles, with width 3 m and height 4 m, and the trough is 10 m long. Water is being pumped into the trough at a rate of $5\ \mathrm{m}^3/\mathrm{min}$. At what rate does the height of the water change when the water is 1 m deep?



Exercise:

Problem:

A tank is shaped like an upside-down square pyramid, with base of 4 m by 4 m and a height of 12 m (see the following figure). How fast does the height increase when the water is 2 m deep if water is being pumped in at a rate of $\frac{2}{3}$ m^3/sec?



Solution:

$\frac{3}{2}$ m/sec

For the following problems, consider a pool shaped like the bottom half of a sphere, that is being filled at a rate of 25 ft^3/min. The radius of the pool is 10 ft.
Exercise:

Problem:

Find the rate at which the depth of the water is changing when the water has a depth of 5 ft.

Exercise:

Problem:

Find the rate at which the depth of the water is changing when the water has a depth of 1 ft.

Solution:

$\frac{25}{19\pi}$ ft/min

Exercise:

Problem:

If the height is increasing at a rate of 1 in./min when the depth of the water is 2 ft, find the rate at which water is being pumped in.

Exercise:

Problem:

Gravel is being unloaded from a truck and falls into a pile shaped like a cone at a rate of 10 ft^3/min. The radius of the cone base is three times the height of the cone. Find the rate at which the height of the gravel changes when the pile has a height of 5 ft.

Solution:

$\frac{2}{45\pi}$ ft/min

Exercise:

Problem:

Using a similar setup from the preceding problem, find the rate at which the gravel is being unloaded if the pile is 5 ft high and the height is increasing at a rate of 4 in./min.

For the following exercises, draw the situations and solve the related-rate problems.

Exercise:

Problem:

You are stationary on the ground and are watching a bird fly horizontally at a rate of 10 m/sec. The bird is located 40 m above your head. How fast does the angle of elevation change when the horizontal distance between you and the bird is 9 m?

Solution:

The angle decreases at $\frac{400}{1681}\ \text{rad/sec}.$

Exercise:

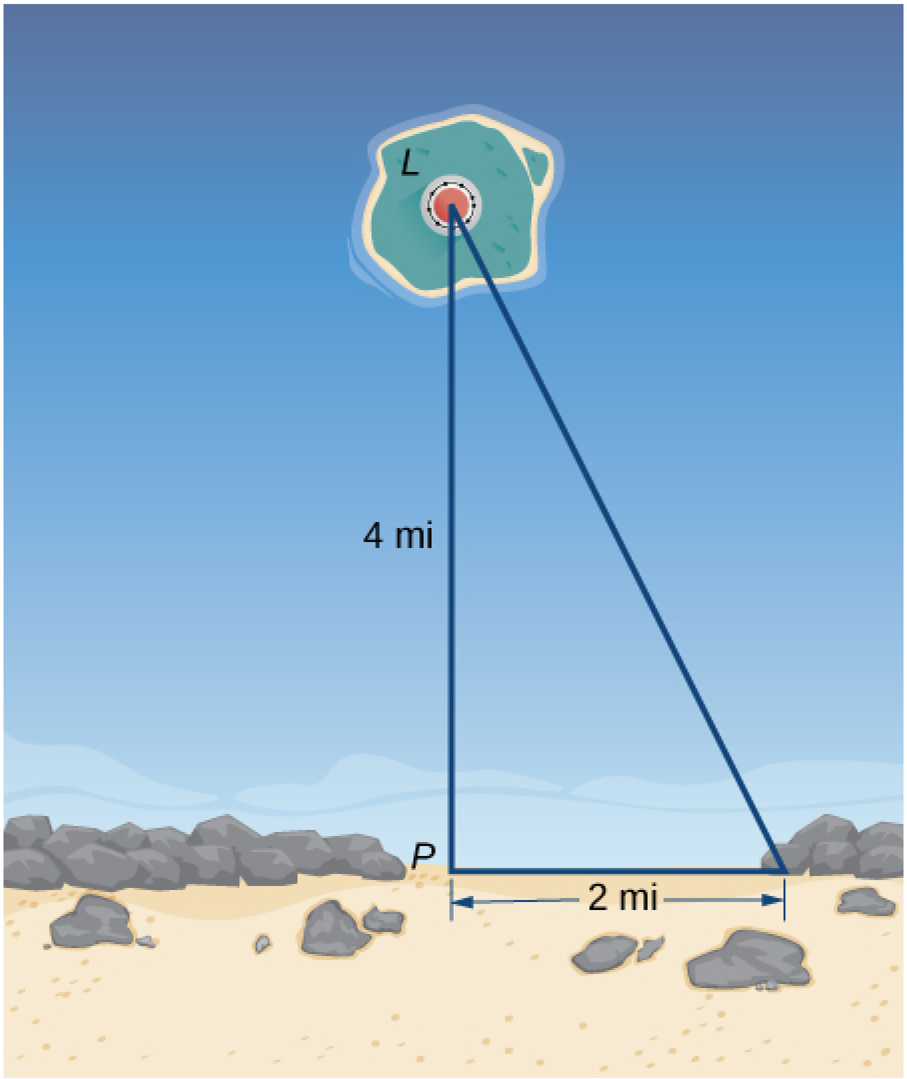
Problem:

You stand 40 ft from a bottle rocket on the ground and watch as it takes off vertically into the air at a rate of 20 ft/sec. Find the rate at which the angle of elevation changes when the rocket is 30 ft in the air.

Exercise:

Problem:

A lighthouse, *L*, is on an island 4 mi away from the closest point, *P*, on the beach (see the following image). If the lighthouse light rotates clockwise at a constant rate of 10 revolutions/min, how fast does the beam of light move across the beach 2 mi away from the closest point on the beach?



Solution:

$100\pi \text{mi/min}$

Exercise:

Problem:

Using the same setup as the previous problem, determine at what rate the beam of light moves across the beach 1 mi away from the closest point on the beach.

Exercise:

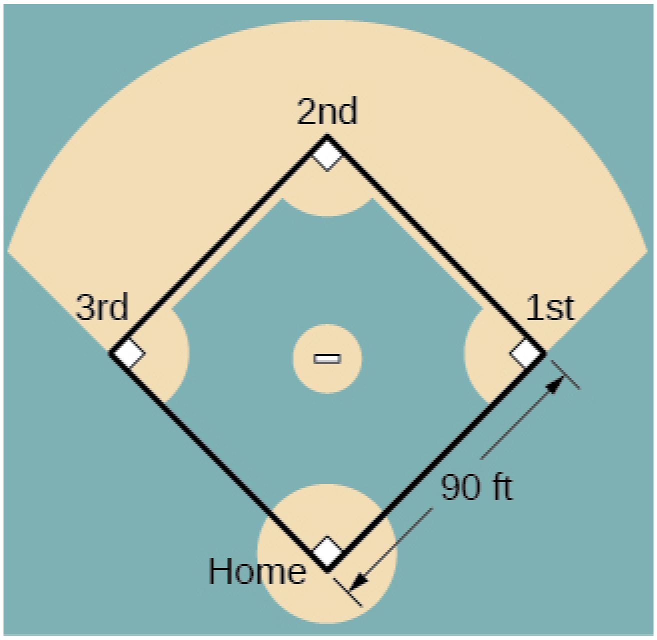
Problem:

You are walking to a bus stop at a right-angle corner. You move north at a rate of 2 m/sec and are 20 m south of the intersection. The bus travels west at a rate of 10 m/sec away from the intersection – you have missed the bus! What is the rate at which the angle between you and the bus is changing when you are 20 m south of the intersection and the bus is 10 m west of the intersection?

Solution:

The angle is changing at a rate of $\frac{11}{25}\ \text{rad/sec}$.

For the following exercises, refer to the figure of baseball diamond, which has sides of 90 ft.



Exercise:

Problem:

[T] A batter hits a ball toward third base at 75 ft/sec and runs toward first base at a rate of 24 ft/sec. At what rate does the distance between the ball and the batter change when 2 sec have passed?

Exercise:

Problem:

[T] A batter hits a ball toward second base at 80 ft/sec and runs toward first base at a rate of 30 ft/sec. At what rate does the distance between the ball and the batter change when the runner has covered one-third of the distance to first base? (*Hint*: Recall the law of cosines.)

Solution:

The distance is increasing at a rate of 62.50 ft/sec.

Exercise:

Problem:

[T] A batter hits the ball and runs toward first base at a speed of 22 ft/sec. At what rate does the distance between the runner and second base change when the runner has run 30 ft?

Exercise:

Problem:

[T] Runners start at first and second base. When the baseball is hit, the runner at first base runs at a speed of 18 ft/sec toward second base and the runner at second base runs at a speed of 20 ft/sec toward third base. How fast is the distance between runners changing 1 sec after the ball is hit?

Solution:

The distance is decreasing at a rate of 11.99 ft/sec.

Glossary

related rates

are rates of change associated with two or more related quantities that are changing over time

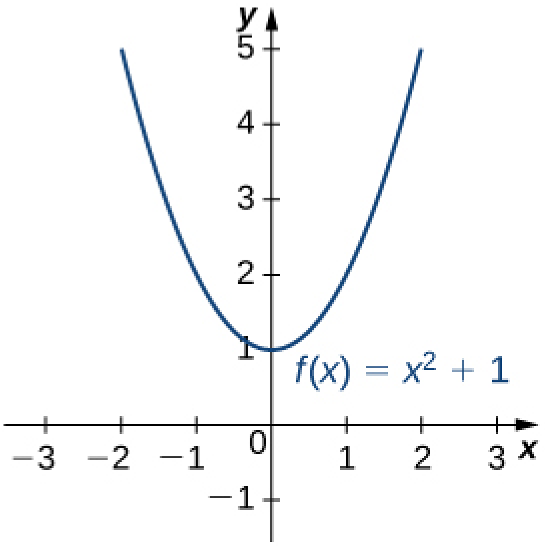
Maxima and Minima

- Define absolute extrema.
- Define local extrema.
- Explain how to find the critical points of a function over a closed interval.
- Describe how to use critical points to locate absolute extrema over a closed interval.

Given a particular function, we are often interested in determining the largest and smallest values of the function. This information is important in creating accurate graphs. Finding the maximum and minimum values of a function also has practical significance because we can use this method to solve optimization problems, such as maximizing profit, minimizing the amount of material used in manufacturing an aluminum can, or finding the maximum height a rocket can reach. In this section, we look at how to use derivatives to find the largest and smallest values for a function.

Absolute Extrema

Consider the function $f(x) = x^2 + 1$ over the interval $(-\infty, \infty)$. As $x \to \pm\infty$, $f(x) \to \infty$. Therefore, the function does not have a largest value. However, since $x^2 + 1 \geq 1$ for all real numbers x and $x^2 + 1 = 1$ when $x = 0$, the function has a smallest value, 1, when $x = 0$. We say that 1 is the absolute minimum of $f(x) = x^2 + 1$ and it occurs at $x = 0$. We say that $f(x) = x^2 + 1$ does not have an absolute maximum (see the following figure).



The given function has an absolute minimum of 1 at $x = 0$. The function does not have an absolute maximum.

Note:
Definition
Let f be a function defined over an interval I and let $c \in I$. We say f has an **absolute maximum** on I at c if $f(c) \geq f(x)$ for all $x \in I$. We say f has an **absolute minimum** on I at c if $f(c) \leq f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an **absolute extremum** on I at c.

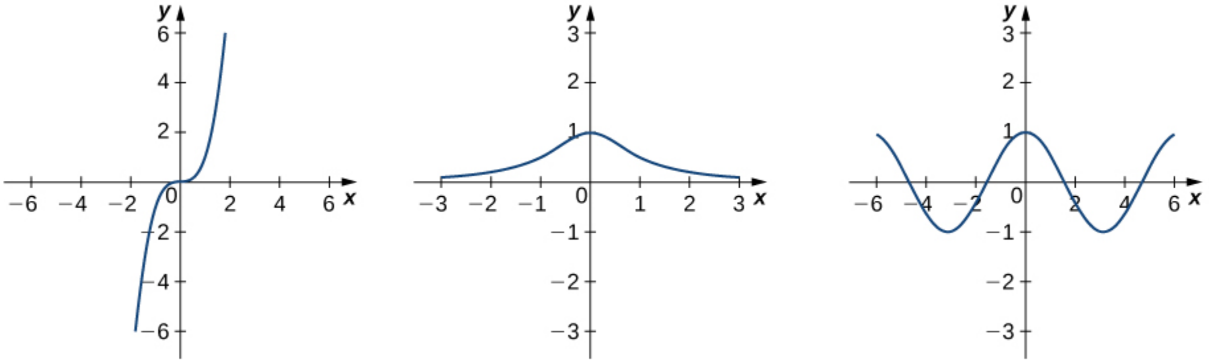
Before proceeding, let's note two important issues regarding this definition. First, the term *absolute* here does not refer to absolute value. An absolute extremum may be positive, negative, or zero. Second, if a function f has an absolute extremum over an interval I at c, the absolute extremum is $f(c)$. The real number c is a point in the domain at which the absolute extremum occurs. For example, consider the function $f(x) = 1/(x^2 + 1)$ over the interval $(-\infty, \infty)$. Since

Equation:

$$f(0) = 1 \geq \frac{1}{x^2 + 1} = f(x)$$

for all real numbers x, we say f has an absolute maximum over $(-\infty, \infty)$ at $x = 0$. The absolute maximum is $f(0) = 1$. It occurs at $x = 0$, as shown in [link](b).

A function may have both an absolute maximum and an absolute minimum, just one extremum, or neither. [link] shows several functions and some of the different possibilities regarding absolute extrema. However, the following theorem, called the **Extreme Value Theorem**, guarantees that a continuous function f over a closed, bounded interval $[a, b]$ has both an absolute maximum and an absolute minimum.



$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

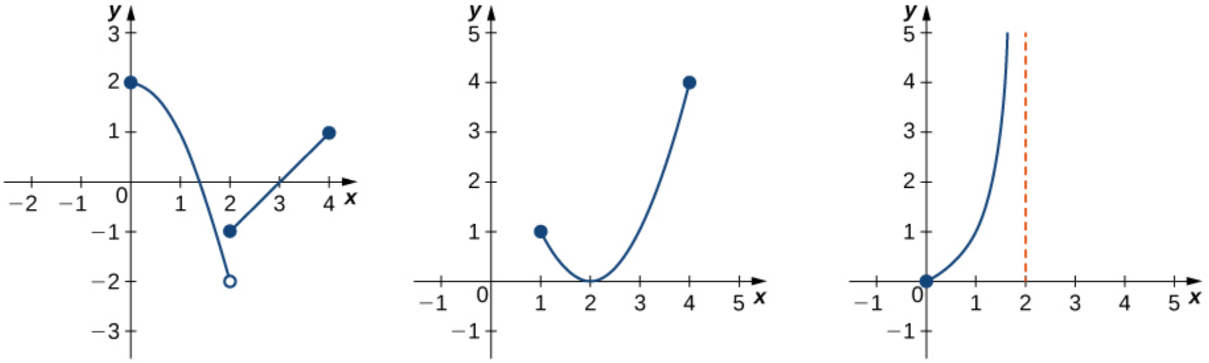
(a)

$f(x) = \frac{1}{x^2+1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)

$f(x) = \cos(x)$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0, \pm 2\pi, \pm 4\pi\ldots$
Absolute minimum of -1 at $x = \pm\pi, \pm 3\pi\ldots$

(c)



$f(x) = \begin{cases} 2 - x^2 & 0 \le x < 2 \\ x - 3 & 2 \le x \le 4 \end{cases}$
Absolute maximum of 2 at $x = 0$
No absolute minimum

(d)

$f(x) = (x-2)^2$ on $[1, 4]$
Absolute maximum of 4 at $x = 4$
Absolute minimum of 0 at $x = 2$

(e)

$f(x) = \frac{x}{2-x}$ on $[0, 2)$
No absolute maximum
Absolute minimum of 0 at $x = 0$

(f)

Graphs (a), (b), and (c) show several possibilities for absolute extrema for functions with a domain of $(-\infty, \infty)$. Graphs (d), (e), and (f) show several possibilities for absolute extrema for functions with a domain that is a bounded interval.

Note:

Extreme Value Theorem

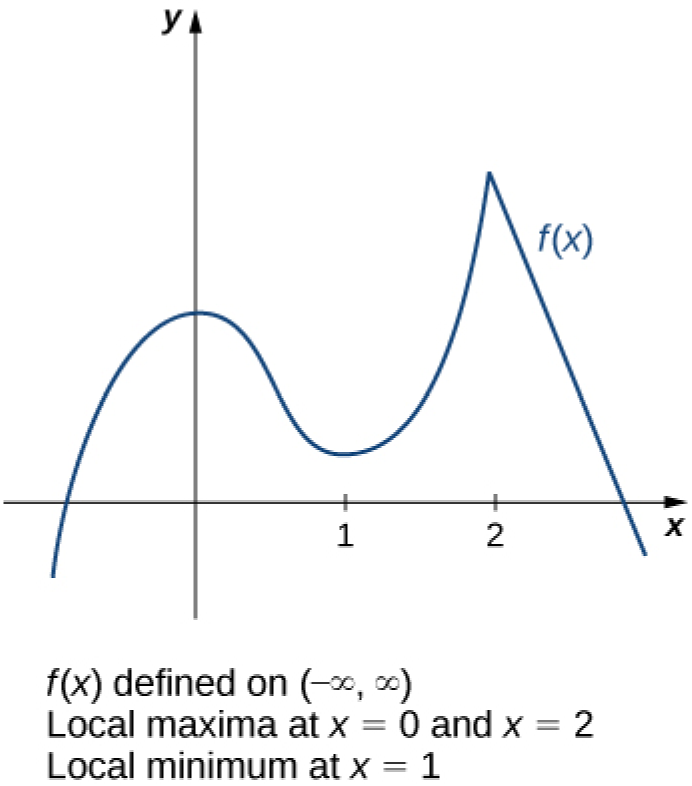
If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$ and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.

The proof of the extreme value theorem is beyond the scope of this text. Typically, it is proved in a course on real analysis. There are a couple of key points to note about the statement of this theorem. For the extreme value theorem to apply, the function must be continuous over a closed, bounded interval. If the interval I is open or the function has even one point of discontinuity, the function may not have an absolute maximum or absolute minimum over I. For example, consider the functions shown in [link](d), (e), and (f). All three of these functions are defined over bounded intervals. However, the function in graph (e) is the only one that has both an absolute maximum and an absolute minimum over its domain. The extreme value theorem cannot be applied to the functions in graphs (d) and (f) because neither of these functions is continuous over a closed, bounded interval. Although the function in graph (d) is defined over the closed interval $[0, 4]$, the function is discontinuous at $x = 2$. The function has an absolute maximum over $[0, 4]$ but does not have an absolute minimum. The function in graph (f) is continuous over the half-open interval $[0, 2)$, but is not defined at $x = 2$, and therefore is not continuous over a closed, bounded interval. The function has an absolute minimum over $[0, 2)$, but does not have an absolute maximum over $[0, 2)$. These two graphs illustrate why a function over a bounded interval may fail to have an absolute maximum and/or absolute minimum.

Before looking at how to find absolute extrema, let’s examine the related concept of local extrema. This idea is useful in determining where absolute extrema occur.

Local Extrema and Critical Points

Consider the function f shown in [link]. The graph can be described as two mountains with a valley in the middle. The absolute maximum value of the function occurs at the higher peak, at $x = 2$. However, $x = 0$ is also a point of interest. Although $f(0)$ is not the largest value of f, the value $f(0)$ is larger than $f(x)$ for all x near 0. We say f has a local maximum at $x = 0$. Similarly, the function f does not have an absolute minimum, but it does have a local minimum at $x = 1$ because $f(1)$ is less than $f(x)$ for x near 1.



This function f has two local maxima and one local minimum. The local maximum at $x = 2$ is also the absolute maximum.

Note:
Definition
A function f has a **local maximum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$. A function f has a **local minimum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$. A function f has a **local extremum** at c if f has a local maximum at c or f has a local minimum at c.

Note that if f has an absolute extremum at c and f is defined over an interval containing c, then $f(c)$ is also considered a local extremum. If an absolute extremum for a function f occurs at an endpoint, we do not consider that to be a local extremum, but instead refer to that as an endpoint extremum.

Given the graph of a function f, it is sometimes easy to see where a local maximum or local minimum occurs. However, it is not always easy to see, since the interesting features on the graph of a function may not be visible because they occur at a very small scale. Also, we may not have a graph of the function. In these cases, how can we use a formula for a function to determine where these extrema occur?

To answer this question, let's look at [link] again. The local extrema occur at $x = 0$, $x = 1$, and $x = 2$. Notice that at $x = 0$ and $x = 1$, the derivative $f\prime(x) = 0$. At $x = 2$, the derivative $f\prime(x)$ does not exist, since the function f has a corner there. In fact, if f has a local extremum at a point $x = c$, the derivative $f\prime(c)$ must satisfy one of the following conditions: either $f\prime(c) = 0$ or $f\prime(c)$ is undefined. Such a value c is known as a critical point and it is important in finding extreme values for functions.

Note:
Definition

Let c be an interior point in the domain of f. We say that c is a **critical point** of f if $f\prime(c) = 0$ or $f\prime(c)$ is undefined.

As mentioned earlier, if f has a local extremum at a point $x = c$, then c must be a critical point of f. This fact is known as **Fermat's theorem**.

Note:
Fermat's Theorem
If f has a local extremum at c and f is differentiable at c, then $f\prime(c) = 0$.

Proof

Suppose f has a local extremum at c and f is differentiable at c. We need to show that $f\prime(c) = 0$. To do this, we will show that $f\prime(c) \geq 0$ and $f\prime(c) \leq 0$, and therefore $f\prime(c) = 0$. Since f has a local extremum at c, f has a local maximum or local minimum at c. Suppose f has a local maximum at c. The case in which f has a local minimum at c can be handled similarly. There then exists an open interval I such that $f\left(c\right) \geq f\left(x\right)$ for all $x \in I$. Since f is differentiable at c, from the definition of the derivative, we know that
Equation:

$$f\prime(c) = \lim_{x \to c} \frac{f\left(x\right) - f\left(c\right)}{x - c}.$$

Since this limit exists, both one-sided limits also exist and equal $f\prime(c)$. Therefore,
Equation:

$$f\prime(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c},$$

and

Equation:

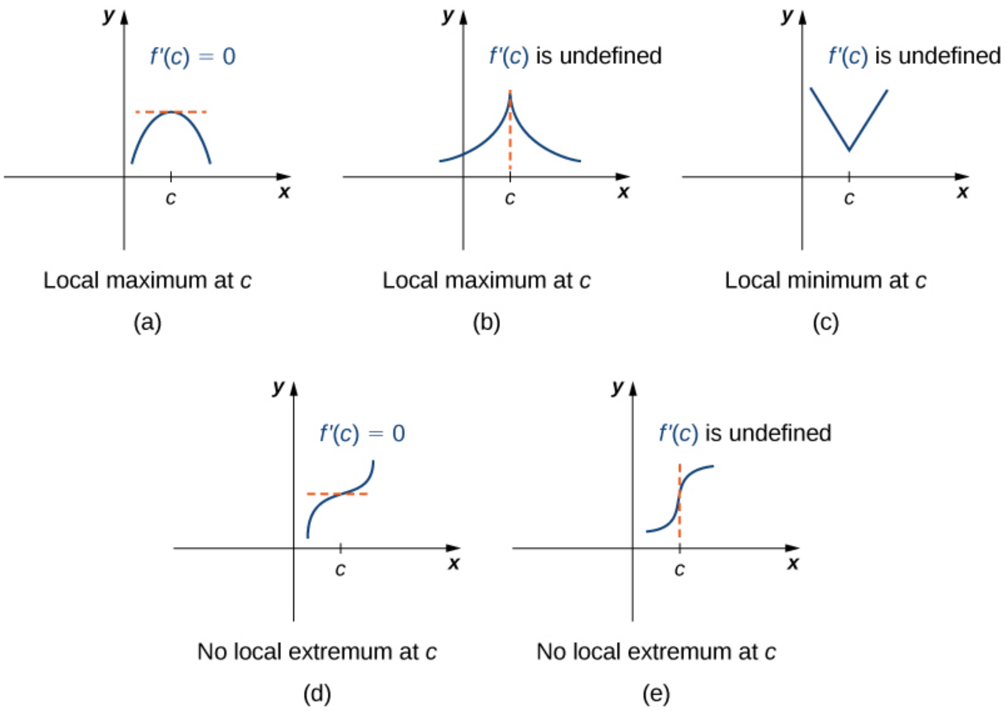
$$f\prime(c)=\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}.$$

Since $f(c)$ is a local maximum, we see that $f(x)-f(c)\le 0$ for x near c. Therefore, for x near c, but $x>c$, we have $\frac{f(x)-f(c)}{x-c}\le 0$. From [link] we conclude that $f\prime(c)\le 0$. Similarly, it can be shown that $f\prime(c)\ge 0$. Therefore, $f\prime(c)=0$.

□

From Fermat’s theorem, we conclude that if f has a local extremum at c, then either $f\prime(c)=0$ or $f\prime(c)$ is undefined. In other words, local extrema can only occur at critical points.

Note this theorem does not claim that a function f must have a local extremum at a critical point. Rather, it states that critical points are candidates for local extrema. For example, consider the function $f\left(x\right)=x^3$. We have $f\prime(x)=3x^2=0$ when $x=0$. Therefore, $x=0$ is a critical point. However, $f(x)=x^3$ is increasing over $(-\infty,\infty)$, and thus f does not have a local extremum at $x=0$. In [link], we see several different possibilities for critical points. In some of these cases, the functions have local extrema at critical points, whereas in other cases the functions do not. Note that these graphs do not show all possibilities for the behavior of a function at a critical point.


y
$f'(c) = 0$
c
x
Local maximum at c
(a)
y
$f'(c)$ is undefined
c
x
Local maximum at c
(b)
y
$f'(c)$ is undefined
c
x
Local minimum at c
(c)
y
$f'(c) = 0$
c
x
No local extremum at c
(d)
y
$f'(c)$ is undefined
c
x
No local extremum at c
(e)

(a–e) A function f has a critical point at c if $f\prime(c) = 0$ or $f\prime(c)$ is undefined. A function may or may not have a local extremum at a critical point.

Later in this chapter we look at analytical methods for determining whether a function actually has a local extremum at a critical point. For now, let's turn our attention to finding critical points. We will use graphical observations to determine whether a critical point is associated with a local extremum.

Example:
Exercise:

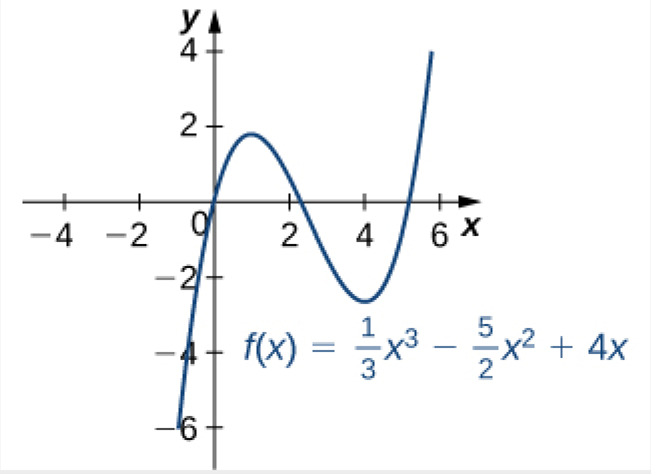
Problem:
Locating Critical Points

For each of the following functions, find all critical points. Use a graphing utility to determine whether the function has a local extremum at each of the critical points.

a. $f\left(x\right)=\frac{1}{3}x^3-\frac{5}{2}x^2+4x$
b. $f\left(x\right)=\left(x^2-1\right)^3$
c. $f\left(x\right)=\frac{4x}{1+x^2}$

Solution:

a. The derivative $f\prime(x)=x^2-5x+4$ is defined for all real numbers x. Therefore, we only need to find the values for x where $f\prime(x)=0$. Since $f\prime(x)=x^2-5x+4=\left(x-4\right)\left(x-1\right)$, the critical points are $x=1$ and $x=4$. From the graph of f in [link], we see that f has a local maximum at $x=1$ and a local minimum at $x=4$.

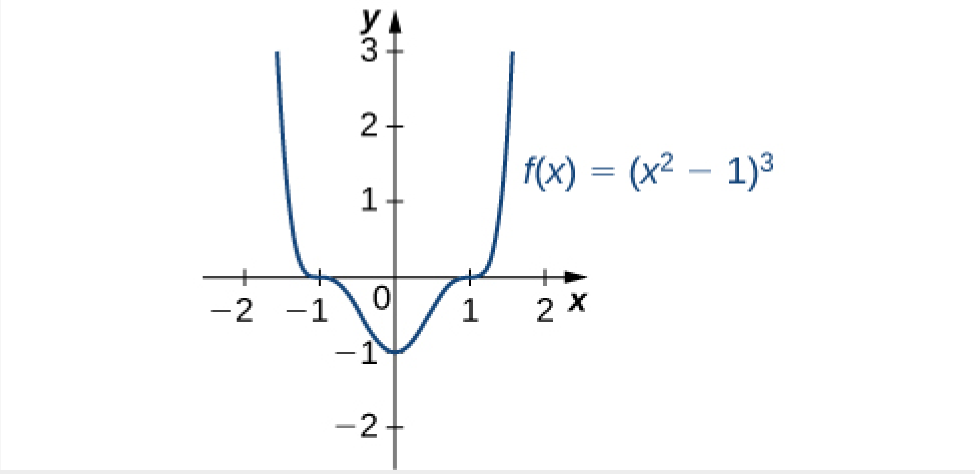


This function has a local maximum and a local minimum.

b. Using the chain rule, we see the derivative is
Equation:

$$f\prime(x) = 3\left(x^2 - 1\right)^2 (2x) = 6x\left(x^2 - 1\right)^2.$$

Therefore, f has critical points when $x = 0$ and when $x^2 - 1 = 0$. We conclude that the critical points are $x = 0, \pm 1$. From the graph of f in [link], we see that f has a local (and absolute) minimum at $x = 0$, but does not have a local extremum at $x = 1$ or $x = -1$.



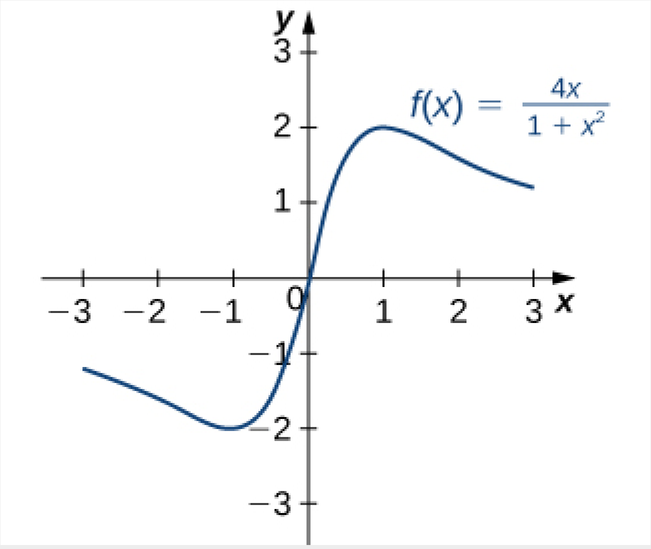
This function has three critical points: $x = 0$, $x = 1$, and $x = -1$. The function has a local (and absolute) minimum at $x = 0$, but does not have extrema at the other two critical points.

c. By the chain rule, we see that the derivative is
Equation:

$$f\prime(x) = \frac{\left(1 + x^2\right)(4) - 4x\,(2x)}{\left(1 + x^2\right)^2} = \frac{4 - 4x^2}{\left(1 + x^2\right)^2}.$$

The derivative is defined everywhere. Therefore, we only need to find values for x where $f\prime(x) = 0$. Solving $f\prime(x) = 0$, we see that $4 - 4x^2 = 0$, which implies $x = \pm 1$. Therefore, the critical points are $x = \pm 1$. From the graph of f in [link], we see that f

has an absolute maximum at $x = 1$ and an absolute minimum at $x = -1$. Hence, f has a local maximum at $x = 1$ and a local minimum at $x = -1$. (Note that if f has an absolute extremum over an interval I at a point c that is not an endpoint of I, then f has a local extremum at c.)



This function has an absolute maximum and an absolute minimum.

Note:
Exercise:

Problem: Find all critical points for $f\left(x\right) = x^3 - \frac{1}{2}x^2 - 2x + 1$.

Solution:

$x = -\frac{2}{3}, x = 1$

Hint

Calculate $f\prime(x)$.

Locating Absolute Extrema

The extreme value theorem states that a continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. As shown in [link], one or both of these absolute extrema could occur at an endpoint. If an absolute extremum does not occur at an endpoint, however, it must occur at an interior point, in which case the absolute extremum is a local extremum. Therefore, by [link], the point c at which the local extremum occurs must be a critical point. We summarize this result in the following theorem.

Note:
Location of Absolute Extrema
Let f be a continuous function over a closed, bounded interval I. The absolute maximum of f over I and the absolute minimum of f over I must occur at endpoints of I or at critical points of f in I.

With this idea in mind, let’s examine a procedure for locating absolute extrema.

Note:
Problem-Solving Strategy: Locating Absolute Extrema over a Closed Interval
Consider a continuous function f defined over the closed interval $[a, b]$.

1. Evaluate f at the endpoints $x = a$ and $x = b$.
2. Find all critical points of f that lie over the interval (a, b) and evaluate f at those critical points.

3. Compare all values found in (1) and (2). From [link], the absolute extrema must occur at endpoints or critical points. Therefore, the largest of these values is the absolute maximum of f. The smallest of these values is the absolute minimum of f.

Now let’s look at how to use this strategy to find the absolute maximum and absolute minimum values for continuous functions.

Example:
Exercise:

Problem:
Locating Absolute Extrema

For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

a. $f(x) = -x^2 + 3x - 2$ over $[1, 3]$.
b. $f(x) = x^2 - 3x^{2/3}$ over $[0, 2]$.

Solution:

a. Step 1. Evaluate f at the endpoints $x = 1$ and $x = 3$.
Equation:

$$f\,(1) = 0 \text{ and } f\,(3) = -2$$

Step 2. Since $f\prime(x) = -2x + 3$, $f\prime$ is defined for all real numbers x. Therefore, there are no critical points where the derivative is undefined. It remains to check where $f\prime(x) = 0$. Since $f\prime(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval

$[1, 3]$, $f\left(\frac{3}{2}\right)$ is a candidate for an absolute extremum of f over $[1, 3]$. We evaluate $f\left(\frac{3}{2}\right)$ and find

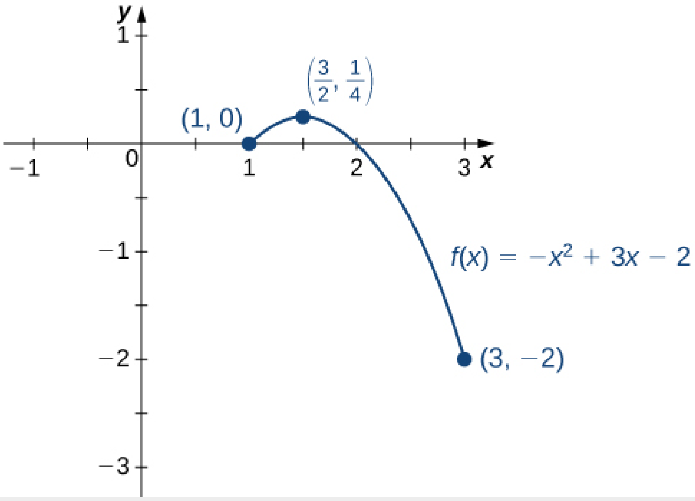
Equation:

$$f\left(\frac{3}{2}\right) = \frac{1}{4}.$$

Step 3. We set up the following table to compare the values found in steps 1 and 2.

x	$f(x)$	**Conclusion**
0	0	
$\frac{3}{2}$	$\frac{1}{4}$	Absolute maximum
3	-2	Absolute minimum

From the table, we find that the absolute maximum of f over the interval [1, 3] is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$. The absolute minimum of f over the interval [1, 3] is -2, and it occurs at $x = 3$ as shown in the following graph.



This function has both an absolute maximum and an absolute minimum.

b. Step 1. Evaluate f at the endpoints $x = 0$ and $x = 2$.
Equation:

$$f(0) = 0 \text{ and } f(2) = 4 - 3\sqrt[3]{4} \approx -0.762$$

Step 2. The derivative of f is given by
Equation:

$$f\prime(x) = 2x - \frac{2}{x^{1/3}} = \frac{2x^{4/3} - 2}{x^{1/3}}$$

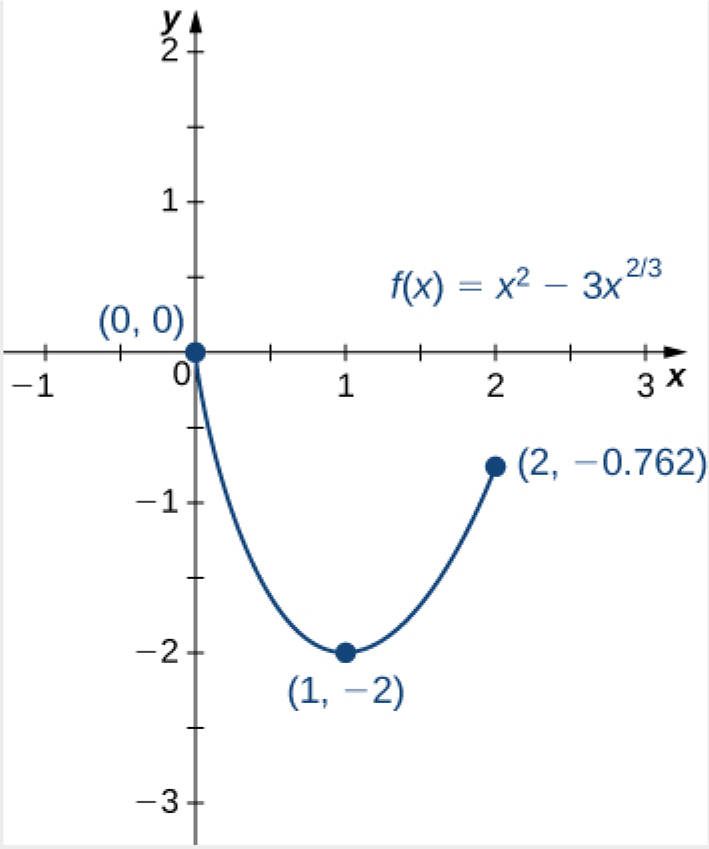
for $x \neq 0$. The derivative is zero when $2x^{4/3} - 2 = 0$, which implies $x = \pm 1$. The derivative is undefined at $x = 0$. Therefore, the critical points of f are $x = 0, 1, -1$. The point $x = 0$ is an endpoint, so we already evaluated $f(0)$ in step 1. The point $x = -1$ is not in the interval of interest, so we need only evaluate $f\,(1)$. We find that
Equation:

$$f(1) = -2.$$

Step 3. We compare the values found in steps 1 and 2, in the following table.

x	$f(x)$	**Conclusion**
0	0	Absolute maximum
1	-2	Absolute minimum
2	-0.762	

We conclude that the absolute maximum of f over the interval [0, 2] is zero, and it occurs at $x = 0$. The absolute minimum is −2, and it occurs at $x = 1$ as shown in the following graph.



This function has an absolute maximum at an endpoint of the interval.

Note:
Exercise:

Problem:

Find the absolute maximum and absolute minimum of $f(x) = x^2 - 4x + 3$ over the interval $[1, 4]$.

Solution:

The absolute maximum is 3 and it occurs at $x = 4$. The absolute minimum is -1 and it occurs at $x = 2$.

Hint

Look for critical points. Evaluate f at all critical points and at the endpoints.

At this point, we know how to locate absolute extrema for continuous functions over closed intervals. We have also defined local extrema and determined that if a function f has a local extremum at a point c, then c must be a critical point of f. However, c being a critical point is not a sufficient condition for f to have a local extremum at c. Later in this chapter, we show how to determine whether a function actually has a local extremum at a critical point. First, however, we need to introduce the Mean Value Theorem, which will help as we analyze the behavior of the graph of a function.

Key Concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or have no absolute maximum or absolute minimum.
- If a function has a local extremum, the point at which it occurs must be a critical point. However, a function need not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and an absolute minimum. Each extremum occurs at a critical point or an endpoint.

Exercise:

Problem:

In precalculus, you learned a formula for the position of the maximum or minimum of a quadratic equation $y = ax^2 + bx + c$, which was $h = -\frac{b}{(2a)}$. Prove this formula using calculus.

Exercise:

Problem:

If you are finding an absolute minimum over an interval $[a, b]$, why do you need to check the endpoints? Draw a graph that supports your hypothesis.

Solution:

Answers may vary

Exercise:

Problem:

If you are examining a function over an interval (a, b), for a and b finite, is it possible not to have an absolute maximum or absolute minimum?

Exercise:

Problem:

When you are checking for critical points, explain why you also need to determine points where $f'(x)$ is undefined. Draw a graph to support your explanation.

Solution:

Answers will vary

Exercise:

Problem:

Can you have a finite absolute maximum for $y = ax^2 + bx + c$ over $(-\infty, \infty)$? Explain why or why not using graphical arguments.

Exercise:

Problem:

Can you have a finite absolute maximum for $y = ax^3 + bx^2 + cx + d$ over $(-\infty, \infty)$ assuming *a* is non-zero? Explain why or why not using graphical arguments.

Solution:

No; answers will vary

Exercise:

Problem:

Let m be the number of local minima and M be the number of local maxima. Can you create a function where $M > m + 2$? Draw a graph to support your explanation.

Exercise:

Problem:

Is it possible to have more than one absolute maximum? Use a graphical argument to prove your hypothesis.

Solution:

Since the absolute maximum is the function (output) value rather than the *x* value, the answer is no; answers will vary

Exercise:

Problem:

Is it possible to have no absolute minimum or maximum for a function? If so, construct such a function. If not, explain why this is not possible.

Exercise:

Problem:

[T] Graph the function $y = e^{ax}$. For which values of a, on any infinite domain, will you have an absolute minimum and absolute maximum?

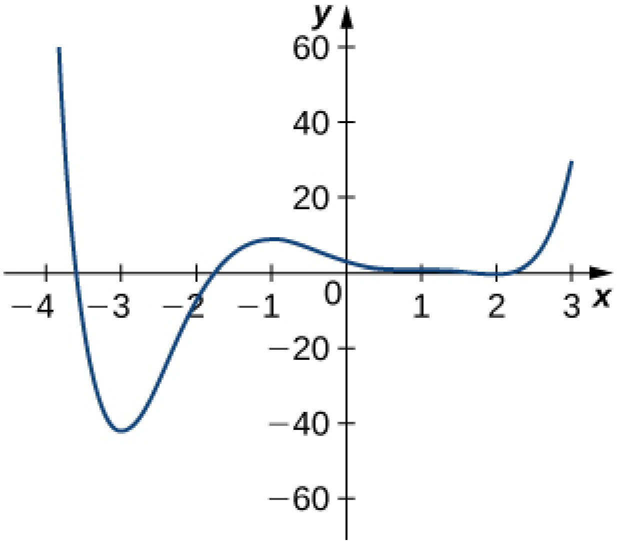
Solution:

When $a = 0$

For the following exercises, determine where the local and absolute maxima and minima occur on the graph given. Assume the graph represents the entirety of each function.

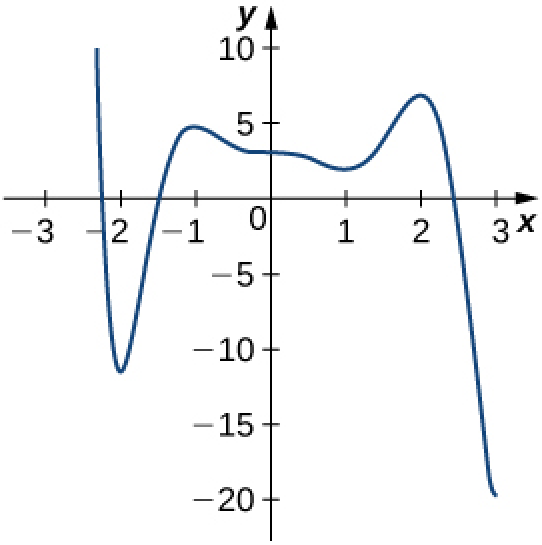
Exercise:

Problem:



Exercise:

Problem:

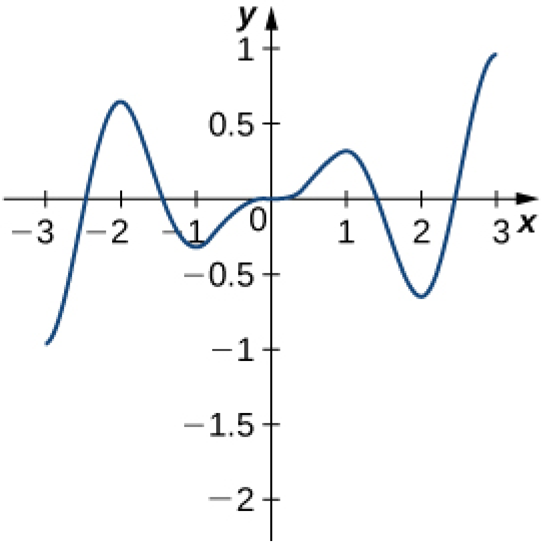


Solution:

Absolute minimum at 3; Absolute maximum at −2.2; local minima at −2, 1; local maxima at −1, 2

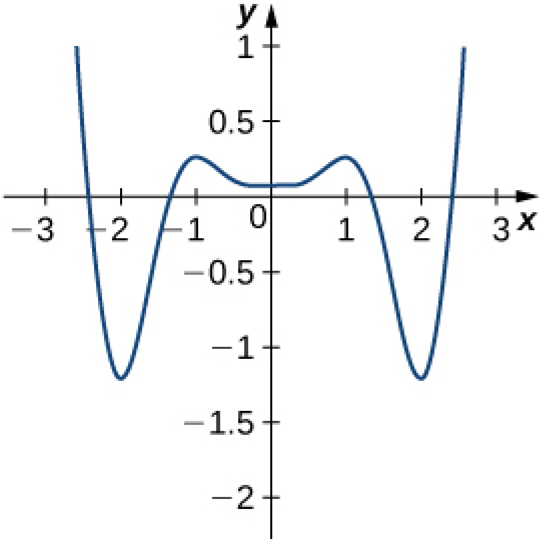
Exercise:

Problem:



Exercise:

Problem:



Solution:

Absolute minima at −2, 2; absolute maxima at −2.5, 2.5; local minimum at 0; local maxima at −1, 1

For the following problems, draw graphs of $f\left(x\right)$, which is continuous, over the interval $\left[-4,4\right]$ with the following properties:

Exercise:

Problem:

Absolute maximum at $x=2$ and absolute minima at $x=\pm 3$

Exercise:

Problem:

Absolute minimum at $x=1$ and absolute maximum at $x=2$

Solution:

Answers may vary.

Exercise:

Problem:

Absolute maximum at $x=4$, absolute minimum at $x=-1$, local maximum at $x=-2$, and a critical point that is not a maximum or minimum at $x=2$

Exercise:

Problem:

Absolute maxima at $x=2$ and $x=-3$, local minimum at $x=1$, and absolute minimum at $x=4$

Solution:

Answers may vary.

For the following exercises, find the critical points in the domains of the following functions.

Exercise:

Problem: $y=4x^3-3x$

Exercise:

Problem: $y=4\sqrt{x}-x^2$

Solution:

$x=1$

Exercise:

Problem: $y=\frac{1}{x-1}$

Exercise:

Problem: $y = \ln(x - 2)$

Solution:

None

Exercise:

Problem: $y = \tan(x)$

Exercise:

Problem: $y = \sqrt{4 - x^2}$

Solution:

$x = 0$

Exercise:

Problem: $y = x^{3/2} - 3x^{5/2}$

Exercise:

Problem: $y = \frac{x^2-1}{x^2+2x-3}$

Solution:

None

Exercise:

Problem: $y = \sin^2(x)$

Exercise:

Problem: $y = x + \frac{1}{x}$

Solution:

$x = -1, 1$

For the following exercises, find the local and/or absolute maxima for the functions over the specified domain.

Exercise:

Problem: $f(x) = x^2 + 3$ over $[-1, 4]$

Exercise:

Problem: $y = x^2 + \frac{2}{x}$ over $[1, 4]$

Solution:

Absolute maximum: $x = 4$, $y = \frac{33}{2}$; absolute minimum: $x = 1$, $y = 3$

Exercise:

Problem: $y = \left(x - x^2\right)^2$ over $[-1, 1]$

Exercise:

Problem: $y = \frac{1}{(x - x^2)}$ over $(0, 1)$

Solution:

Absolute minimum: $x = \frac{1}{2}$, $y = 4$

Exercise:

Problem: $y = \sqrt{9 - x}$ over $[1, 9]$

Exercise:

Problem: $y = x + \sin(x)$ over $[0, 2\pi]$

Solution:

Absolute maximum: $x = 2\pi$, $y = 2\pi$; absolute minimum: $x = 0$, $y = 0$

Exercise:

Problem: $y = \frac{x}{1+x}$ over $[0, 100]$

Exercise:

Problem: $y = |x+1| + |x-1|$ over $[-3, 2]$

Solution:

Absolute maximum: $x = -3$; absolute minimum: $-1 \leq x \leq 1$, $y = 2$

Exercise:

Problem: $y = \sqrt{x} - \sqrt{x^3}$ over $[0, 4]$

Exercise:

Problem: $y = \sin x + \cos x$ over $[0, 2\pi]$

Solution:

Absolute maximum: $x = \frac{\pi}{4}$, $y = \sqrt{2}$; absolute minimum: $x = \frac{5\pi}{4}$, $y = -\sqrt{2}$

Exercise:

Problem: $y = 4\sin\theta - 3\cos\theta$ over $[0, 2\pi]$

For the following exercises, find the local and absolute minima and maxima for the functions over $(-\infty, \infty)$.

Exercise:

Problem: $y = x^2 + 4x + 5$

Solution:

Absolute minimum: $x = -2$, $y = 1$

Exercise:

Problem: $y = x^3 - 12x$

Exercise:

Problem: $y = 3x^4 + 8x^3 - 18x^2$

Solution:

Absolute minimum: $x = -3$, $y = -135$; local maximum: $x = 0$, $y = 0$; local minimum: $x = 1$, $y = -7$

Exercise:

Problem: $y = x^3(1 - x)^6$

Exercise:

Problem: $y = \frac{x^2+x+6}{x-1}$

Solution:

Local maximum: $x = 1 - 2\sqrt{2}$, $y = 3 - 4\sqrt{2}$; local minimum: $x = 1 + 2\sqrt{2}$, $y = 3 + 4\sqrt{2}$

Exercise:

Problem: $y = \frac{x^2-1}{x-1}$

For the following functions, use a calculator to graph the function and to estimate the absolute and local maxima and minima. Then, solve for them explicitly.

Exercise:

Problem: **[T]** $y = 3x\sqrt{1-x^2}$

Solution:

Absolute maximum: $x = \frac{\sqrt{2}}{2}$, $y = \frac{3}{2}$; absolute minimum: $x = -\frac{\sqrt{2}}{2}$, $y = -\frac{3}{2}$

Exercise:

Problem: **[T]** $y = x + \sin(x)$

Exercise:

Problem: **[T]** $y = 12x^5 + 45x^4 + 20x^3 - 90x^2 - 120x + 3$

Solution:

Local maximum: $x = -2$, $y = 59$; local minimum: $x = 1$, $y = -130$

Exercise:

Problem: **[T]** $y = \frac{x^3+6x^2-x-30}{x-2}$

Exercise:

Problem: **[T]** $y = \frac{\sqrt{4-x^2}}{\sqrt{4+x^2}}$

Solution:

Absolute maximum: $x = 0$, $y = 1$; absolute minimum: $x = -2, 2$, $y = 0$

Exercise:

Problem:

A company that produces cell phones has a cost function of $C = x^2 - 1200x + 36{,}400$, where C is cost in dollars and x is number of cell phones produced (in thousands). How many units of cell phone (in thousands) minimizes this cost function?

Exercise:

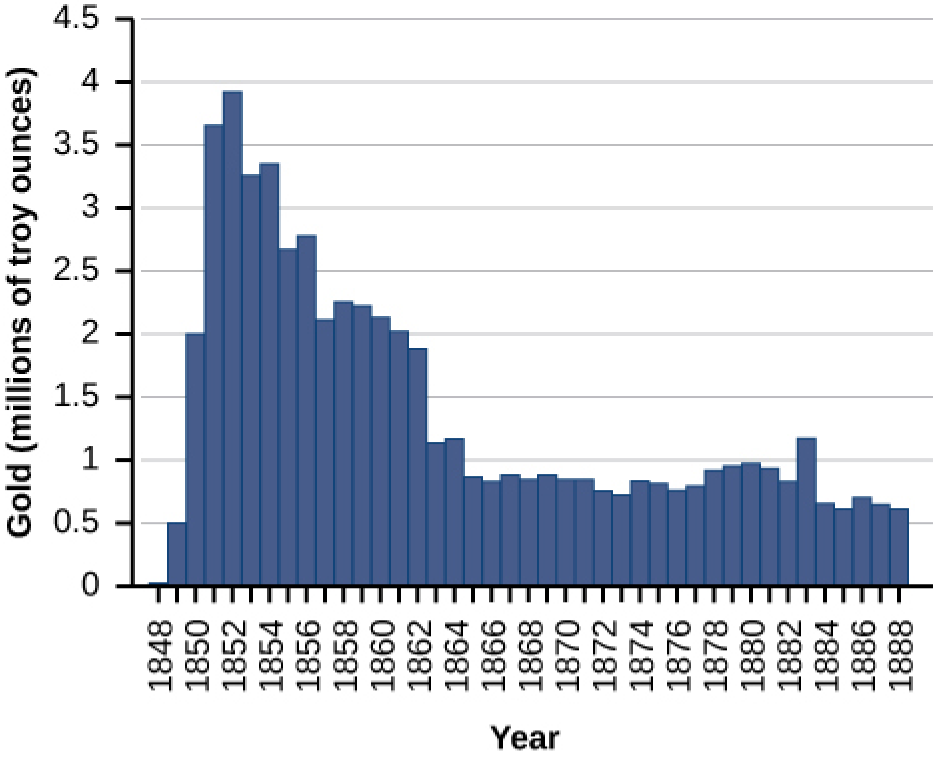
Problem:

A ball is thrown into the air and its position is given by $h\left(t\right) = -4.9t^2 + 60t + 5\ \text{m}$. Find the height at which the ball stops ascending. How long after it is thrown does this happen?

Solution:

$h = \frac{9245}{49}\ \text{m}$, $t = \frac{300}{49}\ \text{s}$

For the following exercises, consider the production of gold during the California gold rush (1848–1888). The production of gold can be modeled by $G\left(t\right) = \frac{(25t)}{(t^2+16)}$, where t is the number of years since the rush began $(0 \le t \le 40)$ and G is ounces of gold produced (in millions). A summary of the data is shown in the following figure.



Exercise:

Problem:

Find when the maximum (local and global) gold production occurred, and the amount of gold produced during that maximum.

Exercise:

Problem:

Find when the minimum (local and global) gold production occurred. What was the amount of gold produced during this minimum?

Solution:

The global minimum was in 1848, when no gold was produced.

Find the critical points, maxima, and minima for the following piecewise functions.

Exercise:

Problem: $y = \begin{cases} x^2 - 4x & 0 \leq x \leq 1 \\ x^2 - 4 & 1 < x \leq 2 \end{cases}$

Exercise:

Problem: $y = \begin{cases} x^2 + 1 & x \leq 1 \\ x^2 - 4x + 5 & x > 1 \end{cases}$

Solution:

Absolute minima: $x = 0$, $x = 2$, $y = 1$; local maximum at $x = 1$, $y = 2$

For the following exercises, find the critical points of the following generic functions. Are they maxima, minima, or neither? State the necessary conditions.

Exercise:

Problem: $y = ax^2 + bx + c$, given that $a > 0$

Exercise:

Problem: $y = (x - 1)^a$, given that $a > 1$ and a is an integer.

Solution:

No maxima/minima if a is odd, minimum at $x = 1$ if a is even

Glossary

absolute extremum

if f has an absolute maximum or absolute minimum at c, we say f has an absolute extremum at c

absolute maximum

if $f(c) \geq f(x)$ for all x in the domain of f, we say f has an absolute maximum at c

absolute minimum

if $f(c) \leq f(x)$ for all x in the domain of f, we say f has an absolute minimum at c

critical point

if $f'(c) = 0$ or $f'(c)$ is undefined, we say that c is a critical point of f

extreme value theorem

if f is a continuous function over a finite, closed interval, then f has an absolute maximum and an absolute minimum

Fermat’s theorem

if f has a local extremum at c, then c is a critical point of f

local extremum

if f has a local maximum or local minimum at c, we say f has a local extremum at c

local maximum

if there exists an interval I such that $f(c) \geq f(x)$ for all $x \in I$, we say f has a local maximum at c

local minimum

if there exists an interval I such that $f(c) \leq f(x)$ for all $x \in I$, we say f has a local minimum at c

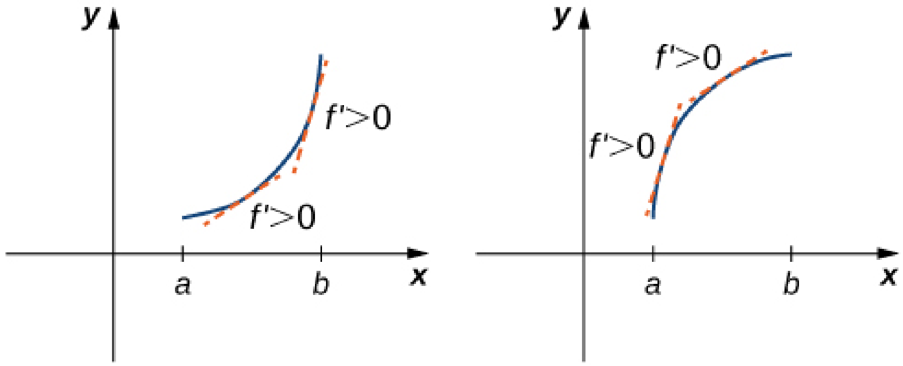
Derivatives and the Shape of a Graph

- Explain how the sign of the first derivative affects the shape of a function’s graph.
- State the first derivative test for critical points.
- Use concavity and inflection points to explain how the sign of the second derivative affects the shape of a function’s graph.
- Explain the concavity test for a function over an open interval.
- Explain the relationship between a function and its first and second derivatives.
- State the second derivative test for local extrema.

Earlier in this chapter we stated that if a function f has a local extremum at a point c, then c must be a critical point of f. However, a function is not guaranteed to have a local extremum at a critical point. For example, $f(x) = x^3$ has a critical point at $x = 0$ since $f'(x) = 3x^2$ is zero at $x = 0$, but f does not have a local extremum at $x = 0$. Using the results from the previous section, we are now able to determine whether a critical point of a function actually corresponds to a local extreme value. In this section, we also see how the second derivative provides information about the shape of a graph by describing whether the graph of a function curves upward or curves downward.

The First Derivative Test

Corollary 3 of the Mean Value Theorem showed that if the derivative of a function is positive over an interval I then the function is increasing over I. On the other hand, if the derivative of the function is negative over an interval I, then the function is decreasing over I as shown in the following figure.

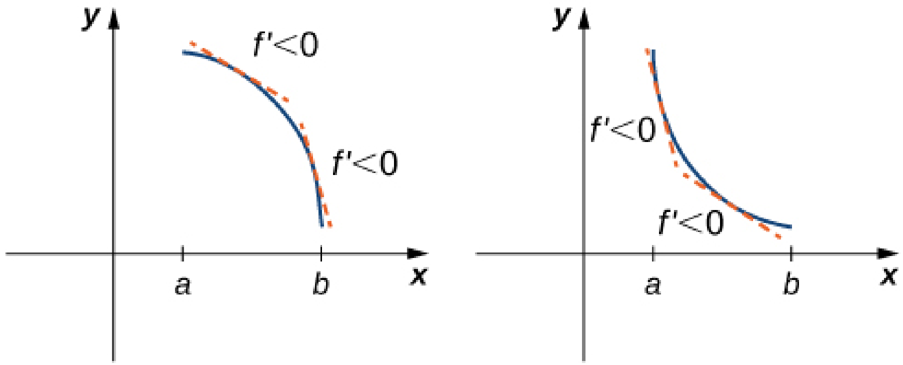


f is increasing

(a)

f is increasing

(b)



f is decreasing

(c)

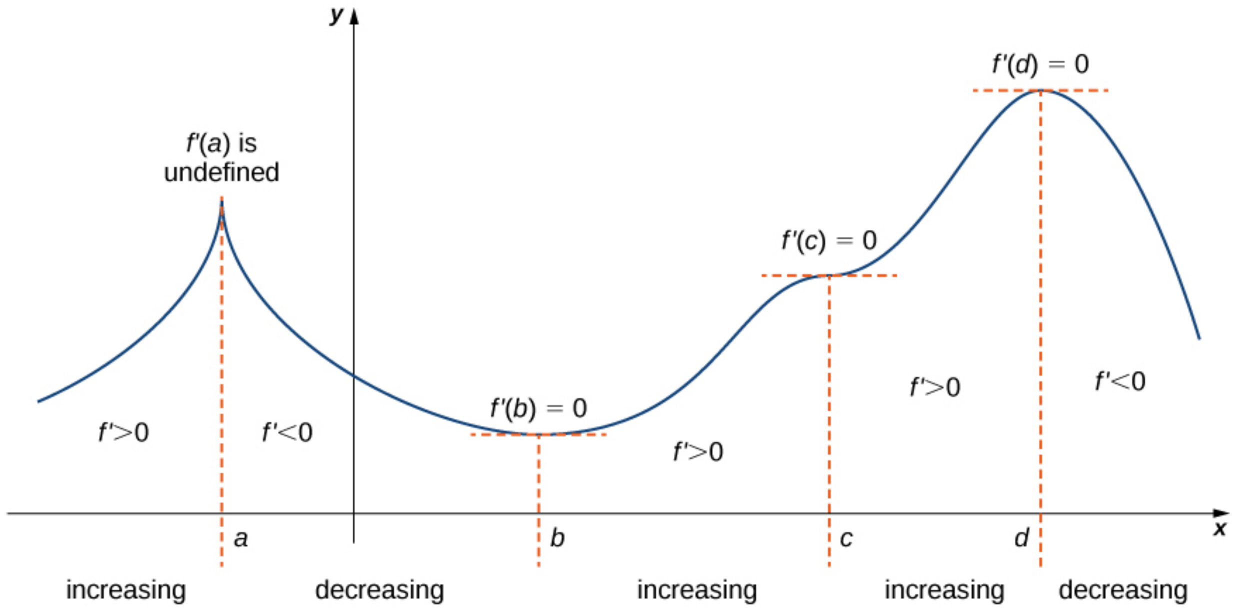
f is decreasing

(d)

Both functions are increasing over the interval (a, b). At each point x, the derivative $f'(x) > 0$. Both functions are decreasing over the interval (a, b). At each point x, the derivative $f'(x) < 0$.

A continuous function f has a local maximum at point c if and only if f switches from increasing to decreasing at point c. Similarly, f has a local minimum at c if and only if f switches from decreasing to increasing at c. If f is a continuous function over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) at point c is if f' changes sign as x increases through c. If f is differentiable at c, the only way that f'. can change sign as x increases through c is if $f'(c) = 0$. Therefore, for a function f that is continuous over an interval I containing c and differentiable over I, except possibly at c, the only way f can switch from increasing to decreasing (or vice versa) is if $f'(c) = 0$ or $f'(c)$ is undefined. Consequently, to locate local extrema for a function f, we look for points c in the domain of f such that $f'(c) = 0$ or $f'(c)$ is undefined. Recall that such points are called critical points of f.

Note that f need not have a local extrema at a critical point. The critical points are candidates for local extrema only. In [link], we show that if a continuous function f has a local extremum, it must occur at a critical point, but a function may not have a local extremum at a critical point. We show that if f has a local extremum at a critical point, then the sign of f' switches as x increases through that point.



The function f has four critical points: $a, b, c,$ and d. The function f has local maxima at a and d, and a local minimum at b. The function f does not have a local extremum at c. The sign of f' changes at all local extrema.

Using [link], we summarize the main results regarding local extrema.

- If a continuous function f has a local extremum, it must occur at a critical point c.

- The function has a local extremum at the critical point c if and only if the derivative f' switches sign as x increases through c.
- Therefore, to test whether a function has a local extremum at a critical point c, we must determine the sign of $f'(x)$ to the left and right of c.

This result is known as the **first derivative test**.

Note:
First Derivative Test
Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:

i. If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.
ii. If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.
iii. If $f\prime$ has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.

We can summarize the first derivative test as a strategy for locating local extrema.

Note:
Problem-Solving Strategy: Using the First Derivative Test
Consider a function f that is continuous over an interval I.

1. Find all critical points of f and divide the interval I into smaller intervals using the critical points as endpoints.
2. Analyze the sign of $f\prime$ in each of the subintervals. If $f\prime$ is continuous over a given subinterval (which is typically the case), then the sign of $f\prime$ in that subinterval does not change and, therefore, can be determined by choosing an arbitrary test point x in that subinterval and by evaluating the sign of $f\prime$ at that test point. Use the sign analysis to determine whether f is increasing or decreasing over that interval.

3. Use [link] and the results of step 2 to determine whether f has a local maximum, a local minimum, or neither at each of the critical points.

Now let’s look at how to use this strategy to locate all local extrema for particular functions.

Example:
Exercise:

Problem:
Using the First Derivative Test to Find Local Extrema

Use the first derivative test to find the location of all local extrema for $f\left(x\right)=x^3-3x^2-9x-1.$ Use a graphing utility to confirm your results.

Solution:

Step 1. The derivative is $f'\left(x\right)=3x^2-6x-9.$ To find the critical points, we need to find where $f'\left(x\right)=0.$ Factoring the polynomial, we conclude that the critical points must satisfy
Equation:

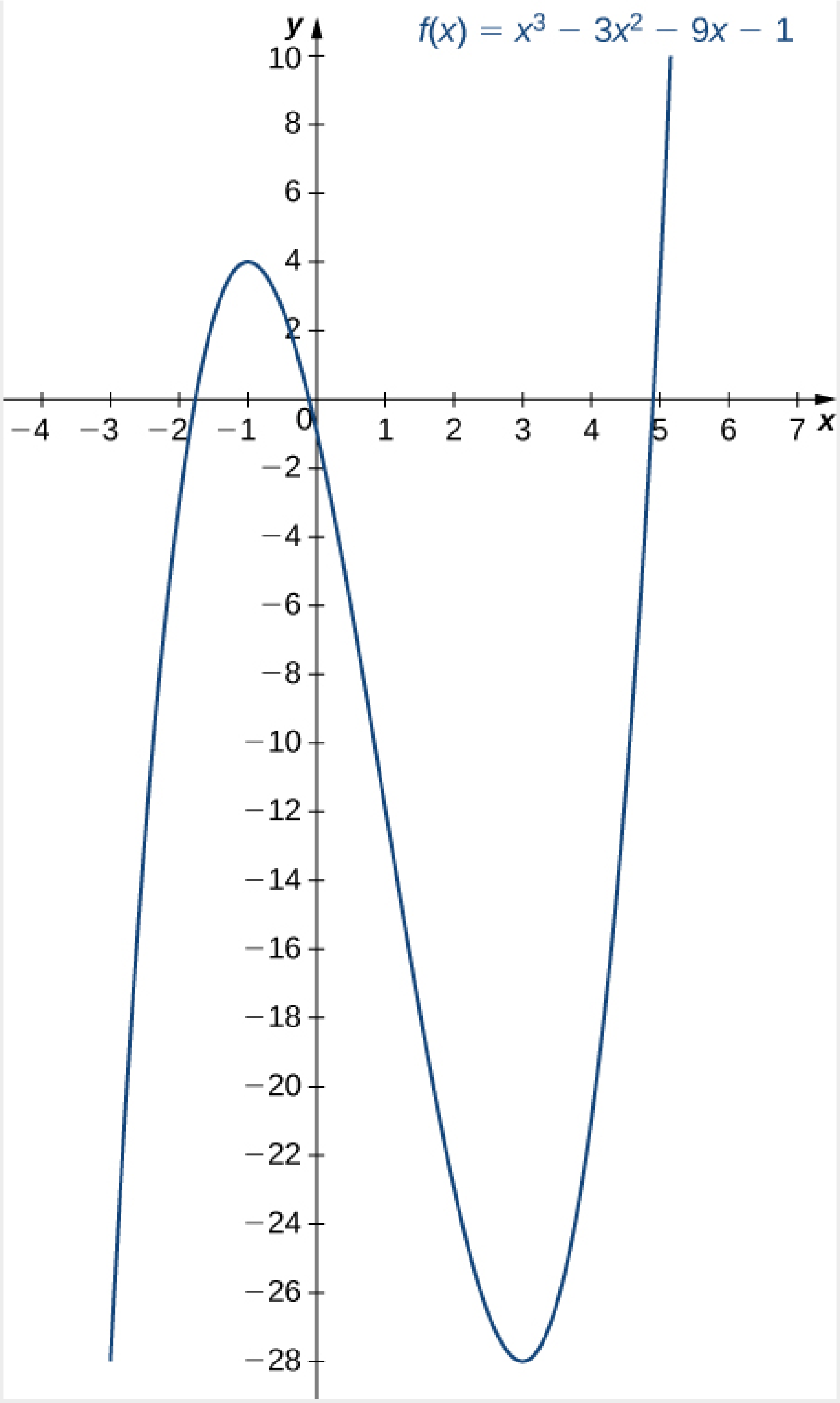
$$3(x^2-2x-3)=3\left(x-3\right)\left(x+1\right)=0.$$

Therefore, the critical points are $x=3,-1.$ Now divide the interval $\left(-\infty,\infty\right)$ into the smaller intervals $\left(-\infty,-1\right),\left(-1,3\right)$ and $\left(3,\infty\right).$

Step 2. Since $f\prime$ is a continuous function, to determine the sign of $f'\left(x\right)$ over each subinterval, it suffices to choose a point over each of the intervals $\left(-\infty,-1\right),\left(-1,3\right)$ and $\left(3,\infty\right)$ and determine the sign of f' at each of these points. For example, let’s choose $x=-2,x=0,$ and $x=4$ as test points.

Interval	**Test Point**	**Sign of $f'(x) = 3(x-3)(x+1)$ at Test Point**	**Conclusion**
$(-\infty, -1)$	$x = -2$	$(+)(-)(-) = +$	f is increasing.
$(-1, 3)$	$x = 0$	$(+)(-)(+) = -$	f is decreasing.
$(3, \infty)$	$x = 4$	$(+)(+)(+) = +$	f is increasing.

Step 3. Since f' switches sign from positive to negative as x increases through 1, f has a local maximum at $x = -1$. Since f' switches sign from negative to positive as x increases through 3, f has a local minimum at $x = 3$. These analytical results agree with the following graph.



The function f has a maximum at $x = -1$ and a minimum at $x = 3$

Note:
Exercise:

Problem:

Use the first derivative test to locate all local extrema for $f\left(x\right)=-x^3+\frac{3}{2}x^2+18x$.

Solution:

f has a local minimum at -2 and a local maximum at 3.

Hint

Find all critical points of f and determine the signs of $f'\left(x\right)$ over particular intervals determined by the critical points.

Example:
Exercise:

Problem:
Using the First Derivative Test

Use the first derivative test to find the location of all local extrema for $f\left(x\right)=5x^{1/3}-x^{5/3}$. Use a graphing utility to confirm your results.

Solution:

Step 1. The derivative is
Equation:

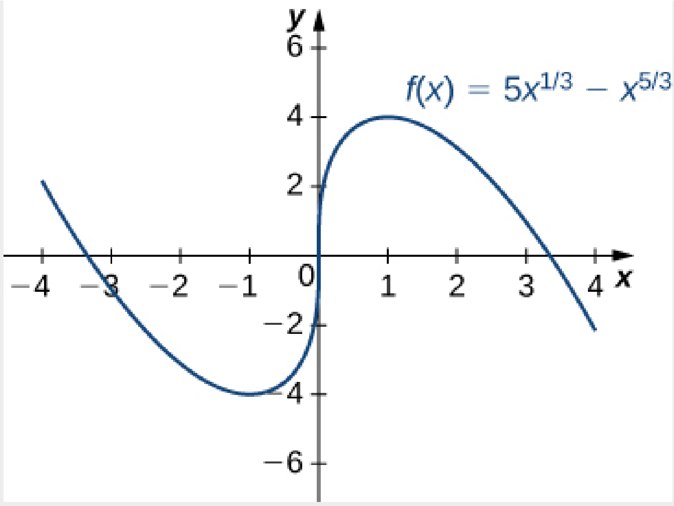
$$f'\left(x\right)=\frac{5}{3}x^{-2/3}-\frac{5}{3}x^{2/3}=\frac{5}{3x^{2/3}}-\frac{5x^{2/3}}{3}=\frac{5-5x^{4/3}}{3x^{2/3}}=\frac{5\left(1-x^{4/3}\right)}{3x^{2/3}}.$$

The derivative $f'\left(x\right)=0$ when $1-x^{4/3}=0$. Therefore, $f'\left(x\right)=0$ at $x=\pm1$. The derivative $f'\left(x\right)$ is undefined at $x=0$. Therefore, we have three critical points: $x=0$, $x=1$, and $x=-1$. Consequently, divide the interval $\left(-\infty,\infty\right)$ into the smaller intervals $\left(-\infty,-1\right)$, $\left(-1,0\right)$, $\left(0,1\right)$, and $\left(1,\infty\right)$.

Step 2: Since f' is continuous over each subinterval, it suffices to choose a test point x in each of the intervals from step 1 and determine the sign of f' at each of these points. The points $x = -2, x = -\frac{1}{2}, x = \frac{1}{2}$, and $x = 2$ are test points for these intervals.

Interval	**Test Point**	**Sign of** $f'(x) = \frac{5(1-x^{4/3})}{3x^{2/3}}$ **at Test Point**	**Conclusion**
$(-\infty, -1)$	$x = -2$	$\frac{(+)(-)}{+} = -$	f is decreasing.
$(-1, 0)$	$x = -\frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(0, 1)$	$x = \frac{1}{2}$	$\frac{(+)(+)}{+} = +$	f is increasing.
$(1, \infty)$	$x = 2$	$\frac{(+)(-)}{+} = -$	f is decreasing.

Step 3: Since f is decreasing over the interval $(-\infty, -1)$ and increasing over the interval $(-1, 0)$, f has a local minimum at $x = -1$. Since f is increasing over the interval $(-1, 0)$ and the interval $(0, 1)$, f does not have a local extremum at $x = 0$. Since f is increasing over the interval $(0, 1)$ and decreasing over the interval $(1, \infty)$, f has a local maximum at $x = 1$. The analytical results agree with the following graph.



The function f has a local minimum at $x = -1$ and a local maximum at $x = 1$.

Note:
Exercise:

Problem:

Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x-1}$.

Solution:

f has no local extrema because f' does not change sign at $x = 1$.

Hint

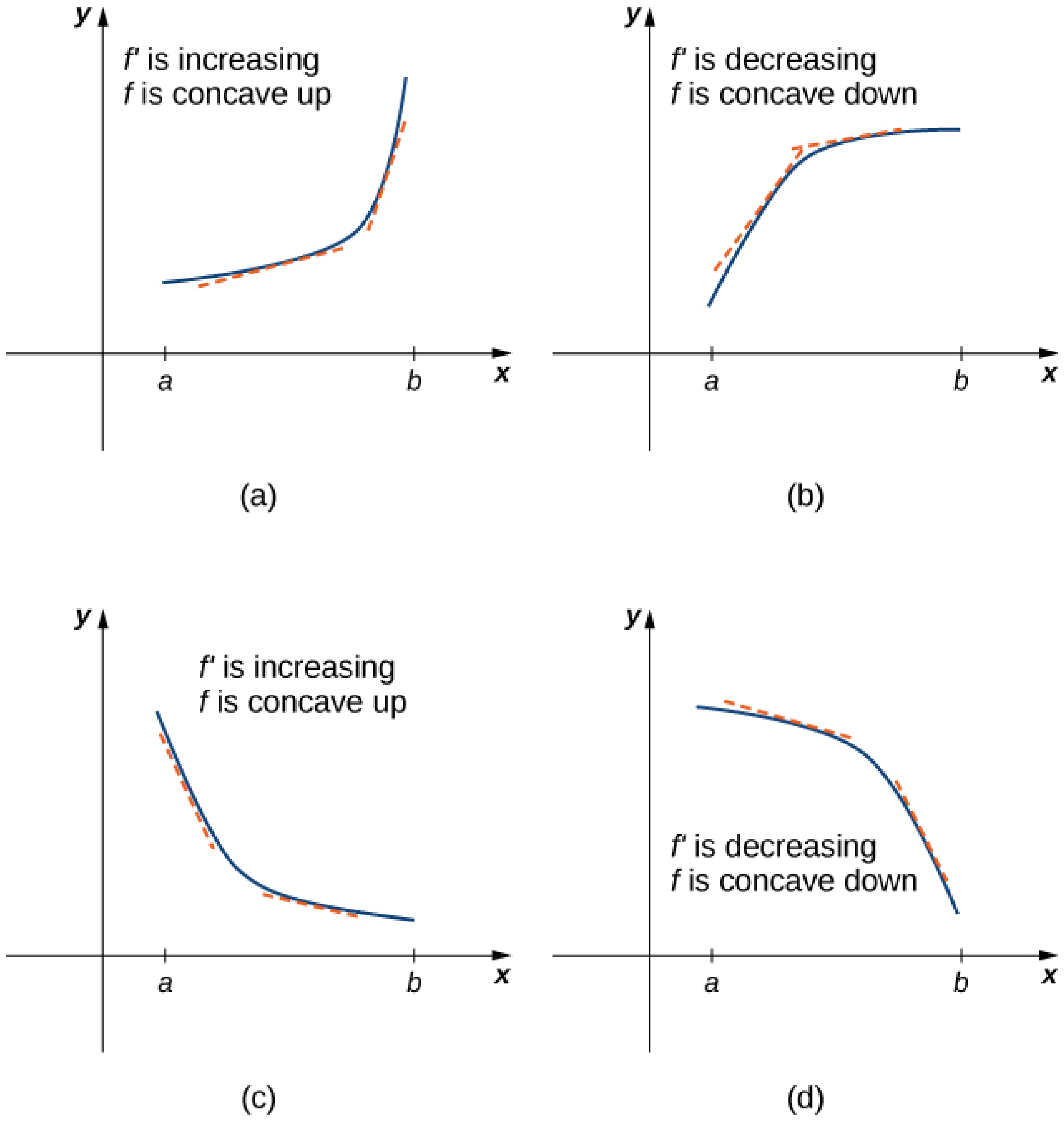
The only critical point of f is $x = 1$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

[link](a) shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. [link](b) shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Note:
Definition
Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.



(a), (c) Since $f\prime$ is increasing over the interval (a, b), we say f is concave up over (a, b). (b), (d) Since $f\prime$ is decreasing over the interval (a, b), we say f is concave down over (a, b).

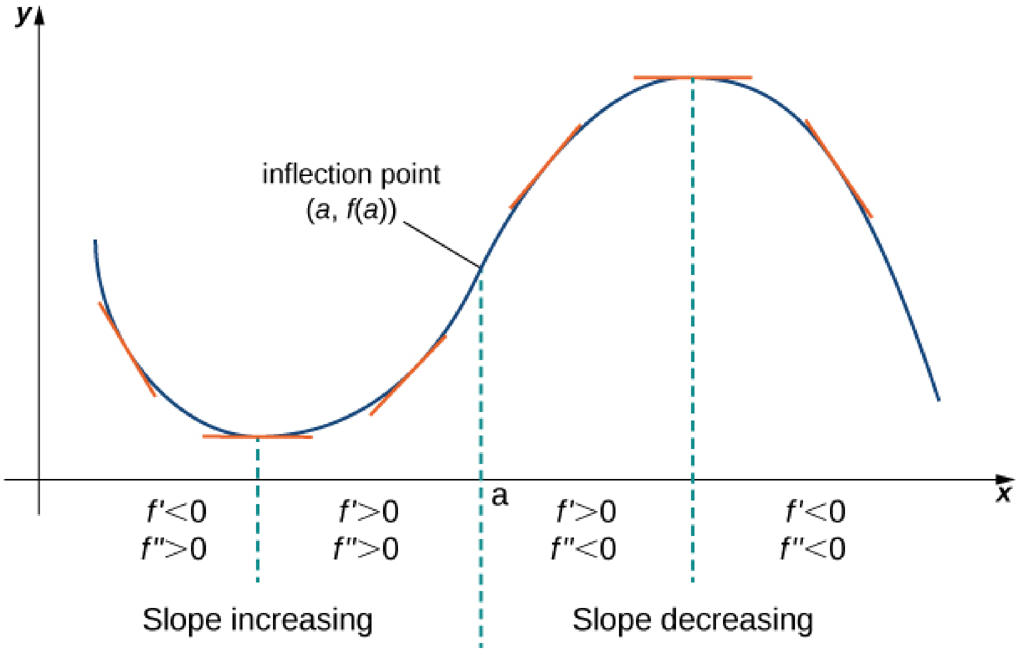
In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f\prime\prime(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the **concavity test**.

Note:
Test for Concavity
Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.
ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity ([link]). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an inflection point of f.

Note:
Definition
If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an **inflection point** of f.



Since $f\prime\prime(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f\prime\prime(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example:
Exercise:

Problem:
Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30$, determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

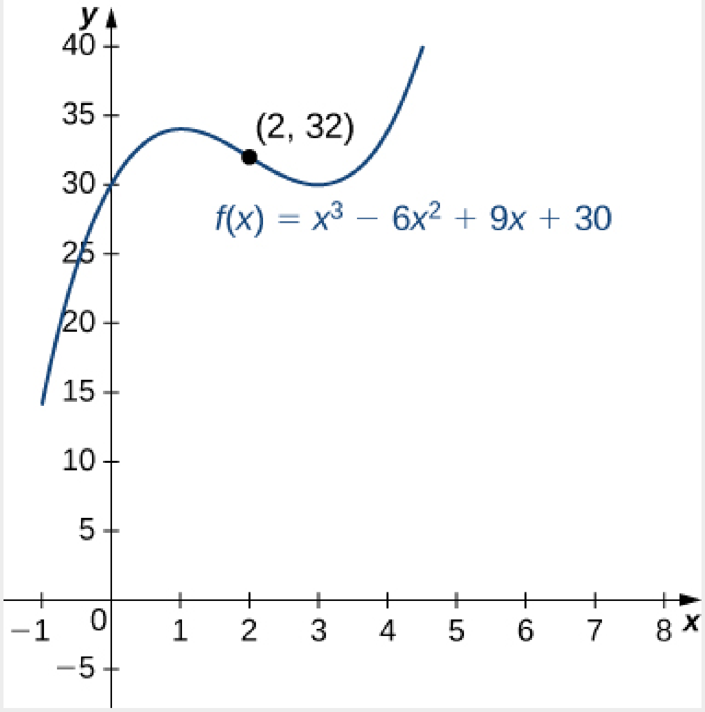
Solution:

To determine concavity, we need to find the second derivative $f\prime\prime(x)$. The first derivative is $f\prime(x) = 3x^2 - 12x + 9$, so the second derivative is $f\prime\prime(x) = 6x - 12$. If the function changes concavity, it occurs either when $f\prime\prime(x) = 0$ or $f\prime\prime(x)$ is undefined. Since $f\prime\prime$ is defined for all real numbers

x, we need only find where $f\prime\prime(x)=0$. Solving the equation $6x-12=0$, we see that $x=2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty,2)$ and $(2,\infty)$ to determine the concavity of f. The points $x=0$ and $x=3$ are test points for these intervals.

Interval	**Test Point**	**Sign of** $f\prime\prime(x)=6x-12$ **at Test Point**	**Conclusion**
$(-\infty,2)$	$x=0$	$-$	f is concave down
$(2,\infty)$	$x=3$	$+$	f is concave up.

We conclude that f is concave down over the interval $(-\infty,2)$ and concave up over the interval $(2,\infty)$. Since f changes concavity at $x=2$, the point $(2,f(2))=(2,32)$ is an inflection point. [link] confirms the analytical results.



The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

Note:
Exercise:

Problem:

For $f\left(x\right)=-x^3+\frac{3}{2}x^2+18x$, find all intervals where f is concave up and all intervals where f is concave down.

Solution:

f is concave up over the interval $\left(-\infty,\frac{1}{2}\right)$ and concave down over the interval $\left(\frac{1}{2},\infty\right)$

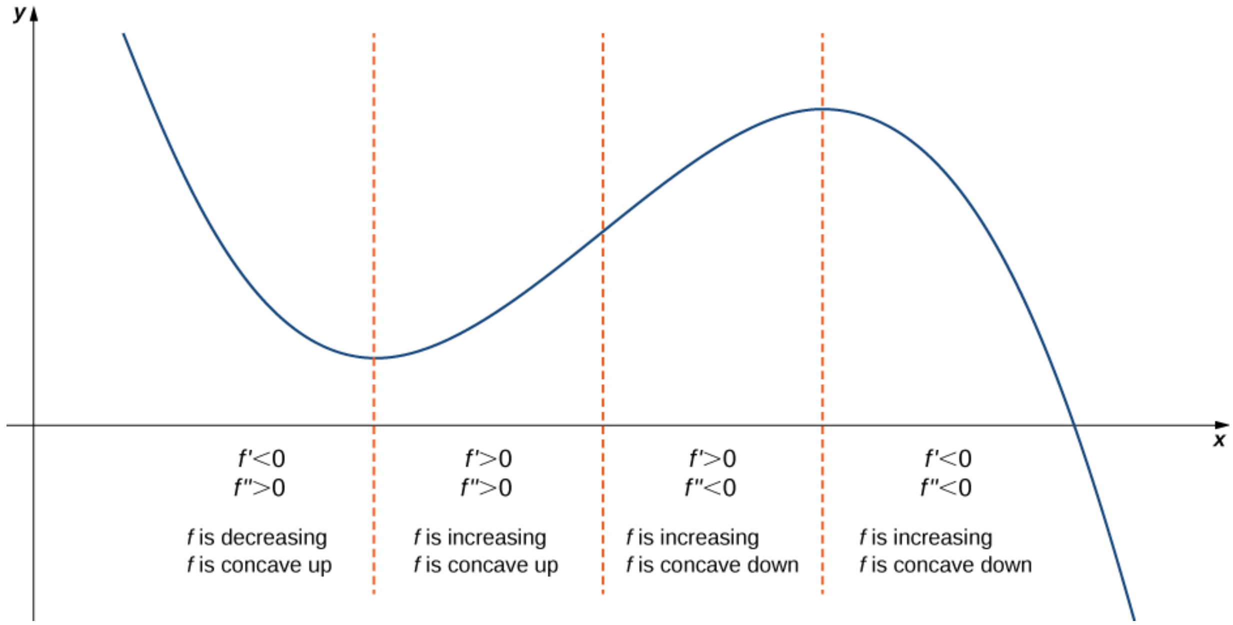
Hint

Find where $f\prime\prime(x) = 0.$

We now summarize, in [link], the information that the first and second derivatives of a function f provide about the graph of f, and illustrate this information in [link].

Sign of $f\prime$	**Sign of $f\prime\prime$**	**Is f increasing or decreasing?**	**Concavity**
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

What Derivatives Tell Us about Graphs


y
x
f'<0
f''>0
f is decreasing
f is concave up
f'>0
f''>0
f is increasing
f is concave up
f'>0
f''<0
f is increasing
f is concave down
f'<0
f''<0
f is increasing
f is concave down

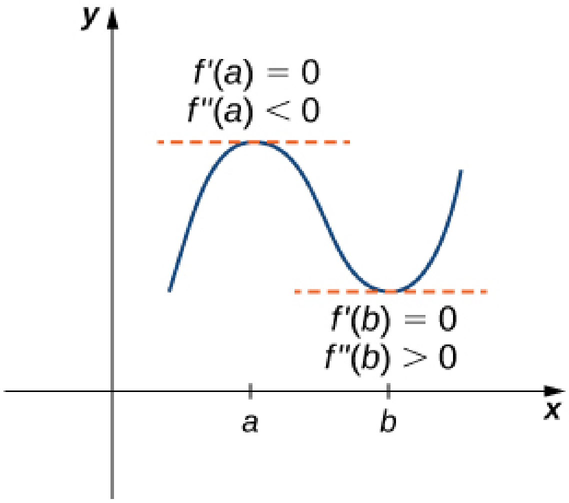
Consider a twice-differentiable function f over an open interval I. If $f'(x) > 0$ for all $x \in I$, the function is increasing over I. If $f'(x) < 0$ for all $x \in I$, the function is decreasing over I. If $f''(x) > 0$ for all $x \in I$, the function is concave up. If $f''(x) < 0$ for all $x \in I$, the function is concave down on I.

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing a. Suppose $f''(a) < 0$. Since f'' is continuous over I, $f''(x) < 0$ for all $x \in I$ ([link]). Then, by Corollary 3, f' is a decreasing function over I. Since $f'(a) = 0$, we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$. On the other hand, suppose there exists a point b such that $f'(b) = 0$ but

$f\prime\prime(b) > 0$. Since $f\prime\prime$ is continuous over an open interval I containing b, then $f\prime\prime(x) > 0$ for all $x \in I$ ([link]). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.



Consider a twice-differentiable function f such that $f\prime\prime$ is continuous. Since $f\prime(a) = 0$ and $f\prime\prime(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f\prime(b) = 0$ and $f\prime\prime(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Note:

Second Derivative Test

Suppose $f'(c) = 0$, $f\prime\prime$ is continuous over an interval containing c.

i. If $f\prime\prime(c) > 0$, then f has a local minimum at c.

ii. If $f''(c) < 0$, then f has a local maximum at c.
iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at x, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let's now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example:
Exercise:

Problem:
Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3$.

Solution:

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0$. The derivative is $f'(x) = 5x^4 - 15x^2$. Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2\left(x^2 - 3\right) = 0$ when $x = 0, \pm\sqrt{3}$.

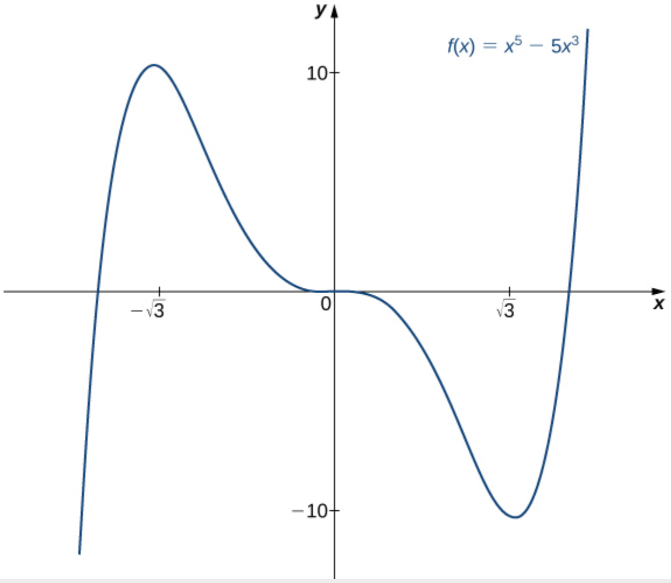
To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is
Equation:

$$f''(x) = 20x^3 - 30x = 10x\left(2x^2 - 3\right).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	**Conclusion**
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x=-\sqrt{3}$ and f has a local minimum at $x=\sqrt{3}$. The second derivative test is inconclusive at $x=0$. To determine whether f has a local extrema at $x=0$, we apply the first derivative test. To evaluate the sign of $f'(x)=5x^2\left(x^2-3\right)$ for $x\in\left(-\sqrt{3},0\right)$ and $x\in\left(0,\sqrt{3}\right)$, let $x=-1$ and $x=1$ be the two test points. Since $f'(-1)<0$ and $f'(1)<0$, we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x=0$ as shown in the following graph.



The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

Note:
Exercise:

Problem:

Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

Solution:

f has a local maximum at -2 and a local minimum at 3.

Hint

$f\prime\prime(x) = 6x - 3$

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.

Key Concepts

- If c is a critical point of f and $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then f has a local maximum at c.
- If c is a critical point of f and $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then f has a local minimum at c.
- If $f\prime\prime(x) > 0$ over an interval I, then f is concave up over I.
- If $f\prime\prime(x) < 0$ over an interval I, then f is concave down over I.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then evaluate $f'(x)$ at a test point x to the left of c and a test point x to the right of c, to determine whether f has a local extremum at c.

Exercise:

Problem:

If c is a critical point of $f(x)$, when is there no local maximum or minimum at c? Explain.

Exercise:

Problem:

For the function $y = x^3$, is $x = 0$ both an inflection point and a local maximum/minimum?

Solution:

It is not a local maximum/minimum because f' does not change sign

Exercise:

Problem: For the function $y = x^3$, is $x = 0$ an inflection point?

Exercise:

Problem:

Is it possible for a point c to be both an inflection point and a local extrema of a twice differentiable function?

Solution:

No

Exercise:

Problem:

Why do you need continuity for the first derivative test? Come up with an example.

Exercise:

Problem:

Explain whether a concave-down function has to cross $y = 0$ for some value of x.

Solution:

False; for example, $y = \sqrt{x}$.

Exercise:

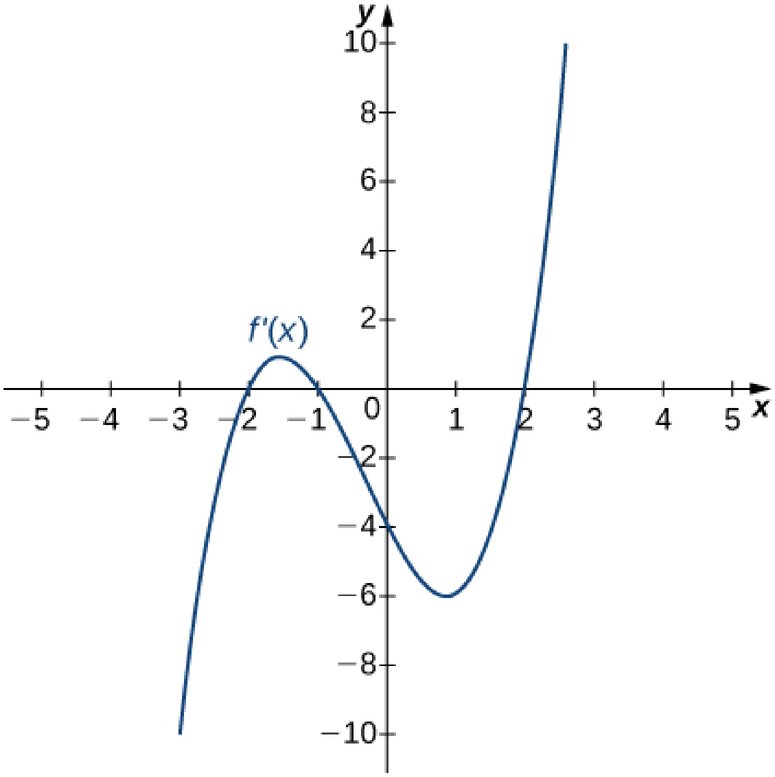
Problem:

Explain whether a polynomial of degree 2 can have an inflection point.

For the following exercises, analyze the graphs of f', then list all intervals where f is increasing or decreasing.

Exercise:

Problem:

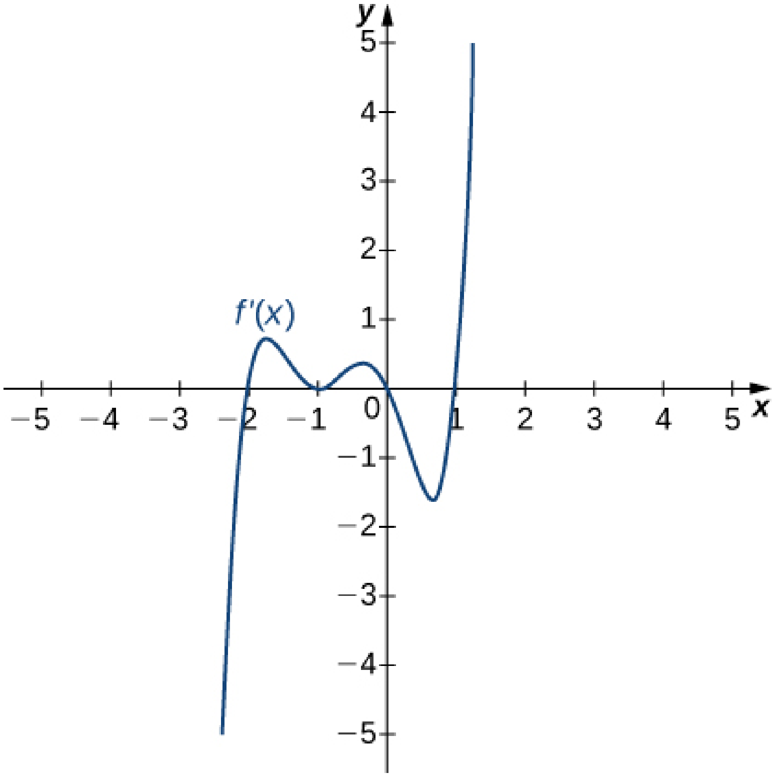


Solution:

Increasing for $-2 < x < -1$ and $x > 2$; decreasing for $x < -2$ and $-1 < x < 2$

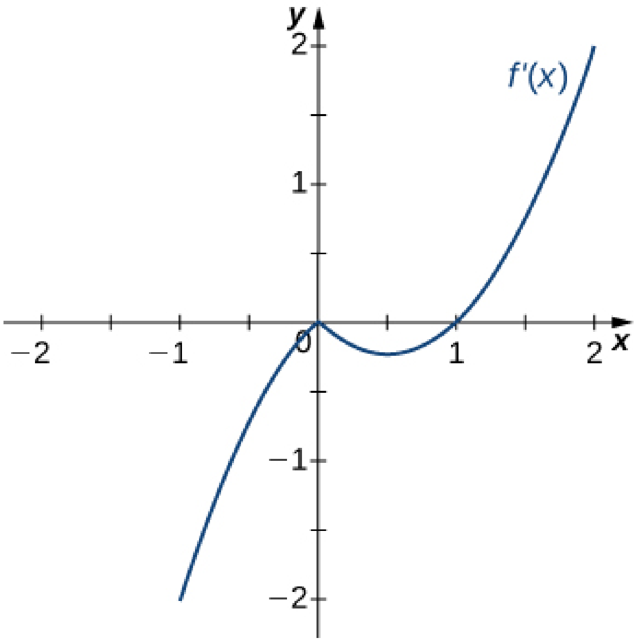
Exercise:

Problem:



Exercise:

Problem:

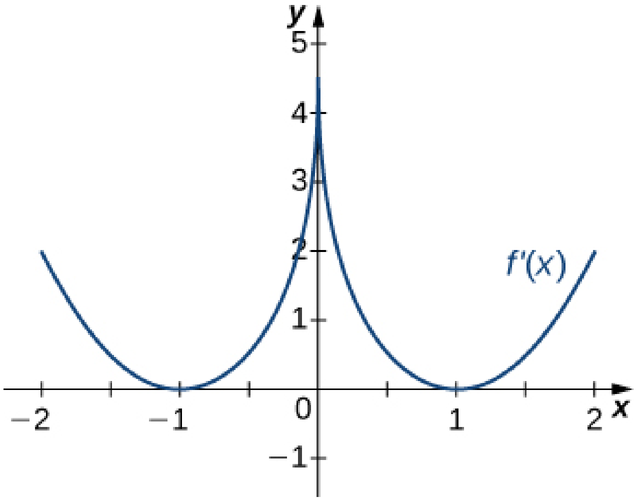


Solution:

Decreasing for $x < 1$, increasing for $x > 1$

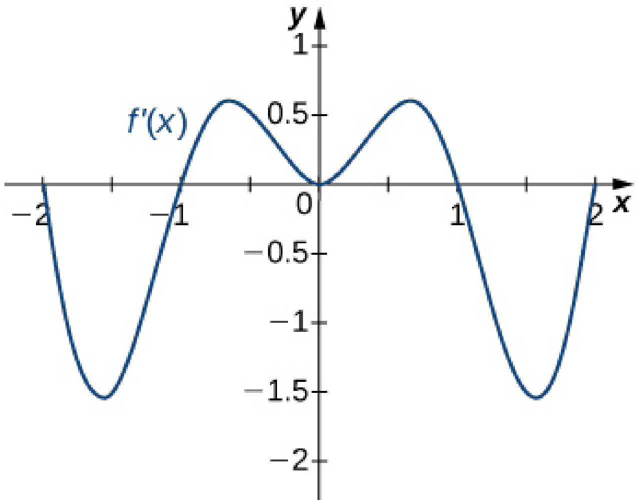
Exercise:

Problem:



Exercise:

Problem:



Solution:

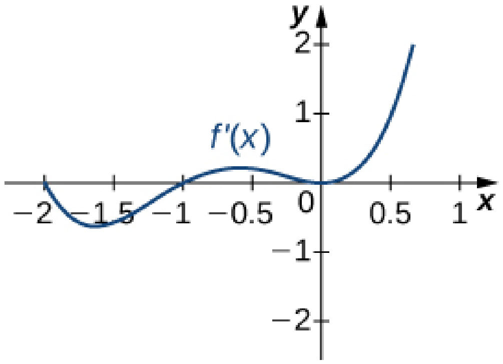
Decreasing for $-2 < x < -1$ and $1 < x < 2$; increasing for $-1 < x < 1$ and $x < -2$ and $x > 2$

For the following exercises, analyze the graphs of f', then list all intervals where

a. f is increasing and decreasing and
b. the minima and maxima are located.

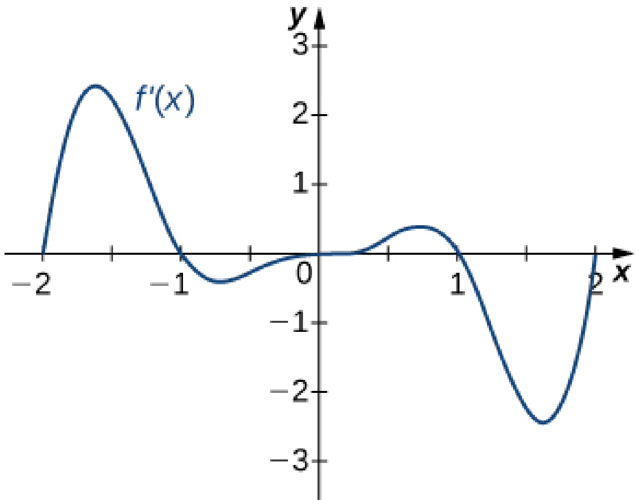
Exercise:

Problem:



Exercise:

Problem:

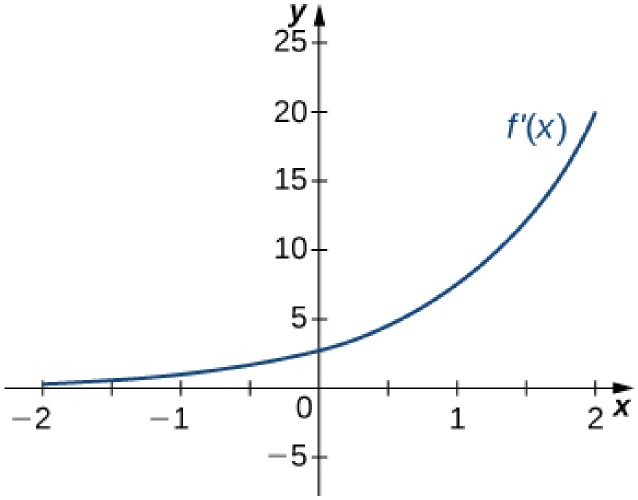


Solution:

a. Increasing over $-2 < x < -1, 0 < x < 1, x > 2$, decreasing over $x < -2$, $-1 < x < 0, 1 < x < 2$; b. maxima at $x = -1$ and $x = 1$, minima at $x = -2$ and $x = 0$ and $x = 2$

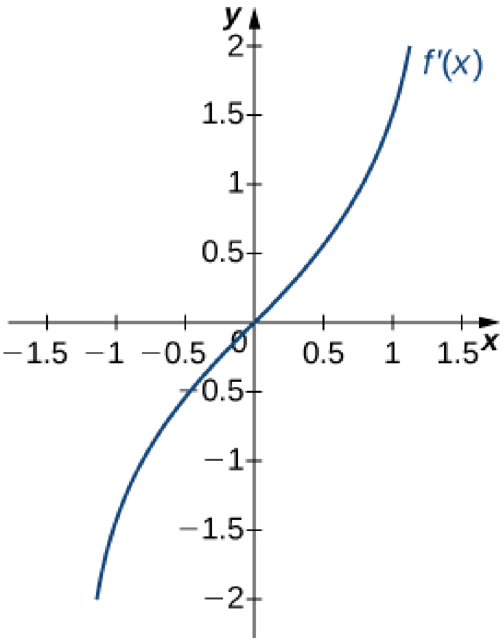
Exercise:

Problem:



Exercise:

Problem:

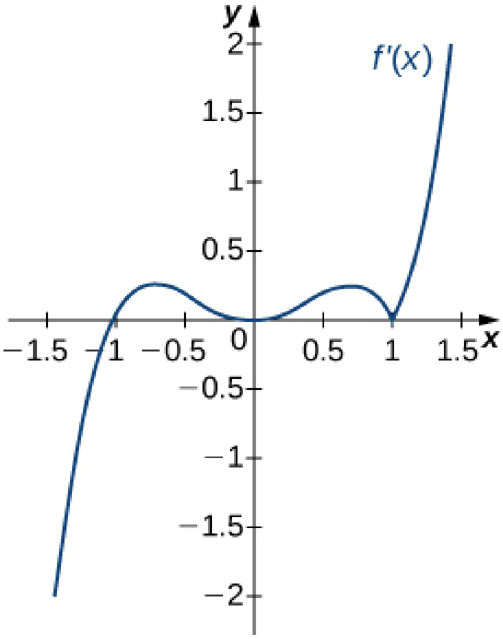


Solution:

a. Increasing over $x > 0$, decreasing over $x < 0$; b. Minimum at $x = 0$

Exercise:

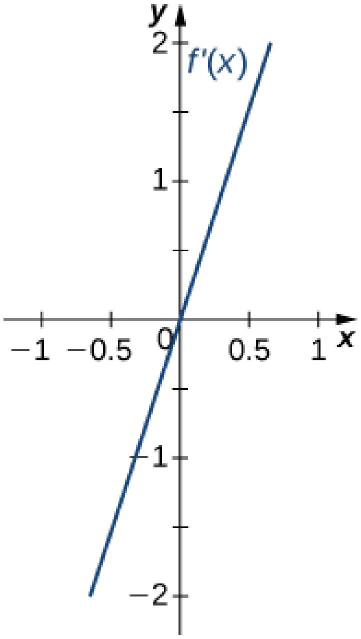
Problem:



For the following exercises, analyze the graphs of f', then list all inflection points and intervals f that are concave up and concave down.

Exercise:

Problem:

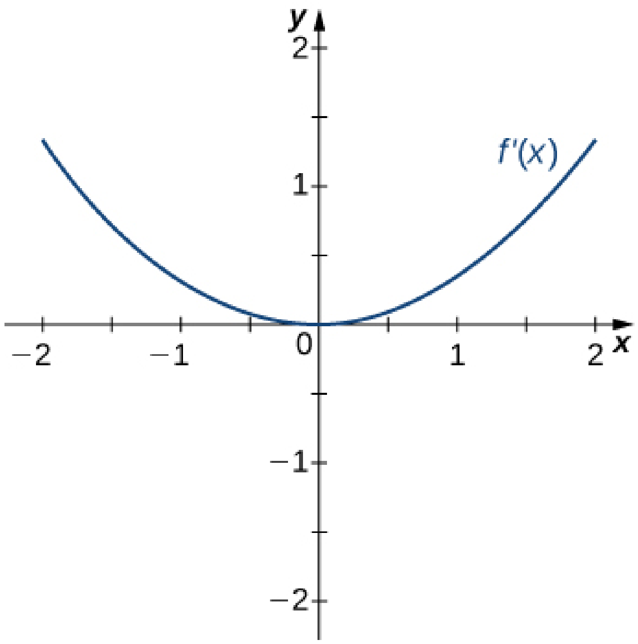


Solution:

Concave up on all x, no inflection points

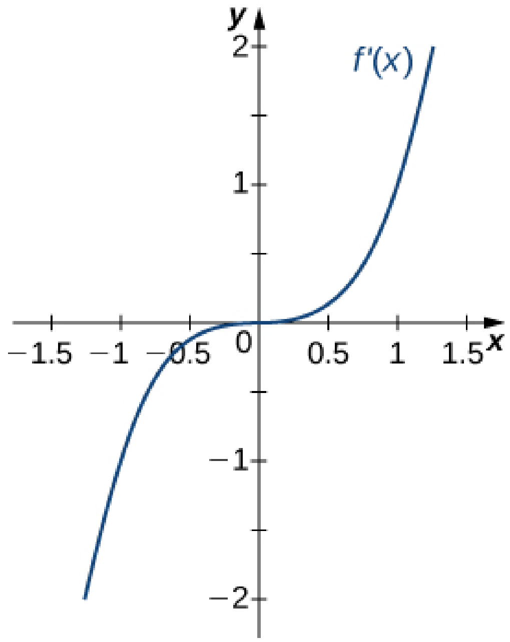
Exercise:

Problem:



Exercise:

Problem:

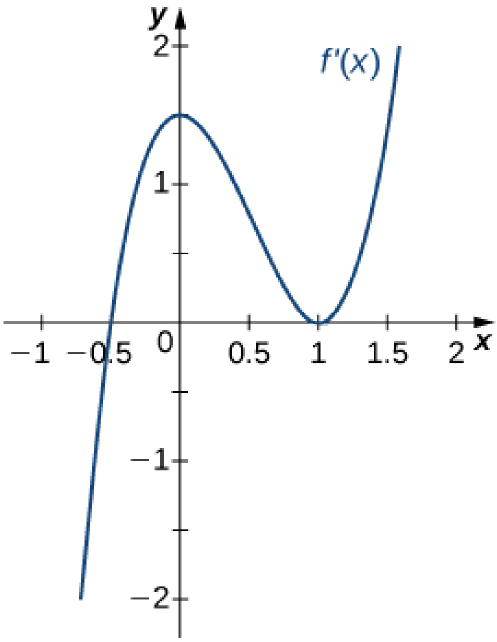


Solution:

Concave up on all x, no inflection points

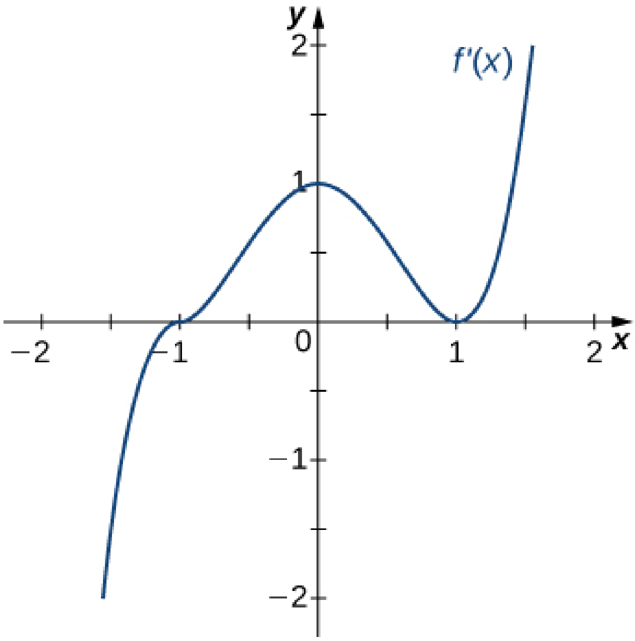
Exercise:

Problem:



Exercise:

Problem:



Solution:

Concave up for $x < 0$ and $x > 1$, concave down for $0 < x < 1$, inflection points at $x = 0$ and $x = 1$

For the following exercises, draw a graph that satisfies the given specifications for the domain $x = [-3, 3]$. The function does not have to be continuous or differentiable.

Exercise:

Problem:

$f(x) > 0$, $f'(x) > 0$ over $x > 1, -3 < x < 0$, $f'(x) = 0$ over $0 < x < 1$

Exercise:

Problem:

$f'(x) > 0$ over $x > 2, -3 < x < -1$, $f'(x) < 0$ over $-1 < x < 2$, $f''(x) < 0$ for all x

Solution:

Answers will vary

Exercise:

Problem:

$f''(x) < 0$ over $-1 < x < 1$, $f''(x) > 0, -3 < x < -1, 1 < x < 3$, local maximum at $x = 0$, local minima at $x = \pm 2$

Exercise:

Problem:

There is a local maximum at $x = 2$, local minimum at $x = 1$, and the graph is neither concave up nor concave down.

Solution:

Answers will vary

Exercise:

Problem:

There are local maxima at $x = \pm 1$, the function is concave up for all x, and the function remains positive for all x.

For the following exercises, determine

a. intervals where f is increasing or decreasing and
b. local minima and maxima of f.

Exercise:

Problem: $f(x) = \sin x + \sin^3 x$ over $-\pi < x < \pi$

Solution:

a. Increasing over $-\frac{\pi}{2} < x < \frac{\pi}{2}$, decreasing over $x < -\frac{\pi}{2}, x > \frac{\pi}{2}$ b. Local maximum at $x = \frac{\pi}{2}$; local minimum at $x = -\frac{\pi}{2}$

Exercise:

Problem: $f(x) = x^2 + \cos x$

For the following exercises, determine a. intervals where f is concave up or concave down, and b. the inflection points of f.
Exercise:

Problem: $f(x) = x^3 - 4x^2 + x + 2$

Solution:

a. Concave up for $x > \frac{4}{3}$, concave down for $x < \frac{4}{3}$ b. Inflection point at $x = \frac{4}{3}$

For the following exercises, determine

a. intervals where f is increasing or decreasing,
b. local minima and maxima of f,

c. intervals where f is concave up and concave down, and
d. the inflection points of f.

Exercise:

Problem: $f\left(x\right)=x^2-6x$

Exercise:

Problem: $f\left(x\right)=x^3-6x^2$

Solution:

a. Increasing over $x<0$ and $x>4$, decreasing over $0<x<4$ b. Maximum at $x=0$, minimum at $x=4$ c. Concave up for $x>2$, concave down for $x<2$ d. Infection point at $x=2$

Exercise:

Problem: $f\left(x\right)=x^4-6x^3$

Exercise:

Problem: $f\left(x\right)=x^{11}-6x^{10}$

Solution:

a. Increasing over $x<0$ and $x>\frac{60}{11}$, decreasing over $0<x<\frac{60}{11}$ b. Minimum at $x=\frac{60}{11}$ c. Concave down for $x<\frac{54}{11}$, concave up for $x>\frac{54}{11}$ d. Inflection point at $x=\frac{54}{11}$

Exercise:

Problem: $f\left(x\right)=x+x^2-x^3$

Exercise:

Problem: $f\left(x\right)=x^2+x+1$

Solution:

a. Increasing over $x > -\frac{1}{2}$, decreasing over $x < -\frac{1}{2}$ b. Minimum at $x = -\frac{1}{2}$ c. Concave up for all x d. No inflection points

Exercise:

Problem: $f(x) = x^3 + x^4$

For the following exercises, determine

a. intervals where f is increasing or decreasing,
b. local minima and maxima of f,
c. intervals where f is concave up and concave down, and
d. the inflection points of f. Sketch the curve, then use a calculator to compare your answer. If you cannot determine the exact answer analytically, use a calculator.

Exercise:

Problem: **[T]** $f(x) = \sin(\pi x) - \cos(\pi x)$ over $x = [-1, 1]$

Solution:

a. Increases over $-\frac{1}{4} < x < \frac{3}{4}$, decreases over $x > \frac{3}{4}$ and $x < -\frac{1}{4}$ b. Minimum at $x = -\frac{1}{4}$, maximum at $x = \frac{3}{4}$ c. Concave up for $-\frac{3}{4} < x < \frac{1}{4}$, concave down for $x < -\frac{3}{4}$ and $x > \frac{1}{4}$ d. Inflection points at $x = -\frac{3}{4}, x = \frac{1}{4}$

Exercise:

Problem: **[T]** $f(x) = x + \sin(2x)$ over $x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Exercise:

Problem: **[T]** $f(x) = \sin x + \tan x$ over $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:

a. Increasing for all x b. No local minimum or maximum c. Concave up for $x > 0$, concave down for $x < 0$ d. Inflection point at $x = 0$

Exercise:

Problem: **[T]** $f\left(x\right)=(x-2)^2(x-4)^2$

Exercise:

Problem: **[T]** $f\left(x\right)=\frac{1}{1-x},x\ne 1$

Solution:

a. Increasing for all x where defined b. No local minima or maxima c. Concave up for $x<1$; concave down for $x>1$ d. No inflection points in domain

Exercise:

Problem: **[T]** $f\left(x\right)=\frac{\sin x}{x}$ over $x=\left[2\pi,0\right)\cup\left(0,2\pi\right]$

Exercise:

Problem: $f\left(x\right)=\sin\left(x\right)e^x$ over $x=\left[-\pi,\pi\right]$

Solution:

a. Increasing over $-\frac{\pi}{4}<x<\frac{3\pi}{4}$, decreasing over $x>\frac{3\pi}{4},x<-\frac{\pi}{4}$ b. Minimum at $x=-\frac{\pi}{4}$, maximum at $x=\frac{3\pi}{4}$ c. Concave up for $-\frac{\pi}{2}<x<\frac{\pi}{2}$, concave down for $x<-\frac{\pi}{2},x>\frac{\pi}{2}$ d. Infection points at $x=\pm\frac{\pi}{2}$

Exercise:

Problem: $f\left(x\right)=\ln x\sqrt{x},x>0$

Exercise:

Problem: $f\left(x\right)=\frac{1}{4}\sqrt{x}+\frac{1}{x},x>0$

Solution:

a. Increasing over $x > 4$, decreasing over $0 < x < 4$ b. Minimum at $x = 4$ c. Concave up for $0 < x < 8\sqrt[3]{2}$, concave down for $x > 8\sqrt[3]{2}$ d. Inflection point at $x = 8\sqrt[3]{2}$

Exercise:

Problem: $f\left(x\right) = \frac{e^x}{x}, x \neq 0$

For the following exercises, interpret the sentences in terms of f, f', and $f\prime\prime$.

Exercise:

Problem: The population is growing more slowly. Here f is the population.

Solution:

$f > 0, f' > 0, f\prime\prime < 0$

Exercise:

Problem:

A bike accelerates faster, but a car goes faster. Here $f =$ Bike's position minus Car's position.

Exercise:

Problem: The airplane lands smoothly. Here f is the plane's altitude.

Solution:

$f > 0, f' < 0, f\prime\prime < 0$

Exercise:

Problem: Stock prices are at their peak. Here f is the stock price.

Exercise:

Problem:

The economy is picking up speed. Here f is a measure of the economy, such as GDP.

Solution:

$f > 0, f' > 0, f'' > 0$

For the following exercises, consider a third-degree polynomial $f\left(x\right)$, which has the properties $f'\left(1\right) = 0, f'\left(3\right) = 0$. Determine whether the following statements are *true or false*. Justify your answer.

Exercise:

Problem: $f\left(x\right) = 0$ for some $1 \le x \le 3$

Exercise:

Problem: $f''\left(x\right) = 0$ for some $1 \le x \le 3$

Solution:

True, by the Mean Value Theorem

Exercise:

Problem: There is no absolute maximum at $x = 3$

Exercise:

Problem: If $f\left(x\right)$ has three roots, then it has 1 inflection point.

Solution:

True, examine derivative

Exercise:

Problem: If $f\left(x\right)$ has one inflection point, then it has three real roots.

Glossary

concave down

if f is differentiable over an interval I and f' is decreasing over I, then f is concave down over I

concave up

if f is differentiable over an interval I and f' is increasing over I, then f is concave up over I

concavity

the upward or downward curve of the graph of a function

concavity test

suppose f is twice differentiable over an interval I; if $f'' > 0$ over I, then f is concave up over I; if $f'' < 0$ over I, then f is concave down over I

first derivative test

let f be a continuous function over an interval I containing a critical point c such that f is differentiable over I except possibly at c; if f' changes sign from positive to negative as x increases through c, then f has a local maximum at c; if f' changes sign from negative to positive as x increases through c, then f has a local minimum at c; if f' does not change sign as x increases through c, then f does not have a local extremum at c

inflection point

if f is continuous at c and f changes concavity at c, the point $(c, f(c))$ is an inflection point of f

second derivative test

suppose $f'(c) = 0$ and f'' is continuous over an interval containing c; if $f''(c) > 0$, then f has a local minimum at c; if $f''(c) < 0$, then f has a local maximum at c; if $f''(c) = 0$, then the test is inconclusive

Applied Optimization Problems

- Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

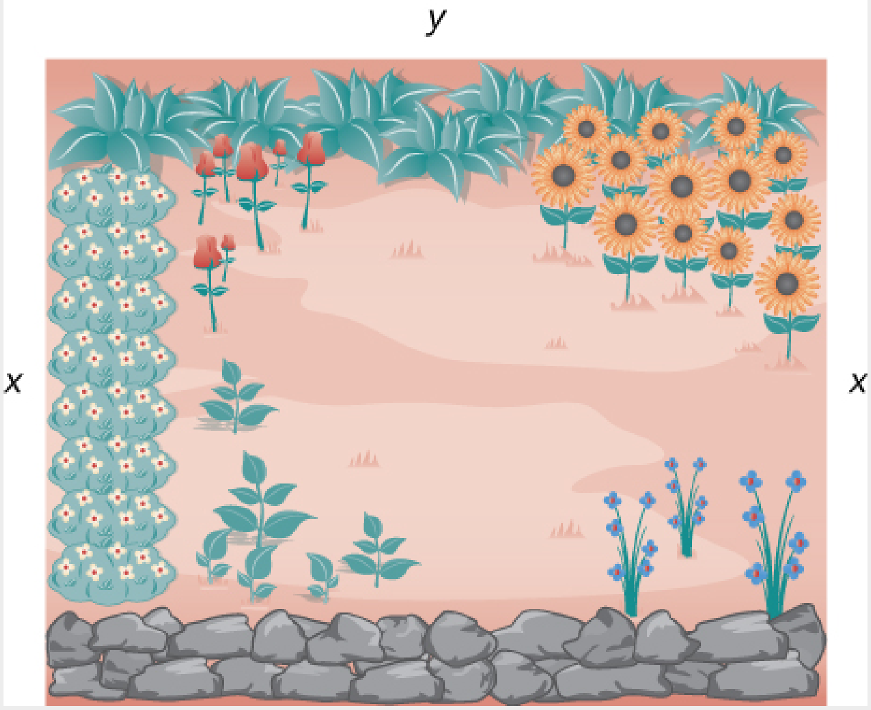
Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in [link], we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

Example:
Exercise:

Problem:
Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides ([link]). Given 100 ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?



We want to determine the measurements x and y that will create a garden with a maximum area using 100 ft of fencing.

Solution:

Let x denote the length of the side of the garden perpendicular to the rock wall and y denote the length of the side parallel to the rock wall. Then the area of the garden is

Equation:

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is $100\,\text{ft}$. From [link], the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

Equation:

$$2x + y = 100.$$

Solving this equation for y, we have $y = 100 - 2x$. Thus, we can write the area as

Equation:

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for x over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value x in the open interval $(0, 50)$ that maximizes the area of the garden. Therefore, we consider the following problem:

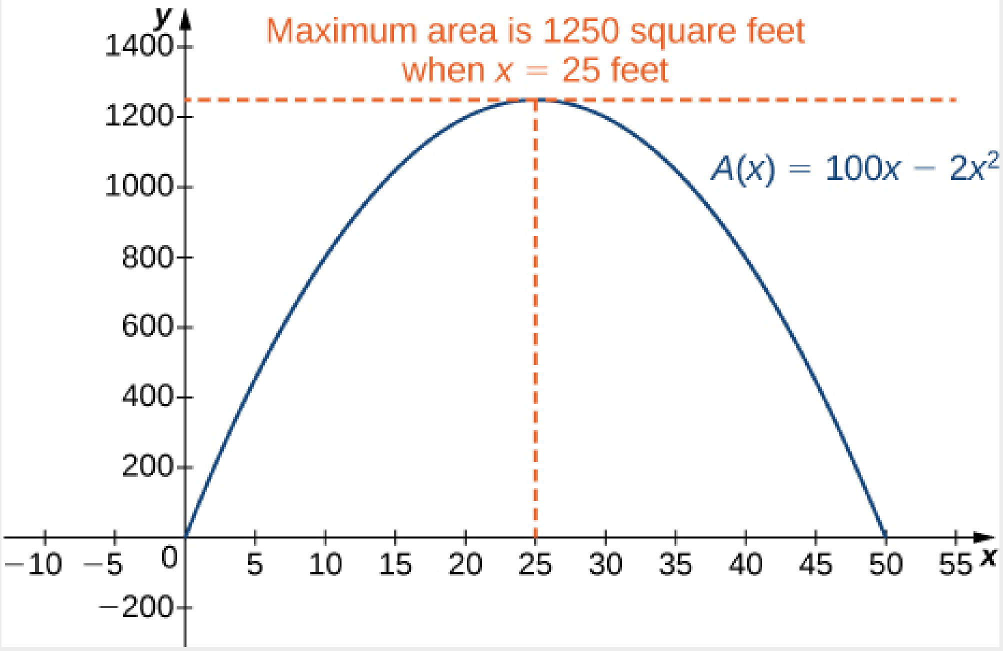
Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since A is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all x in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

Equation:

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ ([link]). We conclude that the maximum area must occur when $x = 25$. Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50\,\text{ft}$. The area of this garden is $1250\,\text{ft}^2$.



To maximize the area of the garden, we need to find the maximum value of the function $A\left(x\right)=100x-2x^2$.

Note:
Exercise:

Problem:

Determine the maximum area if we want to make the same rectangular garden as in [link], but we have 200 ft of fencing.

Solution:

The maximum area is $5000\ \text{ft}^2$.

Hint

We need to maximize the function $A\left(x\right)=200x-2x^2$ over the interval $\left[0,100\right]$.

Now let’s look at a general strategy for solving optimization problems similar to [link].

Note:
Problem-Solving Strategy: Solving Optimization Problems

1. Introduce all variables. If applicable, draw a figure and label all variables.
2. Determine which quantity is to be maximized or minimized, and for what range of values of the other variables (if this can be determined at this time).
3. Write a formula for the quantity to be maximized or minimized in terms of the variables. This formula may involve more than one variable.
4. Write any equations relating the independent variables in the formula from step 3. Use these equations to write the quantity to be maximized or minimized as a function of one variable.
5. Identify the domain of consideration for the function in step 4 based on the physical problem to be solved.
6. Locate the maximum or minimum value of the function from step 4. This step typically involves looking for critical points and evaluating a function at endpoints.

Now let’s apply this strategy to maximize the volume of an open-top box given a constraint on the amount of material to be used.

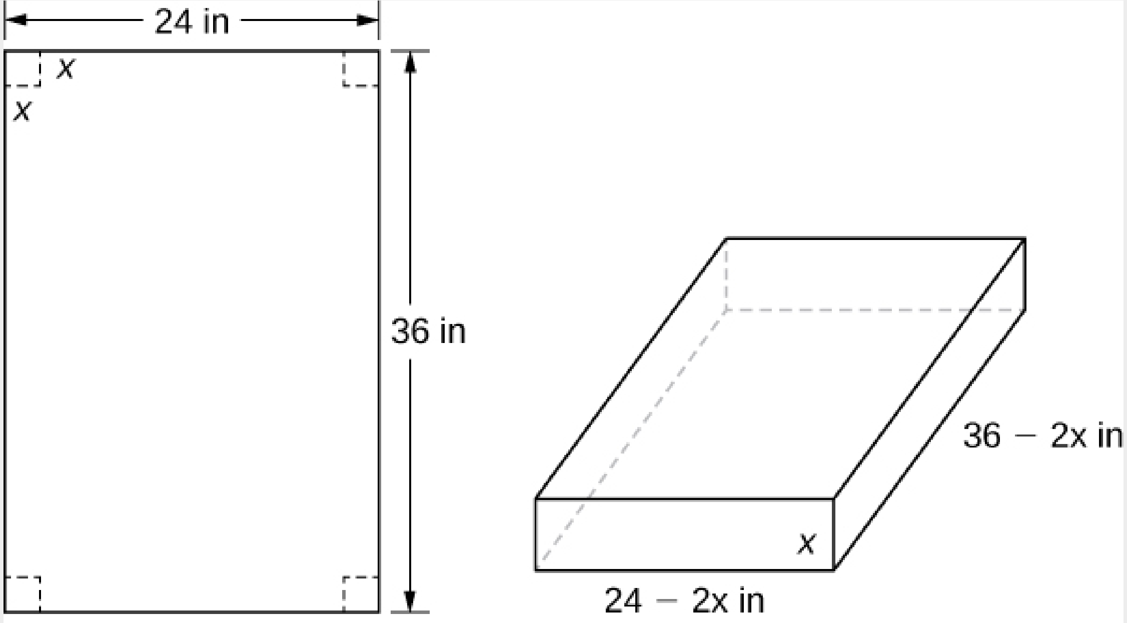
Example:
Exercise:

Problem:
Maximizing the Volume of a Box

An open-top box is to be made from a 24 in. by 36 in. piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What size square should be cut out of each corner to get a box with the maximum volume?

Solution:

Step 1: Let x be the side length of the square to be removed from each corner ([link]). Then, the remaining four flaps can be folded up to form an open-top box. Let V be the volume of the resulting box.



A square with side length x inches is removed from each corner of the piece of cardboard. The remaining flaps are folded to form an open-top box.

Step 2: We are trying to maximize the volume of a box. Therefore, the problem is to maximize V.

Step 3: As mentioned in step 2, are trying to maximize the volume of a box. The volume of a box is $V = L \cdot W \cdot H$, where $L, W,$ and H are the length, width, and height, respectively.

Step 4: From [link], we see that the height of the box is x inches, the length is $36 - 2x$ inches, and the width is $24 - 2x$ inches. Therefore, the volume of the box is

Equation:

$$V\left(x\right) = \left(36 - 2x\right)\left(24 - 2x\right)x = 4x^3 - 120x^2 + 864x.$$

Step 5: To determine the domain of consideration, let’s examine [link]. Certainly, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in.; otherwise, one of the flaps would be completely cut off. Therefore, we are trying to determine whether there is a maximum volume of the box for x over the open interval $(0, 12)$. Since V is a continuous function over the closed interval $[0, 12]$, we know V will have an absolute maximum over the closed interval. Therefore, we consider V over the closed interval $[0, 12]$ and check whether the absolute maximum occurs at an interior point.

Step 6: Since $V(x)$ is a continuous function over the closed, bounded interval $[0, 12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

Equation:

$$V'(x) = 12x^2 - 240x + 864.$$

To find the critical points, we need to solve the equation

Equation:

$$12x^2 - 240x + 864 = 0.$$

Dividing both sides of this equation by 12, the problem simplifies to solving the equation

Equation:

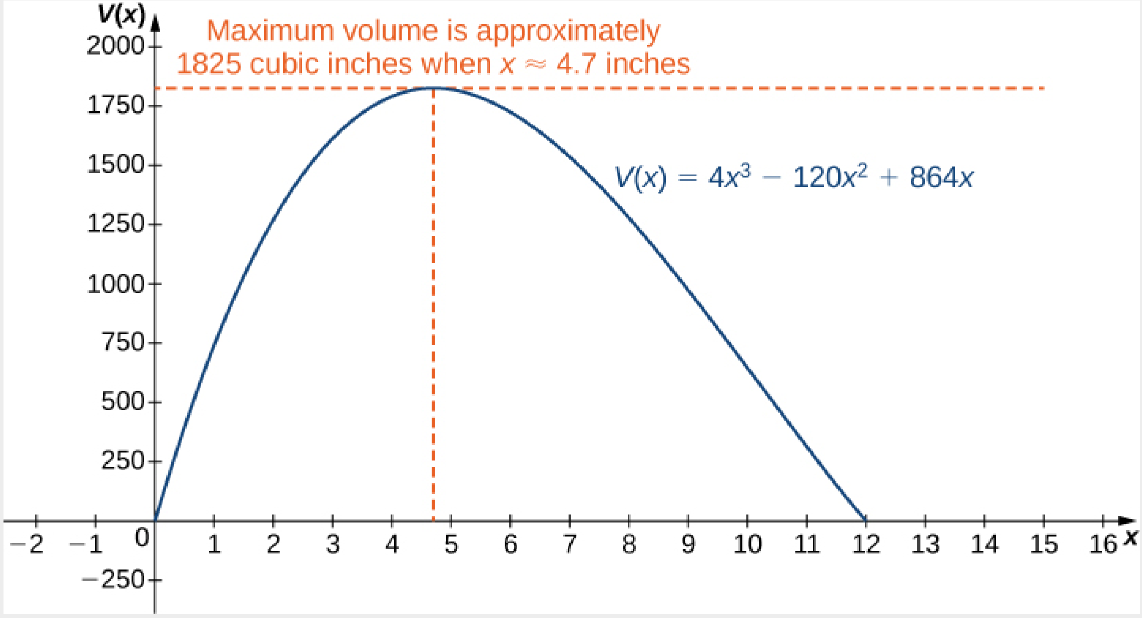
$$x^2 - 20x + 72 = 0.$$

Using the quadratic formula, we find that the critical points are

Equation:

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 4\sqrt{7}}{2} = 10 \pm 2\sqrt{7}.$$

Since $10+2\sqrt{7}$ is not in the domain of consideration, the only critical point we need to consider is $10-2\sqrt{7}$. Therefore, the volume is maximized if we let $x=10-2\sqrt{7}$ in. The maximum volume is $V\left(10-2\sqrt{7}\right)=640+448\sqrt{7}\approx 1825$ in.^{3} as shown in the following graph.



Maximizing the volume of the box leads to finding the maximum value of a cubic polynomial.

Note:
Watch a video about optimizing the volume of a box.

Note:
Exercise:

Problem:

Suppose the dimensions of the cardboard in [link] are 20 in. by 30 in. Let x be the side length of each square and write the volume of the open-top box as a function of x. Determine the domain of consideration for x.

Solution:

$V\left(x\right)=x\left(20-2x\right)\left(30-2x\right)$. The domain is $\left[0,10\right]$.

Hint

The volume of the box is $L\cdot W\cdot H$.

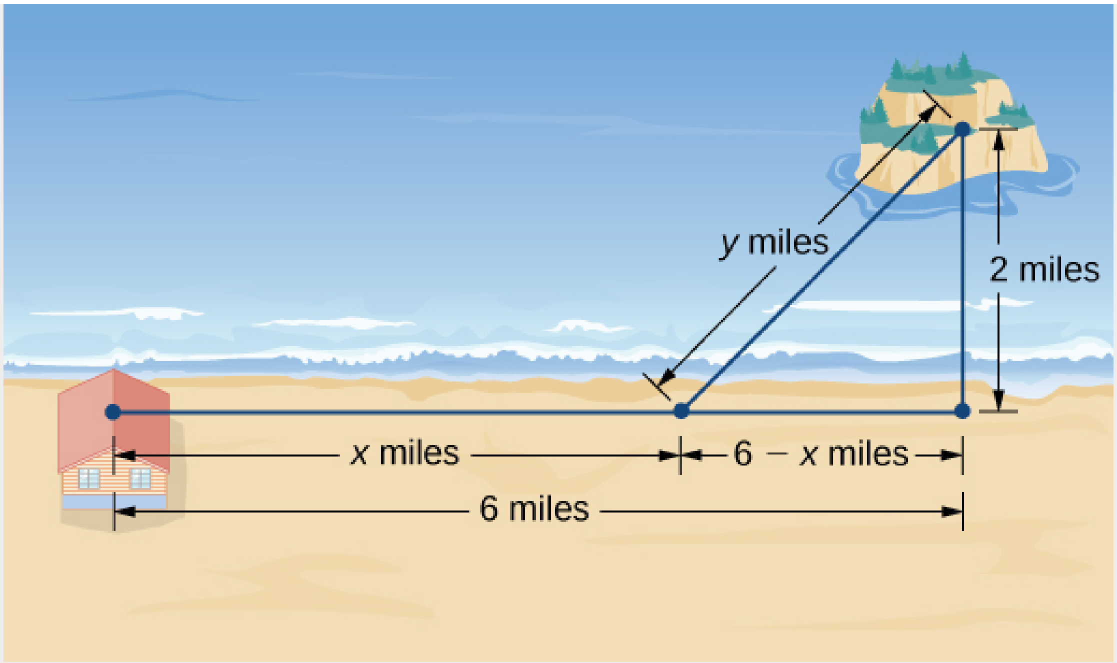
Example:
Exercise:

Problem:
Minimizing Travel Time

An island is $2\phantom{\rule{0.2em}{0ex}}\text{mi}$ due north of its closest point along a straight shoreline. A visitor is staying at a cabin on the shore that is $6\phantom{\rule{0.2em}{0ex}}\text{mi}$ west of that point. The visitor is planning to go from the cabin to the island. Suppose the visitor runs at a rate of $8\phantom{\rule{0.2em}{0ex}}\text{mph}$ and swims at a rate of $3\phantom{\rule{0.2em}{0ex}}\text{mph}$. How far should the visitor run before swimming to minimize the time it takes to reach the island?

Solution:

Step 1: Let x be the distance running and let y be the distance swimming ([link]). Let T be the time it takes to get from the cabin to the island.



How can we choose x and y to minimize the travel time from the cabin to the island?

Step 2: The problem is to minimize T.

Step 3: To find the time spent traveling from the cabin to the island, add the time spent running and the time spent swimming. Since Distance $=$ Rate $\times$ Time $(D = R \times T)$, the time spent running is

Equation:

$$T_{\text{running}} = \frac{D_{\text{running}}}{R_{\text{running}}} = \frac{x}{8},$$

and the time spent swimming is

Equation:

$$T_{\text{swimming}} = \frac{D_{\text{swimming}}}{R_{\text{swimming}}} = \frac{y}{3}.$$

Therefore, the total time spent traveling is

Equation:

$$T = \frac{x}{8} + \frac{y}{3}.$$

Step 4: From [link], the line segment of y miles forms the hypotenuse of a right triangle with legs of length 2 mi and $6 - x\text{ mi}$. Therefore, by the Pythagorean theorem, $2^2 + (6-x)^2 = y^2$, and we obtain $y = \sqrt{(6-x)^2 + 4}$. Thus, the total time spent traveling is given by the function

Equation:

$$T(x) = \frac{x}{8} + \frac{\sqrt{(6-x)^2+4}}{3}.$$

Step 5: From [link], we see that $0 \le x \le 6$. Therefore, $[0, 6]$ is the domain of consideration.

Step 6: Since $T(x)$ is a continuous function over a closed, bounded interval, it has a maximum and a minimum. Let's begin by looking for any critical points of T over the interval $[0, 6]$. The derivative is

Equation:

$$T'(x) = \frac{1}{8} - \frac{1}{2}\frac{\left[(6-x)^2+4\right]^{-1/2}}{3} \cdot 2(6-x) = \frac{1}{8} - \frac{(6-x)}{3\sqrt{(6-x)^2+4}}.$$

If $T'(x) = 0$, then

Equation:

$$\frac{1}{8} = \frac{6-x}{3\sqrt{(6-x)^2+4}}.$$

Therefore,

Equation:

$$3\sqrt{(6-x)^2+4}=8(6-x).$$

Squaring both sides of this equation, we see that if x satisfies this equation, then x must satisfy

Equation:

$$9\left[(6-x)^2+4\right]=64(6-x)^2,$$

which implies

Equation:

$$55(6-x)^2=36.$$

We conclude that if x is a critical point, then x satisfies

Equation:

$$(x-6)^2=\frac{36}{55}.$$

Therefore, the possibilities for critical points are

Equation:

$$x=6\pm\frac{6}{\sqrt{55}}.$$

Since $x=6+6/\sqrt{55}$ is not in the domain, it is not a possibility for a critical point. On the other hand, $x=6-6/\sqrt{55}$ is in the domain. Since we squared both sides of [link] to arrive at the possible critical points, it remains to verify that $x=6-6/\sqrt{55}$ satisfies [link]. Since $x=6-6/\sqrt{55}$ does satisfy that equation, we conclude that $x=6-6/\sqrt{55}$ is a critical point, and it is the only one. To justify that the time is minimized for this value of x, we just need to check the values of

$T(x)$ at the endpoints $x = 0$ and $x = 6$, and compare them with the value of $T(x)$ at the critical point $x = 6 - 6/\sqrt{55}$. We find that $T(0) \approx 2.108\,\text{h}$ and $T(6) \approx 1.417\,\text{h}$, whereas $T\left(6 - 6/\sqrt{55}\right) \approx 1.368\,\text{h}$. Therefore, we conclude that T has a local minimum at $x \approx 5.19\,\text{mi}$.

Note:
Exercise:

Problem:

Suppose the island is $1\,\text{mi}$ from shore, and the distance from the cabin to the point on the shore closest to the island is $15\,\text{mi}$. Suppose a visitor swims at the rate of $2.5\,\text{mph}$ and runs at a rate of $6\,\text{mph}$. Let x denote the distance the visitor will run before swimming, and find a function for the time it takes the visitor to get from the cabin to the island.

Solution:

$$T(x) = \frac{x}{6} + \frac{\sqrt{(15-x)^2+1}}{2.5}$$

Hint

The time $T = T_{\text{running}} + T_{\text{swimming}}$.

In business, companies are interested in maximizing revenue. In the following example, we consider a scenario in which a company has collected data on how many cars it is able to lease, depending on the price it charges its customers to rent a car. Let’s use these data to determine the price the company should charge to maximize the amount of money it brings in.

Example:
Exercise:

Problem:
Maximizing Revenue

Owners of a car rental company have determined that if they charge customers p dollars per day to rent a car, where $50 \leq p \leq 200$, the number of cars n they rent per day can be modeled by the linear function $n\left(p\right) = 1000 - 5p$. If they charge \$50 per day or less, they will rent all their cars. If they charge \$200 per day or more, they will not rent any cars. Assuming the owners plan to charge customers between \$50 per day and \$200 per day to rent a car, how much should they charge to maximize their revenue?

Solution:

Step 1: Let p be the price charged per car per day and let n be the number of cars rented per day. Let R be the revenue per day.

Step 2: The problem is to maximize R.

Step 3: The revenue (per day) is equal to the number of cars rented per day times the price charged per car per day—that is, $R = n \ \times \ p$.

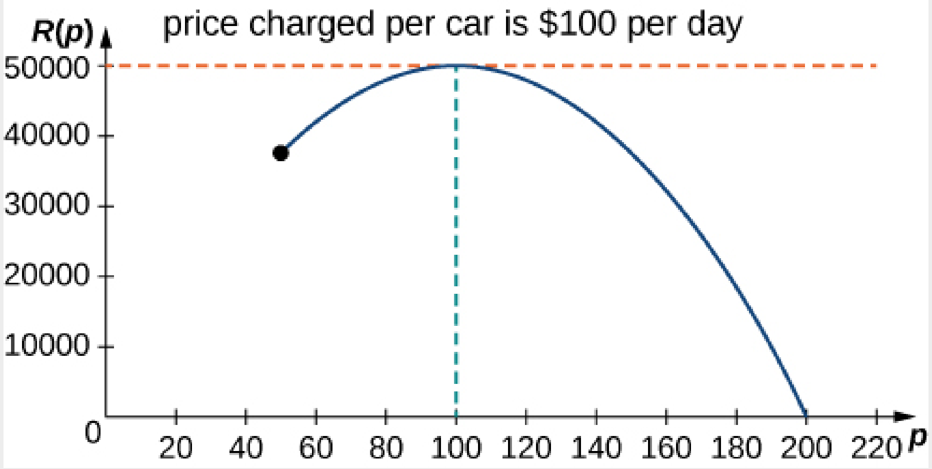
Step 4: Since the number of cars rented per day is modeled by the linear function $n\left(p\right) = 1000 - 5p$, the revenue R can be represented by the function
Equation:

$$R\left(p\right) = n \ \times \ p = (1000 - 5p)p = -5p^2 + 1000p.$$

Step 5: Since the owners plan to charge between \$50 per car per day and \$200 per car per day, the problem is to find the maximum revenue $R\left(p\right)$ for p in the closed interval $[50, 200]$.

Step 6: Since R is a continuous function over the closed, bounded interval $[50, 200]$, it has an absolute maximum (and an absolute minimum) in that interval. To find the maximum value, look for critical points. The derivative is $R'\left(p\right) = -10p + 1000$. Therefore, the critical point is $p = 100$ When $p = 100$, $R\left(100\right) = \$50{,}000$. When $p = 50$, $R\left(p\right) = \$37{,}500$. When $p = 200$, $R\left(p\right) = \$0$. Therefore, the absolute maximum occurs at

$p = \$100$. The car rental company should charge \$100 per day per car to maximize revenue as shown in the following figure.



To maximize revenue, a car rental company has to balance the price of a rental against the number of cars people will rent at that price.

Note:
Exercise:

Problem:

A car rental company charges its customers p dollars per day, where $60 \leq p \leq 150$. It has found that the number of cars rented per day can be modeled by the linear function $n\left(p\right) = 750 - 5p$. How much should the company charge each customer to maximize revenue?

Solution:

The company should charge \$75 per car per day.

Hint

$R\left(p\right) = n\ \times\ p$, where n is the number of cars rented and p is the price charged per car.

Example:
Exercise:

Problem:
Maximizing the Area of an Inscribed Rectangle

A rectangle is to be inscribed in the ellipse

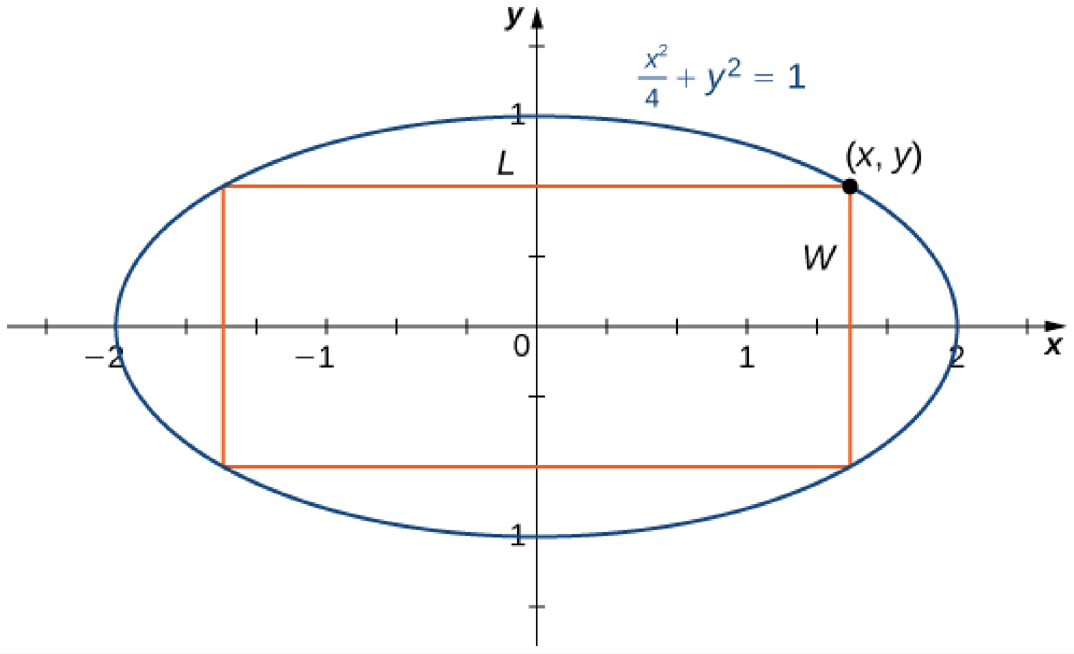
Equation:

$$\frac{x^2}{4} + y^2 = 1.$$

What should the dimensions of the rectangle be to maximize its area? What is the maximum area?

Solution:

Step 1: For a rectangle to be inscribed in the ellipse, the sides of the rectangle must be parallel to the axes. Let L be the length of the rectangle and W be its width. Let A be the area of the rectangle.



We want to maximize the area of a rectangle inscribed in an ellipse.

Step 2: The problem is to maximize A.

Step 3: The area of the rectangle is $A = LW$.

Step 4: Let (x, y) be the corner of the rectangle that lies in the first quadrant, as shown in [link]. We can write length $L = 2x$ and width $W = 2y$. Since $\frac{x^2}{4} + y^2 = 1$ and $y > 0$, we have $y = \sqrt{1 - \frac{x^2}{4}}$. Therefore, the area is

Equation:

$$A = LW = (2x)\,(2y) = 4x\sqrt{1 - \frac{x^2}{4}} = 2x\sqrt{4 - x^2}.$$

Step 5: From [link], we see that to inscribe a rectangle in the ellipse, the x-coordinate of the corner in the first quadrant must satisfy $0 < x < 2$. Therefore, the problem reduces to looking for the maximum value of $A\,(x)$ over the open interval $(0, 2)$. Since $A\,(x)$ will have an absolute maximum (and absolute minimum) over the closed interval $[0, 2]$, we consider $A\,(x) = 2x\sqrt{4 - x^2}$ over the interval $[0, 2]$. If the absolute maximum occurs at an interior point, then we have found an absolute maximum in the open interval.

Step 6: As mentioned earlier, $A\,(x)$ is a continuous function over the closed, bounded interval $[0, 2]$. Therefore, it has an absolute maximum (and absolute minimum). At the endpoints $x = 0$ and $x = 2$, $A\,(x) = 0$. For $0 < x < 2$, $A\,(x) > 0$. Therefore, the maximum must occur at a critical point. Taking the derivative of $A\,(x)$, we obtain

Equation:

$$\begin{aligned} A'(x) &= 2\sqrt{4 - x^2} + 2x \cdot \tfrac{1}{2\sqrt{4 - x^2}}(-2x) \\ &= 2\sqrt{4 - x^2} - \tfrac{2x^2}{\sqrt{4 - x^2}} \\ &= \tfrac{8 - 4x^2}{\sqrt{4 - x^2}}. \end{aligned}$$

To find critical points, we need to find where $A'(x) = 0$. We can see that if x is a solution of

Equation:

$$\frac{8 - 4x^2}{\sqrt{4 - x^2}} = 0,$$

then x must satisfy

Equation:

$$8 - 4x^2 = 0.$$

Therefore, $x^2 = 2$. Thus, $x = \pm\sqrt{2}$ are the possible solutions of [link]. Since we are considering x over the interval $[0, 2]$, $x = \sqrt{2}$ is a possibility for a critical point, but $x = -\sqrt{2}$ is not. Therefore, we check whether $\sqrt{2}$ is a solution of [link]. Since $x = \sqrt{2}$ is a solution of [link], we conclude that $\sqrt{2}$ is the only critical point of $A\left(x\right)$ in the interval $[0, 2]$. Therefore, $A\left(x\right)$ must have an absolute maximum at the critical point $x = \sqrt{2}$. To determine the dimensions of the rectangle, we need to find the length L and the width W. If $x = \sqrt{2}$ then

Equation:

$$y = \overline{1 - \frac{\left(\sqrt{2}\right)^2}{4}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Therefore, the dimensions of the rectangle are $L = 2x = 2\sqrt{2}$ and $W = 2y = \frac{2}{\sqrt{2}} = \sqrt{2}$. The area of this rectangle is $A = LW = \left(2\sqrt{2}\right)\left(\sqrt{2}\right) = 4.$

Note:

Exercise:

Problem:

Modify the area function A if the rectangle is to be inscribed in the unit circle $x^2 + y^2 = 1$. What is the domain of consideration?

Solution:

$A\left(x\right) = 4x\sqrt{1 - x^2}$. The domain of consideration is $\left[0, 1\right]$.

Hint

If $\left(x, y\right)$ is the vertex of the square that lies in the first quadrant, then the area of the square is $A = \left(2x\right)\left(2y\right) = 4xy$.

Solving Optimization Problems when the Interval Is Not Closed or Is Unbounded

In the previous examples, we considered functions on closed, bounded domains. Consequently, by the extreme value theorem, we were guaranteed that the functions had absolute extrema. Let’s now consider functions for which the domain is neither closed nor bounded.

Many functions still have at least one absolute extrema, even if the domain is not closed or the domain is unbounded. For example, the function $f\left(x\right) = x^2 + 4$ over $\left(-\infty, \infty\right)$ has an absolute minimum of 4 at $x = 0$. Therefore, we can still consider functions over unbounded domains or open intervals and determine whether they have any absolute extrema. In the next example, we try to minimize a function over an unbounded domain. We will see that, although the domain of consideration is $\left(0, \infty\right)$, the function has an absolute minimum.

In the following example, we look at constructing a box of least surface area with a prescribed volume. It is not difficult to show that for a closed-top box, by symmetry, among all boxes with a specified volume, a cube will have the smallest surface area. Consequently, we consider the modified problem of determining which open-topped box with a specified volume has the smallest surface area.

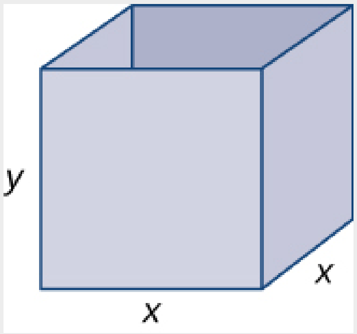
Example:
Exercise:

Problem:
Minimizing Surface Area

A rectangular box with a square base, an open top, and a volume of 216 in.^3 is to be constructed. What should the dimensions of the box be to minimize the surface area of the box? What is the minimum surface area?

Solution:

Step 1: Draw a rectangular box and introduce the variable x to represent the length of each side of the square base; let y represent the height of the box. Let S denote the surface area of the open-top box.



We want to minimize the surface area of a square-based box with a given volume.

Step 2: We need to minimize the surface area. Therefore, we need to minimize S.

Step 3: Since the box has an open top, we need only determine the area of the four vertical sides and the base. The area of each of the four vertical sides is $x \cdot y$. The area of the base is x^2. Therefore, the surface area of the box is

Equation:

$$S = 4xy + x^2.$$

Step 4: Since the volume of this box is x^2y and the volume is given as 216 in.^3, the constraint equation is

Equation:

$$x^2y = 216.$$

Solving the constraint equation for y, we have $y = \frac{216}{x^2}$. Therefore, we can write the surface area as a function of x only:

Equation:

$$S(x) = 4x\left(\frac{216}{x^2}\right) + x^2.$$

Therefore, $S(x) = \frac{864}{x} + x^2$.

Step 5: Since we are requiring that $x^2y = 216$, we cannot have $x = 0$. Therefore, we need $x > 0$. On the other hand, x is allowed to have any positive value. Note that as x becomes large, the height of the box y becomes correspondingly small so that $x^2y = 216$. Similarly, as x becomes small, the height of the box becomes correspondingly large. We conclude that the domain is the open, unbounded interval $(0, \infty)$. Note that, unlike the previous examples, we cannot reduce our problem to looking for an absolute maximum or absolute minimum over a closed, bounded interval. However, in the next step, we discover why this function must have an absolute minimum over the interval $(0, \infty)$.

Step 6: Note that as $x \to 0^+$, $S(x) \to \infty$. Also, as $x \to \infty$, $S(x) \to \infty$. Since S is a continuous function that approaches infinity at the ends, it must have an absolute minimum at some $x \in (0, \infty)$. This minimum must occur at a critical point of S. The derivative is

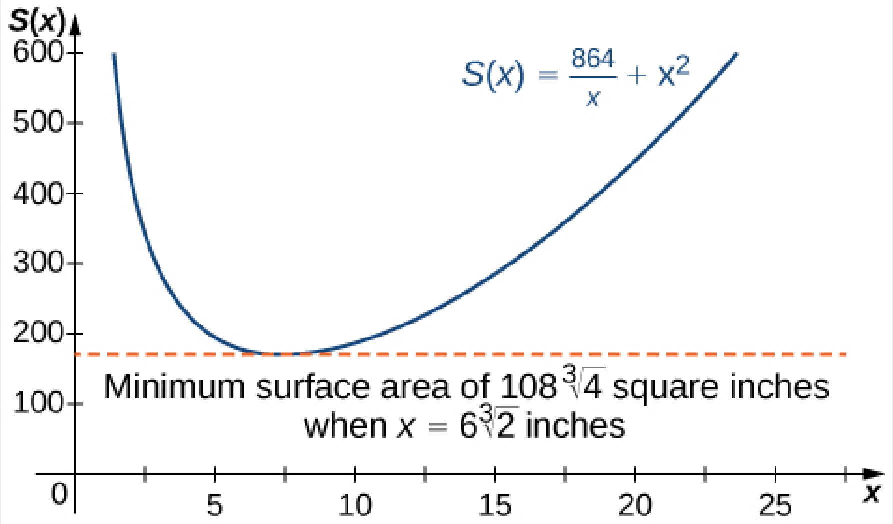
Equation:

$$S'(x) = -\frac{864}{x^2} + 2x.$$

Therefore, $S'(x) = 0$ when $2x = \frac{864}{x^2}$. Solving this equation for x, we obtain $x^3 = 432$, so $x = \sqrt[3]{432} = 6\sqrt[3]{2}$. Since this is the only critical point of S, the absolute minimum must occur at $x = 6\sqrt[3]{2}$ (see [link]). When $x = 6\sqrt[3]{2}$, $y = \frac{216}{\left(6\sqrt[3]{2}\right)^2} = 3\sqrt[3]{2}\,\text{in.}$ Therefore, the dimensions of the box should be $x = 6\sqrt[3]{2}\,\text{in.}$ and $y = 3\sqrt[3]{2}\,\text{in.}$ With these dimensions, the surface area is

Equation:

$$S\left(6\sqrt[3]{2}\right) = \frac{864}{6\sqrt[3]{2}} + \left(6\sqrt[3]{2}\right)^2 = 108\sqrt[3]{4}\,\text{in.}^2$$



We can use a graph to determine the dimensions of a box of given the volume and the minimum surface area.

Note:
Exercise:

Problem:

Consider the same open-top box, which is to have volume 216 in.^3. Suppose the cost of the material for the base is $20¢/\text{in.}^2$ and the cost of the material for the sides is $30¢/\text{in.}^2$ and we are trying to minimize the cost of this box. Write the cost as a function of the side lengths of the base. (Let x be the side length of the base and y be the height of the box.)

Solution:

$c\left(x\right)=\frac{259.2}{x}+0.2x^2$ dollars

Hint

If the cost of one of the sides is $30¢/\text{in.}^2$, the cost of that side is $0.30xy$.

Key Concepts

- To solve an optimization problem, begin by drawing a picture and introducing variables.
- Find an equation relating the variables.
- Find a function of one variable to describe the quantity that is to be minimized or maximized.
- Look for critical points to locate local extrema.

For the following exercises, answer by proof, counterexample, or explanation.

Exercise:

Problem:

When you find the maximum for an optimization problem, why do you need to check the sign of the derivative around the critical points?

Solution:

The critical points can be the minima, maxima, or neither.

Exercise:

Problem:

Why do you need to check the endpoints for optimization problems?

Exercise:

Problem:

True or False. For every continuous nonlinear function, you can find the value x that maximizes the function.

Solution:

False; $y = -x^2$ has a minimum only

Exercise:

Problem:

True or False. For every continuous nonconstant function on a closed, finite domain, there exists at least one x that minimizes or maximizes the function.

For the following exercises, set up and evaluate each optimization problem.

Exercise:

Problem:

To carry a suitcase on an airplane, the length $+\text{width}+$ height of the box must be less than or equal to $62\ \text{in}$. Assuming the height is fixed, show that the maximum volume is $V = h\left(31 - \left(\frac{1}{2}\right)h\right)^2$. What height allows you to have the largest volume?

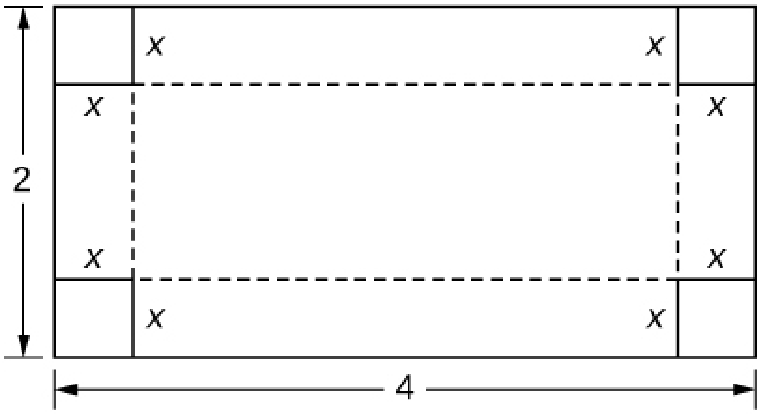
Solution:

$h = \frac{62}{3}$ in.

Exercise:

Problem:

You are constructing a cardboard box with the dimensions 2 m by 4 m. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume?



Exercise:

Problem:

Find the positive integer that minimizes the sum of the number and its reciprocal.

Solution:

1

Exercise:

Problem:

Find two positive integers such that their sum is 10, and minimize and maximize the sum of their squares.

For the following exercises, consider the construction of a pen to enclose an area.

Exercise:

Problem:

You have 400 ft of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize the area?

Solution:

100 ft by 100 ft

Exercise:

Problem:

You have 800 ft of fencing to make a pen for hogs. If you have a river on one side of your property, what is the dimension of the rectangular pen that maximizes the area?

Exercise:

Problem:

You need to construct a fence around an area of $1600\text{ ft}.$ What are the dimensions of the rectangular pen to minimize the amount of material needed?

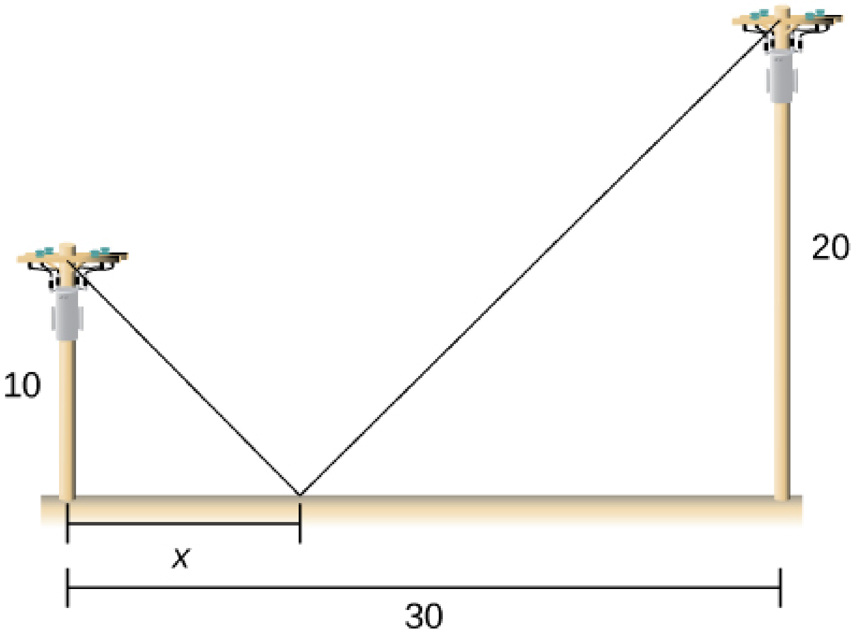
Solution:

40 ft by 40 ft

Exercise:

Problem:

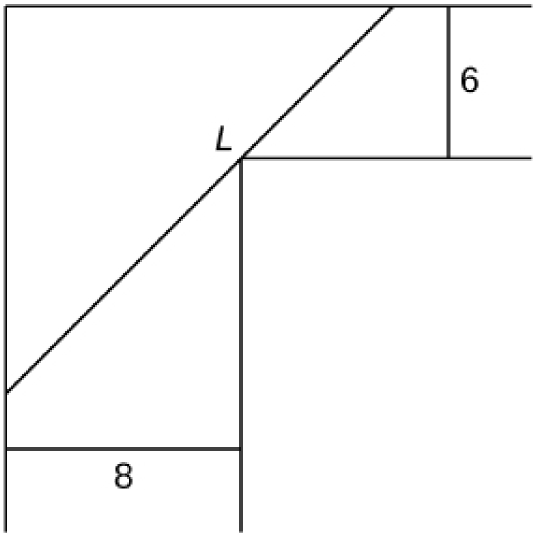
Two poles are connected by a wire that is also connected to the ground. The first pole is 20 ft tall and the second pole is 10 ft tall. There is a distance of 30 ft between the two poles. Where should the wire be anchored to the ground to minimize the amount of wire needed?



Exercise:

Problem:

[T] You are moving into a new apartment and notice there is a corner where the hallway narrows from 8 ft to 6 ft. What is the length of the longest item that can be carried horizontally around the corner?



Solution:

19.73 ft.

Exercise:

Problem:

A patient’s pulse measures 70 bpm, 80 bpm, then 120 bpm. To determine an accurate measurement of pulse, the doctor wants to know what value minimizes the expression $(x-70)^2+(x-80)^2+(x-120)^2$? What value minimizes it?

Exercise:

Problem:

In the previous problem, assume the patient was nervous during the third measurement, so we only weight that value half as much as the others. What is the value that minimizes $(x-70)^2+(x-80)^2+\frac{1}{2}(x-120)^2$?

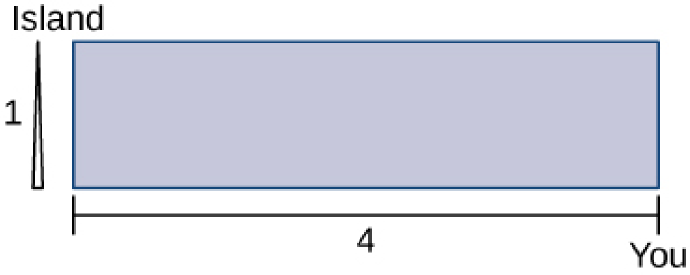
Solution:

84 bpm

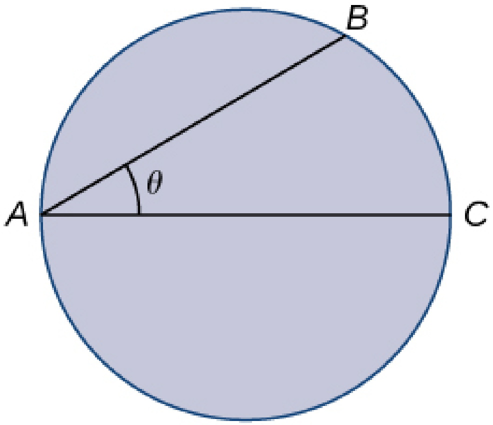
Exercise:

Problem:

You can run at a speed of 6 mph and swim at a speed of 3 mph and are located on the shore, 4 miles east of an island that is 1 mile north of the shoreline. How far should you run west to minimize the time needed to reach the island?



For the following problems, consider a lifeguard at a circular pool with diameter 40 m. He must reach someone who is drowning on the exact opposite side of the pool, at position C. The lifeguard swims with a speed v and runs around the pool at speed $w = 3v$.



Exercise:

Problem:

Find a function that measures the total amount of time it takes to reach the drowning person as a function of the swim angle, θ.

Solution:

$T\left(\theta\right) = \frac{40\theta}{3v} + \frac{40\cos\theta}{v}$

Exercise:

Problem:

Find at what angle θ the lifeguard should swim to reach the drowning person in the least amount of time.

Exercise:

Problem:

A truck uses gas as $g\left(v\right) = av + \frac{b}{v}$, where v represents the speed of the truck and g represents the gallons of fuel per mile. At what speed is fuel consumption minimized?

Solution:

$$v = \sqrt{\frac{b}{a}}$$

For the following exercises, consider a limousine that gets $m\left(v\right) = \frac{\left(120-2v\right)}{5}$ mi/gal at speed v, the chauffeur costs $\$15/\text{h}$, and gas is $\$3.5/\text{gal}$.

Exercise:

Problem: Find the cost per mile at speed v.

Exercise:

Problem: Find the cheapest driving speed.

Solution:

approximately $34.02\,\text{mph}$

For the following exercises, consider a pizzeria that sell pizzas for a revenue of $R\left(x\right) = ax$ and costs $C\left(x\right) = b + cx + dx^2$, where x represents the number of pizzas.

Exercise:

Problem:

Find the profit function for the number of pizzas. How many pizzas gives the largest profit per pizza?

Exercise:

Problem:

Assume that $R\left(x\right) = 10x$ and $C\left(x\right) = 2x + x^2$. How many pizzas sold maximizes the profit?

Solution:

4

Exercise:

Problem:

Assume that $R\left(x\right)=15x$, and $C\left(x\right)=60+3x+\frac{1}{2}x^2$. How many pizzas sold maximizes the profit?

For the following exercises, consider a wire $4\,\text{ft}$ long cut into two pieces. One piece forms a circle with radius r and the other forms a square of side x.

Exercise:

Problem: Choose x to maximize the sum of their areas.

Solution:

0

Exercise:

Problem: Choose x to minimize the sum of their areas.

For the following exercises, consider two nonnegative numbers x and y such that $x+y=10$. Maximize and minimize the quantities.

Exercise:

Problem: xy

Solution:

Maximal: $x=5, y=5$; minimal: $x=0, y=10$ and $y=0, x=10$

Exercise:

Problem: x^2y^2

Exercise:

Problem: $y-\frac{1}{x}$

Solution:

Maximal: $x = 1, y = 9$; minimal: none

Exercise:

Problem: $x^2 - y$

For the following exercises, draw the given optimization problem and solve.

Exercise:

Problem:

Find the volume of the largest right circular cylinder that fits in a sphere of radius 1.

Solution:

$\frac{4\pi}{3\sqrt{3}}$

Exercise:

Problem:

Find the volume of the largest right cone that fits in a sphere of radius 1.

Exercise:

Problem:

Find the area of the largest rectangle that fits into the triangle with sides $x = 0, y = 0$ and $\frac{x}{4} + \frac{y}{6} = 1$.

Solution:

6

Exercise:

Problem:

Find the largest volume of a cylinder that fits into a cone that has base radius R and height h.

Exercise:

Problem:

Find the dimensions of the closed cylinder volume $V = 16\pi$ that has the least amount of surface area.

Solution:

$r = 2, h = 4$

Exercise:

Problem:

Find the dimensions of a right cone with surface area $S = 4\pi$ that has the largest volume.

For the following exercises, consider the points on the given graphs. Use a calculator to graph the functions.

Exercise:

Problem: **[T]** Where is the line $y = 5 - 2x$ closest to the origin?

Solution:

$(2, 1)$

Exercise:

Problem: **[T]** Where is the line $y = 5 - 2x$ closest to point $(1, 1)$?

Exercise:

Problem: **[T]** Where is the parabola $y = x^2$ closest to point $(2, 0)$?

Solution:

$(0.8351, 0.6974)$

Exercise:

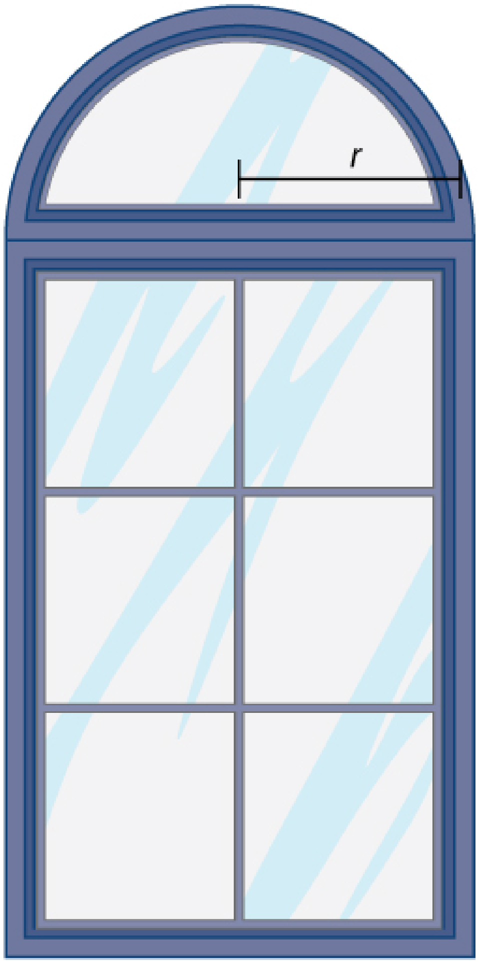
Problem: **[T]** Where is the parabola $y = x^2$ closest to point $(0, 3)$?

For the following exercises, set up, but do not evaluate, each optimization problem.

Exercise:

Problem:

A window is composed of a semicircle placed on top of a rectangle. If you have 20 ft of window-framing materials for the outer frame, what is the maximum size of the window you can create? Use r to represent the radius of the semicircle.



Solution:

$$A = 20r - 2r^2 - \frac{1}{2}\pi r^2$$

Exercise:

Problem:

You have a garden row of 20 watermelon plants that produce an average of 30 watermelons apiece. For any additional watermelon plants planted, the output per watermelon plant drops by one watermelon. How many extra watermelon plants should you plant?

Exercise:

Problem:

You are constructing a box for your cat to sleep in. The plush material for the square bottom of the box costs $\$5/\text{ft}^2$ and the material for the sides costs $\$2/\text{ft}^2$. You need a box with volume $4\ \text{ft}^2$. Find the dimensions of the box that minimize cost. Use x to represent the length of the side of the box.

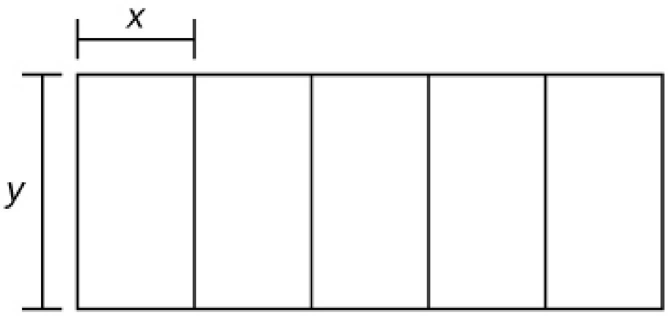
Solution:

$C(x) = 5x^2 + \frac{32}{x}$ Differentiating, setting the derivative equal to zero and solving, we obtain $x = \sqrt[3]{\frac{2}{5}}$ and $h = \sqrt[3]{\frac{25}{4}}$.

Exercise:

Problem:

You are building five identical pens adjacent to each other with a total area of $1000\ \text{m}^2$, as shown in the following figure. What dimensions should you use to minimize the amount of fencing?



Exercise:

Problem:

You are the manager of an apartment complex with 50 units. When you set rent at $\$800/\text{month}$, all apartments are rented. As you increase rent by $\$25/\text{month}$, one fewer apartment is rented. Maintenance costs run $\$50/\text{month}$ for each occupied unit. What is the rent that maximizes the total amount of profit?

Solution:

$P\left(x\right)=\left(50-x\right)\left(800+25x-50\right)$

Glossary

optimization problems

problems that are solved by finding the maximum or minimum value of a function